Covering the integers by arithmetic sequences

by

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1. Introduction. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^+$ the set of positive reals. For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ we call

$$\alpha + \beta \mathbb{Z} = \{ \ldots, \alpha - 2\beta, \alpha - \beta, \alpha, \alpha + \beta, \alpha + 2\beta, \ldots \}$$

an arithmetic sequence with common difference $\beta$. In the case $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^+$, $\alpha + \beta \mathbb{Z}$ is just the residue class $\alpha \mod \beta$ with modulus $\beta$.

Let $m$ be a positive integer. A finite system

$$(1) \quad A = \{ \alpha_s + \beta_s \mathbb{Z} \}_{s=1}^k \quad (\alpha_1, \ldots, \alpha_k \in \mathbb{R} \text{ and } \beta_1, \ldots, \beta_k \in \mathbb{R}^+)$$

of arithmetic sequences is said to be an (exact) $m$-cover of $\mathbb{Z}$ if it covers each integer at least (resp., exactly) $m$ times. Instead of "1-cover" and "exact 1-cover" we use the terms "cover" and "exact cover" respectively.

Since they were introduced by P. Erdős ([5]) in the early 1930's, covers of $\mathbb{Z}$ by (finitely many) residue classes have been studied seriously and many nice applications have been found. (Cf. sections A19, B21, E23, F13 and F14 of R. K. Guy [9].) For problems and results in this area we refer the reader to surveys of Erdős [7, 8], Š. Porubský [13] and Š. Znám [21]. Recently further progress was made by various authors.

If a finite system

$$(2) \quad A = \{ a_s + n_s \mathbb{Z} \}_{s=1}^k \quad (a_1, \ldots, a_k \in \mathbb{Z} \text{ and } n_1, \ldots, n_k \in \mathbb{Z}^+)$$

of residue classes forms an $m$-cover of $\mathbb{Z}$, then $\sum_{s=1}^k 1/n_s \geq m$, and the equality holds if and only if (2) is an exact $m$-cover of $\mathbb{Z}$. This becomes apparent if we calculate

$$\sum_{s=1}^k |\{ 0 \leq x < N : x \equiv a_s \mod n_s \}| = \sum_{x=0}^{N-1} |\{ 1 \leq s \leq k : x \equiv a_s \mod n_s \}|$$

where $N$ is the least common multiple of $n_1, \ldots, n_k$.

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In this paper we investigate properties of $m$-covers of $\mathbb{Z}$ in the form (1). In the next section we shall give three equivalent conditions for (1) to be an $m$-cover of $\mathbb{Z}$. One is that (1) covers $W$ consecutive integers at least $m$ times where

$$W = \left| \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : I \subseteq \{1, \ldots, k\} \right\} \right|$$

([x] and \{x\} stand for the integral and fractional parts of a real $x$ respectively throughout the paper), the other two are finite systems of equalities (not inequalities) involving roots of unity. Our tools used to deduce them include Vandermonde determinants, Stirling numbers, a little analysis and linear algebra.

In Sections 3 and 4 we will derive a number of results including the following ones:

(I) Let (1) be an $m$-cover of $\mathbb{Z}$ and $J \subseteq \{1, \ldots, k\}$. Then

$$\left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\}$$

for some $I \subseteq \{1, \ldots, k\}$ with $I \neq J$,

provided $\sum_{s=1}^k 1/\beta_s = m$ (e.g. (1) is an exact $m$-cover of $\mathbb{Z}$ with $\alpha_s \in \mathbb{Z}$ and $\beta_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$) we have $\sum_{s \in I} 1/\beta_s = \sum_{s \in J} 1/\beta_s$ for some $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ if $\emptyset \neq J \subseteq \{1, \ldots, k\}$, when $\sum_{s \in I} 1/\beta_s = \sum_{s \in J} 1/\beta_s$ for no $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ there are at least $m$ nonzero integers of the form $\sum_{s \in I} 1/\beta_s - \sum_{s \in J} 1/\beta_s$ where $I \subseteq \{1, \ldots, k\}$.

(II) Let $k \geq l \geq 0$ be integers. Then $2^{k-l}(l+1)$ is the smallest $n \in \mathbb{Z}^+$ such that any system of $k$ arithmetic sequences with at least $l$ equal common differences covers an arithmetic sequence at least $m$ times if it covers $n$ consecutive terms in the sequence at least $m$ times.

The last section contains some unsolved problems related to possible extensions.

2. Characterizations of $m$-covers. Let us provide several technical lemmas the first of which serves as the starting point of our new approach.

**Lemma 1.** Let $m \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Then (1) covers $x$ at least $m$ times if and only if

$$\prod_{s=1}^k (1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x)/\beta_s}) = o((1 - r)^{m-1}) \quad (r \to 1).$$

**Proof.** Set $I = \{1 \leq s \leq k : x \in \alpha_s + \beta_s \mathbb{Z}\}$ and $I' = \{1, \ldots, k\} \setminus I$. 

Clearly,
\[
\lim_{r \to 1} \frac{\prod_{s=1}^{k} \left(1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x) / \beta_s}\right)}{(1 - r)^{|I|}} = \lim_{r \to 1} \prod_{s \in I'} \left(1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x) / \beta_s}\right) \cdot \lim_{r \to 1} \prod_{s \in I} \frac{1 - r^{1/\beta_s}}{1 - r} = \prod_{s \in I'} \left(1 - e^{2\pi i (\alpha_s - x) / \beta_s}\right) \cdot \prod_{s \in I} \frac{d}{dr} \left(r^{1/\beta_s}\right)_{r=1} = \prod_{s \in I'} \left(1 - e^{2\pi i (\alpha_s - x) / \beta_s}\right) \cdot \prod_{s \in I} \beta_s^{-1} \neq 0,
\]
and hence
\[
\lim_{r \to 1} \frac{\prod_{s=1}^{k} \left(1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x) / \beta_s}\right)}{(1 - r)^{|I| - m + 1}} = \begin{cases} 0 & \text{if } |I| > m - 1, \\ \infty & \text{if } |I| < m - 1. \end{cases}
\]

Now it is apparent that $|I| \geq m$ if and only if (3) holds. We are done.

**Lemma 2.** Let $\theta_1, \ldots, \theta_n$ be real numbers with distinct fractional parts. For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if
\[
\left|\sum_{t=1}^{n} e^{2\pi i s \theta_t} x_t\right| < \delta
\]
for every $s = 1, \ldots, n$ then $|x_t| < \varepsilon$ for all $t = 1, \ldots, n$.

