Genericity Theory from the Randomness Viewpoint

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Genericity and Randomness

**Definition**

A real $x \in 2^\omega$ is

(i) *weakly $n$-generic* if $x \not\in A$ for all $\Pi^0_n$ meager set $A$;

(ii) *weakly $n$-random* if $x \not\in A$ for all $\Pi^0_n$ null set $A$.

**Definition**

A real $x \in 2^\omega$ is

(i) *$n$-generic* if either $x \in A$ or there exists a finite string $\sigma \prec x$ so that $[\sigma] \cap A = \emptyset$ for all $\Sigma^0_n$ open set $A$;

(ii) *$n$-random* if $x \not\in A$ for set $A$ which is the intersection of a $\Sigma^0_n$-test.
Effective Category and Measure Theory

**Theorem**

Fix a universal $\Sigma^0_n$ set $A \subseteq \omega \times 2^\omega$, then:

1. (Sacks) $\{(i, n) | \mu(A_i) > 2^{-n}\}$ is $\Sigma^0_n$.
2. (Kechris) $\{i | A_i$ is not meager $\}$ is $\Sigma^0_n$.

**Theorem**

For every $\Sigma^0_n$ set $A \subseteq 2^\omega$,

1. (Kurtz) there is a recursive sequence of $\Sigma^0_{n-1}$ closed sets $\{F_m\}_m$ so that $\bigcup_n F_m \subseteq A$ and $\mu(\bigcup_m F_m) = \mu(A)$.
2. (Forklore) $A$ is comeager, then there is an recursive sequence of $\Sigma^0_n$ dense open sets $\{U_m\}_m$ so that $\bigcap_m U_m \subseteq A$. 

Uniformizing the Notions

Corollary (Kurtz)

A real \( x \in 2^\omega \) is

1. weakly \( n \)-generic iff \( x \in U \) for every \( \Sigma^0_n \) dense open set \( U \).
2. weakly \( n + 1 \)-random iff \( x \notin \bigcap_m U_m \) for all recursive sequence of \( \Sigma^0_n \) open sets \( \{U_m\}_m \) with \( \lim_m \mu(U_m) = 0 \).
3. \( n \)-random iff \( x \notin \bigcap_m U_m \) for all recursive sequence of \( \Sigma^0_n \) open sets \( \{U_m\}_m \) with \( \mu(U_m) \leq 2^{-m} \).

So weak \( n + 1 \) randomness \( \implies \) \( n \)-randomness \( \implies \) weak \( n \)-randomness and weak \( n + 1 \)-genericity \( \implies \) \( n \)-genericity \( \implies \) weak \( n \)-genericity
Definition

- A Turing degree is hyperimmune if it contains a function not dominated by any recursive functions. Otherwise, it is hyperimmune-free.
- A degree is recursively traceable if every function computed by it can be traced by a recursive function with identity bound.
- A Turing degree is \textit{DNR} if it contains a function $f$ so that $\forall n (f(n) \neq \Phi_n(n))$.
- A Turing degree is \textit{PA} if it contains a real computing a completion of Peano’s Axioms.
Theorem

1. (Forklore) Every weakly 1-generic real is weakly 1-random.
2. (Kurtz+Jockusch) $x$ has a weakly 1-generic degree iff it has a hyperimmune degree, and every 1-generic real is REA.
   (DNWY+Hirschfeldt, Miller) $x$ has a weakly 2-random degree iff it has a 1-random degree and is incomparable with all of nonrecursive $\Delta^0_2$-degrees.
   (Kurtz) Every 2-random real is REA.
3. (Forklore) Every 1-generic degree is $GL_{1_1}$.
   (Kautz) Every 2-random degree is $GL_{1_1}$.
   (Kucera) If $x \geq_T \emptyset'$, then it has a 1-random degree.
Characterizing Low Levels II

Theorem

- (Forklore) No 1-generic real has DNR-degree.
- (Forklore) Every 1-random degree is a DNR-degree.
- (Stephan) If $x$ is 1-random, then $x$ has a PA degree iff $x \geq_T \emptyset'$.  
- (Yu) A real $x$ is hyperimmune-free, then $x$ is weakly 1-random iff $x$ is weakly 2-random.

Question

Characterizing weakly 1-random degrees.
### Kolmogorov Complexity vs Randomness

#### Theorem

1. *(Schnorr)* $x$ is 1-random iff there is a constant $c$ so that $\forall n(K(x \upharpoonright n) \geq n - c)$.

2. *(Miller and Yu)* $x$ is 1-random iff for every computable function $g$ with $\sum_n 2^{-g(n)} < \infty$, there is a constant $c$ so that $\forall n(C(x \upharpoonright n) \geq n - g(n) - c)$.

3. *(Miller and Yu)* $x \oplus y$ is 1-random iff there is a constant so that $\forall n(K(x \upharpoonright n) + C(y \upharpoonright n) \geq 2n - c)$.

4. *(NST+Miller)* $x$ is 2-random iff there is a constant $c$ so that $\exists \infty n(C(x \upharpoonright n) \geq n - c)$.
Theorem

1. (Nies) There exists a $K$-trivial 1-generic real.
2. (Forklore) $x$ is weakly 2-generic then $x$ is $K$-“very low” and $K$-random infinitely often.

