# SOME QUESTIONS

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ABSTRACT. These are some questions I am every interested in. Your solution to any question in this draft would be welcome. I keep updating this draft.

## 1. Classical recursion theory

We use $\mathcal{R}$ to denote the partial ordering of r.e. degrees and $\mathcal{D}_n$ to denote the partial ordering of $n$-r.e. degrees. Yang and I proved that $\mathcal{R}$ is not a $\Sigma_1$-substructure of $\mathcal{D}_n$.

**Question 1.1** (Khoussainov)**.** *For $n > 1$, is there a function $f : R \to D_n$ so that for any $\Sigma_1$-formula $\varphi(x_1, ..., x_m)$,*

$$\mathcal{D}_1 \models \varphi(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_m}) \text{ iff } \mathcal{D}_n \models \varphi(f(\mathbf{x_1}), f(\mathbf{x_2}), ..., f(\mathbf{x_m})),$$

*where $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_m}$ range over $D_1$?*

**Question 1.2.** *Does there exist a construction of minimal Turing degree without using perfect set forcing?*

Cooper proved that for each $\mathbf{x} \geq_T \emptyset'$, there is a minimal degree $\mathbf{m'} = \mathbf{x}$. Chong and I proved that for each countable set $A$ of reals and a real $x$, there is a minimal cover $z$ of $A$ so that $z'' \geq_T x$.

**Question 1.3.** *Is it true that for each countable set $A$ of reals and a real $x$, there is a minimal cover $z$ of $A$ so that $z' \geq_T x$?*

By a usual Skolem argument, one can show that there is a countable elementary substructure of Turing degrees. Wang and I also proved that each non-principal ideal is a $\Sigma_1$-substructure of r.e. degrees.

**Question 1.4.** *Is there a proper $\Sigma_1$-substructure of $\Delta^0_2$-degrees?*

*Note on 4 July 2006: Slaman pointed out that there does exist one. Roughly speaking, one can embed a countable partial order which satisfies Shoenfield conjecture into $\Delta^0_2$-degrees. But the construction does not give much information about the structure of $\Delta^0_2$-degrees.*

**Question 1.5.** *Is there a reasonable $\Sigma_1$-substructure of $\Delta^0_2$-degrees? Furthermore, for any $\Pi^1_1$-path $T$ through $\mathcal{O}$, is the set $\{\mathbf{a} | \exists n \in T \exists A \in \mathbf{a}(A \in \Sigma^{-1}_n)\}$ a $\Sigma_1$-substructure of $\Delta^0_2$-degrees? Here $\Sigma^{-1}_n$ is the Ershov hierarchy at the level $n$.*

We say a real $x$ is r.e.a some real $y$ if $x$ is $y$-r.e. set and $x >_T y$. It is easy to prove that if a real is r.e.a some real then it is Turing equivalent to a join of two 1-generic reals.

*Date*: June 5, 2008.

**Question 1.6.** *A Turing degree $x$ is r.e.a in some degree $y$ iff there are two 1-generic degrees $g_1$ and $g_2$ so that $x \equiv_T g_1 \oplus g_2$?*

*Note on June 5, 2008: Some progress has been made towarding to solving this question. Ambos-Spies, Ding, Wang and I prove that every degree computing a non-$GL_2$ is REA. Slaman proves that every degree computing a 2-genenric is REA. Shore uniformizes both results by showing that every ANR degree is REA. So it seems that the conjecture holds for all the "natural examples", though the question is still open.*

## 2. Higher recursion theory

Each non-recursive degree cups a 1-generic degree above $\mathbf{0}'$(Posner and Robinson) but there is a non-recursive degree cups no 1-random degree above $\mathbf{0}'$(Nies).