**Proof.** Let $A$ be the matrix $(e^{2\pi i s \theta_t})_{1 \leq s, t \leq n}$. Then
\[
\frac{|A|}{e^{2\pi i \theta_1} e^{2\pi i \theta_2} \ldots e^{2\pi i \theta_n}} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i \theta_1} & e^{2\pi i \theta_2} & \cdots & e^{2\pi i \theta_n} \\ (e^{2\pi i \theta_1})^2 & (e^{2\pi i \theta_2})^2 & \cdots & (e^{2\pi i \theta_n})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (e^{2\pi i \theta_1})^{n-1} & (e^{2\pi i \theta_2})^{n-1} & \cdots & (e^{2\pi i \theta_n})^{n-1} \end{vmatrix}
\]
is a determinant of Vandermonde's type. As $|A| \neq 0$ the inverse matrix of $A$ exists; we denote it by $B = (b_{st})_{1 \leq s, t \leq n}$.

Let $b = \max\{|b_{st}| : s, t = 1, \ldots, n\} > 0$ and $\delta = \varepsilon/(bn)$. Let $x_1, \ldots, x_n$
be any complex numbers, and set
\[ y_s = \sum_{t=1}^{n} e^{2\pi is\theta_t} x_t \quad \text{for } s = 1, \ldots, n. \]

Let
\[ \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \]

Then \( \bar{x} = B \bar{A} \bar{x} = B \bar{y} \). If \( |y_s| < \delta \) for every \( s = 1, \ldots, n \), then
\[ |x_s| = \left| \sum_{t=1}^{n} b_{st} y_t \right| \leq \sum_{t=1}^{n} b_t |y_t| < bn\delta = \epsilon \quad \text{for all } s = 1, \ldots, n. \]

This concludes the proof.

**Lemma 3.** Let \( m \in \mathbb{Z}^+ \). Then
\[ \sum_{n=0}^{m-1} a_n t^{n-m+1} = o(1) \quad (t \to 0) \]
if and only if \( a_0 = \ldots = a_{m-1} = 0 \).

**Proof.** The "if" direction is trivial. When \( a_0, \ldots, a_{m-1} \) are not all zero, for the least \( k \) such that \( a_k \neq 0 \) we have
\[ \sum_{n=0}^{m-1} a_n (x^{-1})^{n-m+1} = \sum_{n=k}^{m-1} a_n x^{m-1-n} \sim a_k x^{m-1-k} \quad (x \to \infty), \]
which contradicts (4). This ends the proof.

**Lemma 4.** Let \( n \geq m > 0 \) be integers and \( a_1, \ldots, a_n \) distinct numbers. Then the system
\[ \begin{align*}
\quad x_1 + \ldots + x_n &= 0, \\
\quad a_1 x_1 + \ldots + a_n x_n &= 0, \\
\quad a_1^2 x_1 + \ldots + a_n^2 x_n &= 0, \\
\phantom{a_1^2 x_1} &\quad \ldots \ldots \ldots \ldots \ldots \ldots \\
\quad a_1^{m-1} x_1 + \ldots + a_n^{m-1} x_n &= 0,
\end{align*} \]

is equivalent to
\[ \begin{align*}
\quad a_{11} x_1 + \ldots + a_{1n} x_n &= 0, \\
\quad a_{21} x_1 + \ldots + a_{2n} x_n &= 0, \\
\phantom{a_{21} x_1} &\quad \ldots \ldots \ldots \ldots \ldots \ldots \\
\quad a_{m1} x_1 + \ldots + a_{mn} x_n &= 0,
\end{align*} \]
where
\[ a_{st} = \prod_{\substack{i=1 \atop i \neq s}}^{m} \frac{a_i - a_t}{a_i - a_s} \quad \text{for } s = 1, \ldots, m \text{ and } t = 1, \ldots, n. \]

Proof. Rewrite (5) in the form
\[
\begin{aligned}
    x_1 + \ldots + x_m &= - \sum_{m < t \leq n} x_t, \\
    a_1 x_1 + \ldots + a_m x_m &= - \sum_{m < t \leq n} a_t x_t, \\
    a_1^2 x_1 + \ldots + a_m^2 x_m &= - \sum_{m < t \leq n} a_t^2 x_t, \\
    \vdots \\
    a_1^{m-1} x_1 + \ldots + a_m^{m-1} x_m &= - \sum_{m < t \leq n} a_t^{m-1} x_t.
\end{aligned}
\]

By Cramer's rule, this says that
\[
x_s = \begin{vmatrix}
1 & \ldots & 1 & -\sum_{m < t \leq n} x_t & 1 & \ldots & 1 \\
1 & \ldots & a_{s-1} & -\sum_{m < t \leq n} a_t x_t & a_{s+1} & \ldots & a_m \\
a_1 & \ldots & a_{s-1} & -\sum_{m < t \leq n} a_t^2 x_t & a_{s+1}^2 & \ldots & a_m^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_1^{m-1} & \ldots & a_{s-1}^{m-1} & -\sum_{m < t \leq n} a_t^{m-1} x_t & a_{s+1}^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}^{-1}
\times
\begin{vmatrix}
1 & \ldots & 1 \\
1 & \ldots & a_m \\
a_1 & \ldots & a_m^2 \\
\vdots & \vdots & \vdots \\
a_1^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}
\]

\[
= - \sum_{m < t \leq n} x_t
\begin{vmatrix}
1 & \ldots & 1 & 1 & \ldots & 1 \\
1 & \ldots & a_{s-1} & a_t & a_{s+1} & \ldots & a_m \\
1 & \ldots & a_{s-1}^2 & a_t^2 & a_{s+1}^2 & \ldots & a_m^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \ldots & a_{s-1}^{m-1} & a_t^{m-1} & a_{s+1}^{m-1} & \ldots & a_m^{m-1}
\end{vmatrix}^{-1}
\times
\begin{vmatrix}
1 & \ldots & 1 \\
1 & \ldots & a_m \\
1 & \ldots & a_m^2 \\
\vdots & \vdots & \vdots \\
1 & \ldots & a_m^{m-1}
\end{vmatrix}
\]
\[
\prod_{1 \leq i < s} (a_t - a_i) \cdot \prod_{s < i \leq m} (a_i - a_t) \cdot \prod_{1 \leq i < j \leq m, i, j \neq s} (a_j - a_i) \\
= - \sum_{m < t \leq n} x_t \prod_{1 \leq i < s} (a_s - a_i) \cdot \prod_{s < i \leq m} (a_i - a_s) \cdot \prod_{1 \leq i < j \leq m, i, j \neq s} (a_j - a_i) \\
= - \sum_{m < t \leq n} a_{st} x_t \quad \text{(Vandermonde)}
\]

for every \( s = 1, \ldots, m \), i.e.

\[
\sum_{t=1}^{m} \delta_{st} x_t + \sum_{m < t \leq n} a_{st} x_t = 0 \quad \text{for } s = 1, \ldots, m
\]

where \( \delta_{st} \) is the Kronecker delta. Since \( a_{st} = \delta_{st} \) for \( s, t = 1, \ldots, m \), we have finished the proof.