Question

Finding out a complexity characterization of weak 2-randomness and genericity.
**Lowness for Randomness and Genericity I**

<table>
<thead>
<tr>
<th>Definition</th>
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<tr>
<td>Given a notion $G$ and its relativization $G^x$, a real $x$ is low for $G$ if $G = G^x$.</td>
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<td>1. <em>(Stephan and Yu)</em> A real is low for weakly 1-random then it must have a degree strictly between recursively traceability and hyperimmune-freeness.</td>
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<td>2. <em>(Stephan and Yu)</em> A real is low for weakly 1-generic iff it hyperimmune-free and non-DNR.</td>
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Lowness for Randomness and Genericity II

**Theorem**

1. *(Hirschfeldt and Nies)* A real \( x \) is low for 1-random iff it is there exists a constant \( c \) so that \( \forall n(K(x \upharpoonright n) \leq K(n) + c) \).

2. *(Greenberg, Miller + Yu)* A real is low for 1-generic iff \( x \) is recursive.

3. *(DNWY+Miller + Nies)* A real \( x \) is low for weakly 2-random iff it is there exists a constant \( c \) so that \( \forall n(K(x \upharpoonright n) \leq K(n) + c) \).
Forcing vs Genericity and Randomness

**Definition**
Let \((P_n, \leq)\) be a forcing notation where
\[ P_n = \{ A \subseteq 2^\omega | A \in \Pi^0_n \land \mu(A) > 0 \} \] and \(\leq = \subseteq\). \(A \Vdash \varphi\) if \(\varphi(x)\) is true for all \(x \in A\). \(x\) is Solovay \(n\)-generic if for every \(\Pi^0_n\)-formula, there is a condition \(x \in A\), \(A\) decides \(\varphi\).

**Theorem**
1. *(Jockusch)* \(x\) is \(n\)-generic iff \(x\) forces all of \(\Sigma^0_n\) sentences in the Cohen forcing sense.
2. *(Kurtz)* \(x\) is weakly \(n\)-random iff \(x\) is Solovay \(n\)-generic.
Van Lambalgen’s Theorem

Theorem

1. (van Lambalgen) \( x \oplus y \) is \( n \)-random iff \( x \) is \( n \)-random and \( y \) is \( n \)-\( x \)-random.

2. (Forklore) \( x \oplus y \) is \( n \)-generic iff \( x \) is \( n \)-generic and \( y \) is \( n \)-\( x \)-generic.
Relativized Randomness

Theorem

1. (Miller and Yu) For any real $z$ and 1-random reals $x \leq_T y$, if $y$ is 1-$z$-random then $x$ is 1-$z$-random.

2. (CDGHM) M-Y Theorem holds for $n$-genericity if $n \geq 2$, but fails for 1-genericity.
**Definition**

Given a class of sets of reals $T$,

1. A real $x$ is $T$-random if $x$ is not in any null set in $T$.
2. A real $x$ is $T$-generic if $x$ is in every dense set in $T$ where $T$ is also a class of open sets.

**Theorem**

1. (Sacks+Hjorth, Nies+Chong, Nies, Yu) $\Pi^1_1$-randomness $\subset \Pi^1_1$-Martin-Löf randomness $\subset \Delta^1_1$-randomness $= \Delta^1_1$-Martin-Löf randomness.
2. $\Pi^1_1$-genericity $= \Delta^1_1$-genericity.
Traceability

**Definition**

(i) Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A $\Pi^1_1$-trace/$\Delta^1_1$-trace with bound $h$ is a uniformly $\Pi^1_1$/uniformly $\Delta^1_1$ sequence $(T_e)_{e \in \omega}$ such that $|T_e| \leq h(e)$ for each $e$.

(ii) $A \subseteq \omega$ is $\Pi^1_1$-traceable/$\Delta^1_1$-traceable if there is $h \in \Delta^1_1$ such that, for each $f \leq_h A$, there is a $\Pi^1_1$-trace/$\Delta^1_1$-trace with bound $h$ such that, for each $e$, $f(e) \in T_e$.

**Proposition (Chong, Nies and Yu)**

*If $x$ is $\Pi^1_1$-traceable, then $x$ is $\Delta^1_1$-traceable.*
Lowness properties

Theorem

1. (Chong, Nies and Yu) Lowness for \( \Delta^1_1 \) randomness
   \( = \Delta^1_1 \)-traceability.

2. (Hjorth and Nies) Lowness for \( \Pi^1_1 \)-Martin-Löf randomness
   \( = \) Hyperarithmetic.

3. (Harrington, Nies and Slaman) Lowness for \( \Pi^1_1 \)-randomness
   \( = \) Lowness for \( \Delta^1_1 \) randomness +
   non-random-cuppable.

4. (Yu) Lowness for \( \Delta^1_1 \)-genericity \( \supseteq \) \( \Delta^1_1 \)-traceability.
Beyond Recursion Theory

Theorem

Assume PD if $n \geq 1$.

- (Kechris) There exists a largest $\Pi^{1}_{2n+1}$ and $\Sigma^{1}_{2n}$ null set.
- (Kechris) There exists a largest $\Pi^{1}_{2n+1}$ and $\Sigma^{1}_{2n}$ meager set.
- (Sacks+Tanaka+Kechris) Each non-null $\Pi^{1}_{2n+1}$ set contains a $\Delta^{1}_{2n}$ real.
- (Hinman+Kechris) Each non-meager $\Pi^{1}_{2n+1}$ set contains a $\Delta^{1}_{2n}$ real.
Some questions

Question

1. How far can genericity and randomness theory go under PD?
2. Finding out an inner model to develop higher genericity and randomness theory.
Thank you