**Question 2.1.** *Can each non-hyperarithmetic degree cup a minimal hyper-degree above $\mathcal{O}$?*

## 3. Randomness theory

**Question 3.1.** *For every $Z \in 2^\omega$, is there a 1-random real $X \geq_K Z$?*

**Question 3.2.** *For 1-random reals $X, Y \in 2^\omega$, does $X \equiv_K Y$ imply $X \equiv_T Y$?*

*Note on 6 Oct, 2006: Frank negatively answered this question based on Nies' results. Just considering a hyperimmune-free 1-random set $X$ and a nonrecursive set $Z$ which is low for $K$. Take $Y = X \triangle Z$. Then $X$ and $Y$ are Turing incomparable but $K$-equivalent.*

Note it is known that $X \geq_K Y$ does not imply $X \geq_T Y$ although both are 1-random reals

**Question 3.3.** *Are there maximal (1-random) $K$-degrees?*

The questions above can also be asked for $\leq_C$. In addition, very little known about the relationship between $\leq_K$ and $\leq_C$. One thing that is known is that $X \equiv_K Y$ does not, in general, imply $X \equiv_C Y$.[1] Other basic questions remain open.

**Question 3.4.** *Does $X \leq_C Y$ imply $X \leq_K Y$?*

**Question 3.5.** *Do $\leq_K$ and $\leq_C$ differ for 1-random reals?*

Rettingen proved that all of random d.c.e reals have the same $K$-degree.

**Question 3.6.** *What's the maximal field in which all of random reals have the same $K$-degree?*

Note by Zorn Lemma, there must be a maximal field in which all of random reals have the same $K$-degree. I want to know whether there is a natural one.

A real $x$ is said to be Kurtz random if $x \in U$ for all $\Sigma^0_1$ set $U$ with $\mu(U) = 1$. Stephan and I prove that lowness for Kurtz random is strictly between hyperimmune-freeness and recursive tracebility.

**Question 3.7.** *Finding out a characterization of lowness for Kurtz randomness.*

[1]By the work of Solovay, Downey, Hirschfeldt, Nies and Stephan, there exists a non-recursive real $X$ so that $X \equiv_K 0^\omega$. However, Chaitin prove that if $X \equiv_C 0^\omega$, then $X$ is recursive. Hence $X \not\equiv_C 0^\omega$.

*Note on June 5, 2008: Miller and Greenberg prove that every low for Kurtz randomness real is not DNR. So this question was answered.*

Ding and I proved that no largest *sw*-degree in r.e. reals.

**Question 3.8.** *Is there a maximal sw-degree in r.e. reals? Is every r.e. random sw-degree maximal in r.e. reals?*

## 4. Descriptive set theory aspects of recursion theory

Yu has constructed a non-measurable antichain using randomness theory. But some interesting questions left. Given a set $X \subseteq 2^\omega$, define $\mathcal{U}(x) = \{y \mid \exists x \in X(y \geq_T x)\}$.

**Question 4.1** (Jockusch)**.** *Does there exist an antichain $X$ in the Turing degrees for which $\mu(X) = 0$ and $\mu(\mathcal{U}(X)) = 1$?*

Yu proved that if $\mu(\mathcal{U}(X)) = 1$ then $\mu(X) = 0$.

**Question 4.2.** *Is it true that for any locally countable $\Pi^1_1$ partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$?*

Yu proved that for any locally countable $\Sigma^1_1$ partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$

We say that a set $X \subset 2^\omega$ is a *quasi-antichain* in the Turing degrees if it satisfies the following properties:

(1) $\forall x \in X \forall y (x \equiv_T y \implies y \in X)$.
(2) $\forall x \in X \forall y \in X (x \not\equiv_T y \implies x \not\leq_T y)$.

It is not hard to see that there is a nonmeasurable quasi-antichain in the Turing degrees.

**Question 4.3** (Jockusch)**.** *Is every maximal quasi-antichain in the Turing degrees nonmeasurable?*

*Note on 8 May 2006: Chong and I negatively answered this question under the assumption CH.*

**Question 4.4.** *Is there a $\Sigma^1_1$-maximal antichain in the Turing degrees? Even is there a perfect antichain in the Turing degrees?*

## 5. set theory

Terwijn asked whether it is a theorem of $ZFC$ that there is a chain of size $2^{2^{\aleph_0}}$ in the Medvedev degrees. This problem can be reduced to a pure set theory question which is related a classical set theory problem, generalized Kurepa-trees.

**Question 5.1.** *Is there a model of $ZFC$ in which $2^{\aleph_2} > 2^{\aleph_1} > \aleph_2 = 2^{\aleph_0}$ and there is no a binary tree of size $\aleph_2$ having $2^{\aleph_2}$-many branches.*