Now we are ready to present

**THEOREM 1.** Let \( A = \{ \alpha_s + \beta_s \mathbb{Z} \}_{s=1}^{k} \), where \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \) and \( \beta_1, \ldots, \beta_k \in \mathbb{R}^+ \). Let \( m \in \mathbb{Z}^+ \) and

\[
S = \left\{ 0 \leq \theta < 1 : \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \theta \text{ for some } I \subseteq \{1, \ldots, k\} \right\}.
\]

Let

\[
V(\theta) = \left\{ \sum_{s \in I} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\} \text{ and } \sum_{s \in I} \frac{1}{\beta_s} - \theta \in \mathbb{Z} \right\}
\]

and \( U(\theta) \) be a set of \( m \) distinct numbers comparable with \( V(\theta) \) (i.e. \( |U(\theta)| = m \), and either \( U(\theta) \subseteq V(\theta) \) or \( U(\theta) \supseteq V(\theta) \)). Then the following statements are equivalent:

(a) \( A \) is an \( m \)-cover of \( \mathbb{Z} \).
(b) \( A \) covers \( |S| \) consecutive integers at least \( m \) times.
(c) For each \( \theta \in S \),

\[
\sum_{\substack{I \subseteq \{1, \ldots, k\} \\ \{\Sigma_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \left( \sum_{s \in I} \frac{1}{\beta_s} \right)^{n} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = 0
\]

holds for every \( n = 0, 1, \ldots, m - 1 \). (As usual \( \binom{x}{n} \) denotes \( \frac{x(x-1) \cdots (x-n+1)}{1 \cdot 2 \cdots (n-1)n} \).)

(d) For any \( \theta \in S \),

\[
\sum_{v \in V(\theta)} a_{uv} f(v) = 0 \quad \text{for all } u \in U(\theta),
\]
where

\[ a_{uv} = \prod_{x \in U(\theta), x \neq u} \frac{x - v}{x - u} \quad \text{and} \quad f(v) = \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} e^{2\pi i \sum_{s \in I} \alpha_s / \beta_s}. \]

**Proof.** (a)⇒(b). This is obvious.

(b)⇒(c). Suppose that each of \( x + 1, \ldots, x + |S| \) is covered by \( A \) at least \( m \) times, where \( x \) is an integer. By Lemma 1 for every \( n = 1, \ldots, |S| \) we have

\[
0 = \lim_{r \to 1} \frac{\prod_{s=1}^{k} (1 - r^{1/\beta_s} e^{2\pi i (\alpha_s - x - n) / \beta_s})}{(1 - r)^{m-1}} \\
= \lim_{r \to 1} \left(1 - r\right)^{1-m} \\
\times \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} \prod_{s \in I} \frac{\sum_{\Sigma_s \in I} e^{2\pi i \sum_{s \in I} (\alpha_s - x) / \beta_s} e^{-2\pi i n \Sigma_s / \beta_s}}{(1 - r)^{m-1}} \\
= \lim_{r \to 1} \sum_{\theta \in S} F(r, \theta) e^{-2\pi i \theta},
\]

where

\[ F(r, \theta) = \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} \prod_{s \in I} e^{2\pi i \sum_{s \in I} \alpha_s / \beta_s} e^{-2\pi i \theta / (1 - r)^{m-1}}. \]

Let \( \varepsilon \) be an arbitrary positive number. By Lemma 2 there is an \( \eta > 0 \) such that if

\[
\left| \sum_{\theta \in S} e^{-2\pi i \theta} x_{\theta} \right| < \eta
\]

for every \( n = 1, \ldots, |S| \) then \( |x_{\theta}| < \varepsilon \) for all \( \theta \in S \). Since

\[
\sum_{\theta \in S} F(r, \theta) e^{-2\pi i \theta} = o(1) \quad (r \to 1) \quad \text{for} \quad n = 1, \ldots, |S|,
\]

there exists a \( \delta > 0 \) such that whenever \( |r - 1| < \delta \),

\[
\left| \sum_{\theta \in S} F(r, \theta) e^{-2\pi i \theta} \right| < \eta \quad \text{for all} \quad n = 1, \ldots, |S|
\]

and hence by the above \( |F(r, \theta)| < \varepsilon \) for each \( \theta \in S \). This shows that

\[
\lim_{r \to 1} F(r, \theta) = 0 \quad \text{for every} \quad \theta \in S.
\]
For any \( \theta \in S \) we have

\[
0 = \lim_{t \to 0} \sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s} \frac{1}{(1 - t)^m}
\]

\[
= \lim_{t \to 0} \sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} (1 - t)^{-m} e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s} t^1
\]

\[
= \lim_{t \to 0} \sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} \left( \sum_{n=0}^{m-1} \left( \sum_{n=0}^{m-1} \left( \frac{1}{n+1} \right)^n \right) t^n \right)^m
\]

\[
= \lim_{t \to 0} \sum_{n=0}^{m-1} (-1)^n \left( \sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} \left( \frac{1}{n+1} \right)^n \right) e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s} t^n
\]

In view of Lemma 3, (7) holds for every \( n = 0, 1, \ldots, m - 1 \). Therefore part (c) follows.

(c) \Rightarrow (d). Fix \( \theta \in S \). For each \( n = 0, 1, \ldots, m - 1 \),

\[
x^n = \sum_{j=0}^{n} S(n, j)x(x - 1) \ldots (x - j + 1)
\]

where \( S(n, j) \) (\( 0 \leq j \leq n \)) are Stirling numbers of the second kind, so by (c) we have

\[
\sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} \left( \sum_{s \in I} 1/\beta_s \right)^n e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s}
\]

\[
= \sum_{j=0}^{n} j! S(n, j) \sum_{I \subseteq \{1, \ldots, k\}, (\Sigma_{\alpha \in I} \alpha_{\beta}) = \theta} (-1)^{|I|} \left( \frac{1}{j} \right)^n e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s} = 0,
\]

i.e.

\[
\sum_{\nu \in \mathcal{V}(\theta)} \sum_{I \subseteq \{1, \ldots, k\}, \Sigma_{\alpha \in I} \alpha_{\beta} = \nu} (-1)^{|I|} e^{2\pi i \Sigma_{\alpha \in I} \alpha_{\beta}/\beta_s} \nu^n = 0.
\]
Case 1: $|V(\theta)| \leq m$. In this case
\[ \sum_{v \in V(\theta)} [v]^n f(v) = 0 \quad \text{for every } n = 0, 1, \ldots, |V(\theta)| - 1. \]

Hence (8) holds since \( f(v) = 0 \) for all \( v \in V(\theta) \) (Vandermonde).

Case 2: $|V(\theta)| > m$. In this case, \( U(\theta) \subset V(\theta) \) and
\[ \sum_{v \in U(\theta)} [v]^n f(v) + \sum_{v \in V(\theta) \setminus U(\theta)} [v]^n f(v) = 0 \]
for each \( n = 0, 1, \ldots, m - 1 \). According to Lemma 4,
\[ \sum_{v \in V(\theta)} a_{uv} f(v) = \sum_{v \in V(\theta)} \left( \prod_{x \in U(\theta) \setminus \{u\}} \frac{[x] - [v]}{[x] - [u]} \right) f(v) = 0 \quad \text{for all } u \in U(\theta). \]

So in either case we have (8).

(d) \( \Rightarrow \) (a). Assume that (d) holds. Let \( \theta \in S \). For \( u, v \in U(\theta) \),
\[ a_{uv} = \prod_{x \in U(\theta) \setminus \{u\}} \frac{x - v}{x - u} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases} \]

Case 1: $|V(\theta)| \leq m$. In this case \( V(\theta) \subseteq U(\theta) \). As
\[ f(u) = \sum_{v \in V(\theta)} a_{uv} f(v) = 0 \quad \text{for each } u \in V(\theta), \]
we get
\[ \sum_{v \in V(\theta)} f(v) [v]^n = 0 \quad \text{for all } n = 0, 1, 2, \ldots \]

Case 2: $|V(\theta)| > m$. In this case \( U(\theta) \subset V(\theta) \), so for any \( u \in U(\theta) \) and \( v \in V(\theta) \) we have \( \{u\} = \{v\} = \theta \) and hence \([u] - [v] = u - v\). Since
\[ \sum_{v \in V(\theta)} \left( \prod_{x \in U(\theta) \setminus \{u\}} \frac{[x] - [v]}{[x] - [u]} \right) f(v) = \sum_{v \in V(\theta)} a_{uv} f(v) = 0 \]
for every \( u \in U(\theta) \), it follows from Lemma 4 that
\[ \sum_{v \in V(\theta)} f(v) [v]^n = \sum_{v \in U(\theta)} [v]^n f(v) + \sum_{v \in V(\theta) \setminus U(\theta)} [v]^n f(v) = 0 \]
for all \( n = 0, 1, \ldots, m - 1 \).

In both cases,
\[ \sum_{v \in V(\theta)} f(v) [v]^n = 0 \quad \text{for } n = 0, 1, \ldots, m - 1. \]
Thus for each nonnegative integer \( n < m \),
\[
\sum_{v \in V(\theta)} f(v) \binom{[v]}{n} = \sum_{v \in V(\theta)} f(v) \sum_{j=0}^{n} \frac{(-1)^{n-j}s(n, j) \sum_{v \in V(\theta)} [v]^j}{j!},
\]
\[
= \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j}s(n, j) \sum_{v \in V(\theta)} [v]^j = 0,
\]
where \( s(n, j) \) (\( 0 \leq j \leq n \)) are Stirling numbers of the first kind, i.e.
\[
\sum_{\{\Sigma_{s \in I}/\beta_s\}_{\Sigma_s \in I}/\beta_s = \theta} (-1)^{|I|} \left( \sum_{s \in I} \binom{[v]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \right) = 0.
\]

Therefore by the proof of (b) \( \Rightarrow \) (c),
\[
\lim_{\tau \to 1} \sum_{\{\Sigma_{s \in I}/\beta_s\}_{\Sigma_s \in I}/\beta_s = \theta} (-1)^{|I|} \tau^{\Sigma_{s \in I}/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} / (1 - \tau)^{m-1}
\]
\[
= \lim_{\tau \to 0} \sum_{n=0}^{m-1} (-1)^n \left( \sum_{\{\Sigma_{s \in I}/\beta_s\}_{\Sigma_s \in I}/\beta_s = \theta} (-1)^{|I|} \left( \sum_{s \in I} \binom{[v]}{n} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s} \right) \right) t^{n-m+1}
\]
\[
= 0.
\]

Now for every integer \( x \),
\[
\prod_{s=1}^{k} (1 - \tau^{1/\beta_s} e^{2\pi i (\alpha_s - x)/\beta_s})
\]
\[
= \sum_{\{\Sigma_{s \in I}/\beta_s\}_{\Sigma_s \in I}/\beta_s = \theta} (-1)^{|I|} \tau^{\Sigma_{s \in I}/\beta_s} e^{2\pi i \Sigma_{s \in I} (\alpha_s - x)/\beta_s}
\]
\[
= \sum_{\theta \in S} e^{-2\pi i x \theta} \sum_{\{\Sigma_{s \in I}/\beta_s\}_{\Sigma_s \in I}/\beta_s = \theta} (-1)^{|I|} \tau^{\Sigma_{s \in I}/\beta_s} e^{2\pi i \Sigma_{s \in I} \alpha_s/\beta_s}
\]
\[
= \sum_{\theta \in S} e^{-2\pi i x \theta} o((1 - \tau)^{m-1}) = o((1 - \tau)^{m-1}) \quad (\tau \to 1).
\]

Applying Lemma 1 we then obtain part (a).

The proof of Theorem 1 is now complete.
3. Reciprocals of common differences. In 1989 M. Z. Zhang [19] showed the following surprising result analytically: Provided that (2) is a cover of \( \mathbb{Z} \), \( \sum_{s \in I} 1/n_s \in \mathbb{Z}^+ \) for some \( I \subseteq \{1, \ldots, k\} \). Here we give

**Theorem 2.** Let (1) be a cover of \( \mathbb{Z} \). Then for any \( J \subseteq \{1, \ldots, k\} \) there is an \( I \subseteq \{1, \ldots, k\} \) with \( I \neq J \) such that

\[
\sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z}.
\]

**Proof.** Set \( \theta = \{ \sum_{s \in J} 1/\beta_s \} \). By Theorem 1,

\[
\sum_{\{s \in I \subseteq \{1, \ldots, k\} \mid \Sigma_{s \in I} 1/\beta_s = \theta\}} (-1)^{|I|} e^{2\pi i \sum_{s \in I} \alpha_s / \beta_s} = 0,
\]

that is,

\[
\sum_{\{s \notin I \subseteq \{1, \ldots, k\} \mid \Sigma_{s \notin I} 1/\beta_s = \theta\}} (-1)^{|I|} e^{2\pi i \sum_{s \notin I} \alpha_s / \beta_s} = -(-1)^{|J|} e^{2\pi i \sum_{s \in J} \alpha_s / \beta_s}.
\]

Therefore

\[
\left\{ I \subseteq \{1, \ldots, k\} : I \neq J \text{ and } \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} = \theta \right\} \neq \emptyset.
\]

We are done.

In the case \( J = \emptyset \), Theorem 2 yields a generalization of Zhang’s result ([19]).

Provided that (1) is an \( m \)-cover of \( \mathbb{Z} \) with \( m \in \mathbb{Z}^+ \), Theorem 2 asserts that for any \( J \subseteq \{1, \ldots, k\} \),

\[
S(J) = \left\{ I \subseteq \{1, \ldots, k\} : I \neq J \text{ and } \sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z} \right\}
\]

is nonempty. This becomes trivial if

\[
\sum_{s \in I} \frac{1}{\beta_s} = \sum_{s \in J} \frac{1}{\beta_s}
\]

for some \( I \subseteq \{1, \ldots, k\} \) with \( I \neq J \).

What can we say about

\[
Z(J) = \left\{ \sum_{s \in I} \frac{1}{\beta_s} - \sum_{s \in J} \frac{1}{\beta_s} : I \in S(J) \right\}
\]

if it does not contain zero? The following theorem gives us more information.

**Theorem 3.** Assume that (1) is an \( m \)-cover of \( \mathbb{Z} \). Let \( J \) be a subset of \( \{1, \ldots, k\} \) such that (11) fails, i.e. \( 0 \notin Z(J) \) where \( S(J) \) and \( Z(J) \) are given by (10) and (12). Then
(i) \(|Z(J)| \geq m\) and hence

\[
\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \text{md}(J) + \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} \geq m,
\]

where \(d(J)\) is the least positive integer that can be written as the difference of two (distinct) numbers of the form

\[
\sum_{s \in I} \frac{1}{\beta_s} \in \mathbb{Z} + \sum_{s \in J} \frac{1}{\beta_s}, \quad \text{where } I \subseteq \{1, \ldots, k\}.
\]

(ii) When \(d(J) \geq \lfloor \sum_{s=1}^{k} 1/\beta_s \rfloor / m\), \(d(J)\) equals \(\lceil \sum_{s=1}^{k} 1/\beta_s \rceil / m\) and divides \(\lfloor \sum_{s \in J} 1/\beta_s \rfloor\), and for every \(n = 0, 1, \ldots, m\) there exist at least

\[
\binom{m}{n} / \binom{\lfloor \sum_{s \in J} 1/\beta_s \rfloor}{\lceil \sum_{s=1}^{k} 1/\beta_s \rceil}
\]

subsets \(I\) of \(\{1, \ldots, k\}\) such that

\[
\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left\lfloor \sum_{s=1}^{k} \frac{1}{\beta_s} \right\rfloor + \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\},
\]

hence

\[
|S(J)| \geq 2^m / \left( m \left\lfloor \sum_{s \in J} 1/\beta_s \right\rfloor / \left\lceil \sum_{s=1}^{k} 1/\beta_s \right\rceil \right) - 1 \quad \text{and} \quad |Z(J)| = m.
\]

Proof. Let \(\theta = \{\sum_{s \in J} 1/\beta_s\}, V(\theta), U(\theta)\) and \(f(x)\) be as in Theorem 1. If \(|V(\theta)| \leq m\), then \(V(\theta) \subseteq U(\theta)\), hence by Theorem 1 for all \(u \in V(\theta) \subseteq U(\theta)\),

\[
f(u) = \sum_{v \in V(\theta)} \left( \prod_{x \in U(\theta)} \frac{x - v}{x - u} \right) f(v) = 0,
\]

which is impossible since \(0 \notin Z(J)\) and

\[
f\left( \sum_{s \in J} \frac{1}{\beta_s} \right) = (-1)^{|J|} e^{2\pi i \sum_{s \in J} \alpha_s / \beta_s} \neq 0.
\]

Thus \(|V(\theta)| > m\).

(i) Let \(v_0 < v_1 < \ldots < v_m\) be the first \(m + 1\) elements of \(V(\theta)\) in ascending order. Clearly

\[
1 + |Z(J)| = |Z(J) \cup \{0\}| = \left| \left\{ v - \sum_{s \in J} \frac{1}{\beta_s} : v \in V(\theta) \right\} \right| = |V(\theta)| \geq m + 1
\]

and

\[
\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m = \sum_{i=0}^{m-1} (v_{i+1} - v_i) + v_0 \geq \text{md}(J) + \theta.
\]
(ii) If $|V(\theta)| > m + 1$ then

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m + 1 \geq 1 + md(J) + \theta.$$ 

Now suppose that $d(J) \geq \lfloor \sum_{s=1}^{k} 1/\beta_s \rfloor / m$. Then we must have $|V(\theta)| = m + 1$, thus $V(\theta) = \{v_0, v_1, \ldots, v_m\}$ and $|Z(J)| = |V(\theta)| - 1 = m$. As

$$md(J) \geq \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right] \geq [v_m] = v_0 - \theta + \sum_{i=0}^{m-1} (v_{i+1} - v_i) \geq [v_0] + md(J),$$

$$md(J) = \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right], \quad [v_0] = 0$$

and

$$[v_n] = v_0 - \theta + \sum_{i=0}^{n-1} (v_{i+1} - v_i) = 0 + \sum_{i=0}^{n-1} d(J) = nd(J)$$

for $n = 1, \ldots, m$.

Choose $0 \leq j \leq m$ such that $v_j = \sum_{s \in J} 1/\beta_s$. Then

$$j = \frac{[v_j]}{d(J)} = m \left[ \sum_{s \in J} \frac{1}{\beta_s} \right] / \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right].$$

Set

$$U'(\theta) = \{v_i : 0 \leq i \leq m, i \neq j\}.$$ 

By Theorem 1, for any $n = 0, 1, \ldots, m$ with $n \neq j$,

$$0 = \sum_{v \in V(\theta)} \left( \prod_{x \in U'(\theta)} \frac{x - v}{x - v_n} \right) f(v) = \sum_{t=0}^{m} \left( \prod_{i=0}^{m} \frac{v_i - v_t}{v_i - v_n} \right) f(v_t)$$

$$= \sum_{t=0}^{m} \left( \prod_{i=0}^{m} \frac{id(J) + \theta - (td(J) + \theta)}{id(J) + \theta - (nd(J) + \theta)} \right) f(v_t)$$

$$= \sum_{t=0}^{m} \left( \prod_{i=0}^{m} \frac{i - t}{i - n} \right) f(v_t)$$

$$= f(v_n) + \left( \prod_{i=0}^{m} \frac{i - j}{i - n} \right) f(v_j).$$
Since
\[
\prod_{\substack{i=0, i \neq j, n \neq j, n}}^{m} \frac{i - j}{i - n} = \prod_{i=0, i \neq j}^{m} \frac{i - j}{i - j} / \prod_{i=0, i \neq n}^{m} \frac{i - n}{j - n}
\]
\[
= - \frac{\prod_{i=0}^{j-1} (i - j) \cdot \prod_{i=j+1}^{m} (i - j)}{\prod_{i=0}^{n-1} (i - n) \cdot \prod_{i=n+1}^{m} (i - n)}
\]
\[
= - \frac{(-1)^j j! (m - j)!}{(-1)^nn! (m - n)!} = (-1)^{j - n + 1} \binom{m}{n} / \binom{m}{j},
\]
we have
\[
\sum_{\substack{I \subseteq \{1, \ldots, k\} \atop \Sigma_{s \in I} 1/\beta_s = nd(J) + \theta}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s}
\]
\[
= f(v_n) = (-1)^{j - n + 1} \binom{m}{n} \binom{m}{j}^{-1} f\left(\sum_{s \in J} \frac{1}{\beta_s}\right)
\]
\[
= (-1)^{j - n} \binom{m}{n} \binom{m}{j}^{-1} (-1)^{|J|} e^{2\pi i \Sigma_{s \in J} \alpha_s / \beta_s},
\]
and hence
\[
\left| \left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} \right\} \right|
\]
\[
= \sum_{\substack{I \subseteq \{1, \ldots, k\} \atop \Sigma_{s \in I} 1/\beta_s = nd(J) + \theta}} 1
\]
\[
\geq \sum_{\substack{I \subseteq \{1, \ldots, k\} \atop \Sigma_{s \in I} 1/\beta_s = nd(J) + \theta}} (-1)^{|I|} e^{2\pi i \Sigma_{s \in I} \alpha_s / \beta_s} = \binom{m}{n} / \binom{m}{j};
\]
therefore
\[
1 + |S(J)| = \left| \left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{1}{\beta_s} \in V(\theta) \right\} \right|
\]
\[
= \sum_{n=0}^{m} \left| \left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{1}{\beta_s} = v_n = nd(J) + \theta \right\} \right|
\]
\[
\geq \sum_{n=0}^{m} \binom{m}{n} / \binom{m}{j} = 2^m / \binom{m}{j}.
\]
This ends the proof.

Now let us apply Theorem 3 to those m-covers (1) with \(\sum_{s=1}^{k} 1/\beta_s = m\).
Theorem 4. Let (1) be an $m$-cover of $\mathbb{Z}$ with $\sum_{s=1}^{k} 1/\beta_s = m \in \mathbb{Z}^+$, which happens if (1) is an exact $m$-cover of $\mathbb{Z}$ by residue classes. Then

(i) For every $l = 1, \ldots, k - 1$ we have

$\sum_{s=l+1}^{k} \frac{1}{\beta_s} \geq \frac{1}{\max\{\beta_1, \ldots, \beta_l\}}$.

(ii) For any $\emptyset \neq J \subset \{1, \ldots, k\}$ there exists an $I \subset \{1, \ldots, k\}$ with $I \neq J$ such that

$\sum_{s \in I} \frac{1}{\beta_s} = \sum_{s \in J} \frac{1}{\beta_s}$,

furthermore when $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$ there are at least

$\left(\frac{m}{\sum_{s \in J} 1/\beta_s}\right) \geq m > 1$

subsets $I$ of $\{1, \ldots, k\}$ satisfying (16).

Proof. (i) For $l = 1, \ldots, k - 1$ (15) follows from part (ii) in the case $J = \{l + 1, \ldots, k\}$, so we proceed to the proof of part (ii).

(ii) If (11) fails then by part (i) of Theorem 3 and the equality $\sum_{s=1}^{k} 1/\beta_s = m$ we must have

$\left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} = 0$, i.e. $\sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z}$.

Observe that

$0 < \sum_{s \in J} \frac{1}{\beta_s} < \sum_{s=1}^{k} \frac{1}{\beta_s} = m$.

If $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$, then $m > 1$ and $\sum_{s \in J} 1/\beta_s = n$ for some $n = 1, \ldots, m - 1$, by part (ii) of Theorem 3 there are at least $\binom{m}{n}/\binom{m}{n} = \binom{m}{n} \geq m$ subsets $I$ of $\{1, \ldots, k\}$ such that

$\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \left\{ \sum_{1 \leq s \leq k} \frac{1}{\beta_s} \right\} = n = \sum_{s \in J} \frac{1}{\beta_s}$.

We are done.

Remark. In 1992 Z. W. Sun ([17]) proved that if (2) is an exact $m$-cover of $\mathbb{Z}$ then for each $n = 1, \ldots, m$ there exist at least $\binom{m}{n}$ subsets $I$ of $\{1, \ldots, k\}$ such that $\sum_{s \in I} 1/\beta_s$ equals $n$. The lower bounds $\binom{m}{n} (1 \leq n \leq m)$ are best possible, and the Riemann zeta function was used in the proof.

From Theorem 3 we can also deduce the following theorem which extends Zhang’s result ([19]) and the theorem of Sun [17] even in the case $l = k$. 
THEOREM 5. Let (1) be an m-cover of \( Z \) and \( l \) a positive integer not exceeding \( k \) such that

\[
\min \left\{ \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_l} \right\} > \sum_{l \leq t \leq k} \frac{1}{\beta_t},
\]

where \( \sum_{l \leq t \leq k} 1/\beta_t \) is considered to be zero for \( l = k \). Then

(i) There are at least \( m \) positive integers representable by

\[
\sum_{s \in I} \frac{1}{\beta_s} = \sum_{l \leq t \leq k} \frac{1}{\beta_t}, \quad \text{where } I \subseteq \{1, \ldots, k\},
\]

thus we have

\[
\sum_{s=1}^{l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \sum_{l \leq t \leq k} \frac{1}{\beta_t} \geq m.
\]

(ii) Provided that any positive integer less than \( \lceil \sum_{s=1}^{k} 1/\beta_s \rceil / m \) cannot be expressed as the difference of two integers of the form (18), \( \lceil \sum_{s=1}^{k} 1/\beta_s \rceil \) is divisible by \( m \) and for each \( n = 0, 1, \ldots, m \) there are at least \( \binom{m}{n} \) subsets \( I \) of \( \{1, \ldots, k\} \) such that

\[
\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \sum_{l \leq t \leq k} \frac{1}{\beta_t},
\]

hence there exist at least \( 2^m - 1 \) subsets \( I \) of \( \{1, \ldots, k\} \) with

\[
\sum_{s \in I} \frac{1}{\beta_s} \in \mathbb{Z}^+ + \sum_{l \leq t \leq k} \frac{1}{\beta_t}.
\]

Proof. Let \( J = \{1 \leq t \leq k : t > l\} \). By (17),

\[
\left[ \sum_{t \in J} \frac{1}{\beta_t} \right] = 0 \quad \text{and} \quad \left\{ \sum_{t \in J} \frac{1}{\beta_t} \right\} = \sum_{l \leq t \leq k} \frac{1}{\beta_t}.
\]

For any \( I \subseteq \{1, \ldots, k\} \), if \( I \subset J \) then

\[
0 < \sum_{t \in J} \frac{1}{\beta_t} - \frac{1}{\beta_s} < 1,
\]

and if \( I \not\subset J \) then

\[
\sum_{s \in I} \frac{1}{\beta_s} - \sum_{t \in J} \frac{1}{\beta_t} \geq \min \left\{ \frac{1}{\beta_s} : 1 \leq s \leq l \right\} - \sum_{l \leq t \leq k} \frac{1}{\beta_t} > 0.
\]

So (11) fails, moreover \( Z(J) \) given by (12) contains only positive integers. Applying Theorem 3 we obtain the desired results.
Erdős conjectured (before 1950) that if (2) is a cover of $\mathbb{Z}$ with $1 < n_1 < n_2 < \ldots < n_k$ then $\sum_{s=1}^{k} 1/n_s > 1$. H. Davenport, L. Mirsky, D. Newman and R. Rado confirmed this conjecture (independently) by proving that if (2) is an exact cover of $\mathbb{Z}$ with $1 < n_1 \leq \ldots \leq n_{k-1} \leq n_k$ then $n_{k-1} = n_k$. For further improvements see Znám [20], M. Newman [10], Porubský [11, 12], M. A. Berger, A. Felzenbaum and A. S. Fraenkel [1]. The best record in this direction is the following result due to the author which is partially announced in [15] and completely proved in [16]: Let $\lambda_1, \ldots, \lambda_k$ be complex numbers and $n_0 \in \mathbb{Z}^+$ a period of the function

$$\sigma(x) = \sum_{x \equiv a_s \pmod{n_s}}^{k} \lambda_s,$$

If $d \in \mathbb{Z}^+$ does not divide $n_0$ and

$$\sum_{s=1}^{k} \frac{\lambda_s}{n_s} \neq 0$$

for some integer $a$,

then

$$|\{a_s \mod d : 1 \leq s \leq k, \ d \mid n_s\}| \geq \min_{0 \leq s \leq k} \frac{d}{\gcd(d, n_s)} \geq p(d),$$

where $p(d)$ is the least prime divisor of $d$. Here we have

**Theorem 6.** Let (1) be an m-cover of $\mathbb{Z}$ with $\beta_1 \leq \ldots \leq \beta_{k-1} < \beta_{k-1+1} = \ldots = \beta_k$ where $1 \leq l < k$. Then either

(21) \hspace{1cm} l \geq \beta_k/\max\{1, \beta_{k-l}\} \ (> 1 \text{ if } \beta_k > 1),

or there are at least m positive integers in the form

(22) \hspace{1cm} \sum_{s \in I} \frac{1}{\beta_s} - \frac{l}{\beta_k}, \quad \text{where } I \subseteq \{1, \ldots, k\},

and hence

(23) \hspace{1cm} \sum_{s=1}^{k} \frac{1}{\beta_s} > \sum_{s=1}^{k-l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \frac{l}{\beta_k} \geq m.

(Also, $\sum_{s=1}^{k} 1/\beta_s > \sum_{s=1}^{k} 1/\beta_k \geq k \geq m$ if $\beta_k \leq 1$.)

**Proof.** Clearly $l < \beta_k/\max\{1, \beta_{k-l}\}$ if and only if

$$\min \left\{ \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{k-l}} \right\} > \sum_{k-l \leq t \leq k} \frac{1}{\beta_t} \ (= l/\beta_k).$$

Therefore Theorem 6 follows from part (i) of Theorem 5.
Note that when $\beta_{k-l} \geq 1$ and $\beta_k / \beta_{k-l} \in \mathbb{Z}$

$$\beta_k / \max\{1, \beta_{k-l}\} = \beta_k / \beta_{k-l} \geq p(\beta_k / \beta_{k-l}) \quad (\geq p(\beta_k) \text{ if } \beta_{k-l}, \beta_k \in \mathbb{Z}).$$

that whenever the residue classes in (2) are pairwise disjoint and the moduli
$n_1, \ldots, n_k > 1$ are distinct there exists an integer $x$ with $1 \leq x \leq 2^k$
such that $x$ is not covered by (2). Erdős [6] confirmed this conjecture with $k \cdot 2^k$
instead of $2^k$. Since the Davenport–Miksy–Newman–Radó result indicates
that an exact cover of $\mathbb{Z}$ by (finitely many) residue classes cannot have its
moduli distinct and greater than one, Erdős proposed the stronger conjecture
that any system of $k$ residue classes not covering all the integers omits
a positive integer not exceeding $2^k$. Both conjectures have some local-global
their positive answer to the stronger conjecture. Later in [3] a long indirect
and awkward proof was given for $k \geq 20$, the authors concluded the paper
with the statements: “Of course it remains to show the conjecture is true
for $k < 20$. This may be checked by more special arguments.”

In 1970 Crittenden and Vanden Eynden [4] conjectured further that if
all the moduli $n_s$ in (2) are greater than an integer $0 \leq l < k$ then (2) is
a cover of $\mathbb{Z}$ if it covers all the integers in the interval $[1, 2^{k-l}(l+1)]$. In
contrast with the Crittenden–Vanden Eynden conjecture we give

**Theorem 7.** For any $m \in \mathbb{Z}^+$, (1) is an $m$-cover of $\mathbb{Z}$ if it covers
$2^{k-M}(M+1)$ consecutive integers at least $m$ times, where

$$M = \max_{1 \leq i \leq k} \{|1 \leq s \leq k : \beta_s = \beta_i\}|.$$  \hspace{1cm} (24)

**Proof.** Let $\beta > 0$ be a number such that $J = \{1 \leq s \leq k : \beta_s = \beta\}$ has
cardinality $M$. As

$$\left|\left\{\frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\}\right\}\right|$$

$$\leq \left|\left\{\sum_{s \in I \cap J} \frac{1}{\beta_s} + \sum_{s \in I \setminus J} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\}\right\}\right|$$

$$\leq \left|\left\{\sum_{s \in I} \frac{1}{\beta} : I \subseteq J\right\}\right| \cdot \left|\left\{\sum_{s \in I} \frac{1}{\beta_s} : I \subseteq \{1, \ldots, k\} \setminus J\right\}\right|$$

$$\leq \left|\left\{\frac{|I|}{\beta} : I \subseteq J\right\}\right| \cdot |\{I : I \subseteq \{1, \ldots, k\} \setminus J\}|$$

$$= (|J| + 1) \cdot 2^{k-|J|} = 2^{k-M}(M+1),$$

Theorem 1 implies Theorem 7.
The following example noted by Crittenden and Vanden Eynden [4] shows that the number \( g(k, M) = 2^{k-M}(M + 1) \) in Theorem 7 is best possible.

**Example.** Let \( M = n - 1 \in \mathbb{Z}^+ \). Consider the system \( A \) consisting of the following \( k \geq M \) residue classes:

\[
1 + n\mathbb{Z}, \quad 2 + n\mathbb{Z}, \quad \ldots, \quad M + n\mathbb{Z}, \\
\quad n + 2n\mathbb{Z}, \quad 2n + 2^2n\mathbb{Z}, \quad \ldots, \quad 2^{k-M-1}n + 2^{k-M}n\mathbb{Z}.
\]

Observe that \( A \) together with \( 2^{k-M}n\mathbb{Z} \) forms an exact cover of \( \mathbb{Z} \). So \( A \) covers positive integers from 1 to \( 2^{k-M}(M + 1) - 1 \), but it does not cover all the integers.

Result (II) stated in Section 1 follows from Theorem 7 and Example, since (1) covers \( \alpha + \beta x \) (where \( \alpha \in \mathbb{R}, \beta \in \mathbb{R}^+ \) and \( x \in \mathbb{Z} \)) at least \( m \) times if and only if \( \left\{ \frac{\alpha s - \alpha}{\beta} + \frac{\beta x}{\beta} \mathbb{Z} \right\}^k_{s=1} \) covers \( x \) at least \( m \) times, and \( 2^{k-l}(l + 1) \geq 2^{k-M}(M + 1) \) if \( k \geq M \geq l > 0 \). (The case \( l = 0 \) can be reduced to the case \( l = 1 \).)

**5. Several open problems.** Theorem 1 tells us that (2) is a cover of \( \mathbb{Z} \) if it covers integers from 1 to

\[
\left| \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq \{1, \ldots, k\} \right| \leq 2^k \leq 2^{n_1 + \ldots + n_k}.
\]

This suggests

**Problem 1.** Can we find a polynomial \( P \) with integer coefficients such that a finite system (2) of residue classes forms a cover of \( \mathbb{Z} \) whenever it covers all positive integers not exceeding \( P(n_1 + \ldots + n_k) \)?

In 1973 L. J. Stockmeyer and A. R. Meyer proved that the problem whether there exists an integer not covered by a given (2) is NP-complete. In 1991 S. P. Tung [18] extended this result to algebraic integer rings. If the required \( P \) in Problem 1 exists, then there is a polynomial time algorithm to decide whether (2) covers all the integers or not. So a positive answer to Problem 1 would imply that NP = P.

By appearances Theorems 2–7 involve no roots of unity. Perhaps vast generalizations of them could be made.

**Problem 2.** Let \( A_1, \ldots, A_k \) be sets of natural numbers having positive densities \( d(A_1), \ldots, d(A_k) \) respectively. If no \( A_s \) contains \( m_s \in \mathbb{Z}^+ \) consecutive integers, does \( \bigcup_{s=1}^k A_s \) have density 1 when it covers \( m_1 \ldots m_k \) arbitrarily large consecutive integers? Suppose that \( \{A_s\}_{s=1}^k \) covers all the natural numbers; does there exist, for any \( J \subseteq \{1, \ldots, k\} \), an \( I \subseteq \{1, \ldots, k\} \...
with \( I \neq J \) such that
\[
\sum_{s \in I} d(A_s) - \sum_{s \in J} d(A_s) \in \mathbb{Z}?
\]

**Problem 3.** Let \( K \) be an algebraic number field and \( O_K \) the ring of algebraic integers in \( K \). Let \( a_1, \ldots, a_k \in O_K \) and \( A_1, \ldots, A_k \) be ideals of \( O_K \) with norms \( N(A_1), \ldots, N(A_k) \) respectively. If \( \{a_s + A_s\}_{s=1}^k \) forms an exact \( m \)-cover of \( O_K \) for some \( m \in \mathbb{Z}^+ \), is it true that for any \( \emptyset \neq J \subseteq \{1, \ldots, k\} \) there exists a subset \( I \) of \( \{1, \ldots, k\} \) with \( I \neq J \) such that
\[
\sum_{s \in I} \frac{1}{N(A_s)} = \sum_{s \in J} \frac{1}{N(A_s)}?
\]

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