

OPEN CONJECTURES ON CONGRUENCES

ZHI-WEI SUN

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
<http://math.nju.edu.cn/~zwsun>

ABSTRACT. We collect here various conjectures on congruences made by the author in a series of papers, some of which involve binary quadratic forms and other advanced theories. Part A consists of 72 unsolved conjectures of the author while conjectures in Part B have been recently confirmed. We hope that this material will interest number theorists and stimulate further research. Number theorists are welcome to work on those open conjectures.

INTRODUCTION

Congruences modulo primes have been widely investigated since the time of Fermat. However, we find that there are still lots of new challenging congruences that cannot be easily solved. They appeal for new powerful tools or advanced theory.

Here we collect various conjectures of the author on congruences, which can be found in the author's papers available from [arxiv](http://arxiv.org) or his homepage. We use two sections to state conjectures and related remarks. Part A contains 72 unsolved conjectures of the author while Part B consists of conjectures that have been recently confirmed. Most of the congruences here are *super* congruences in the sense that they happen to hold modulo some higher power of p . The topic of super congruences is related to the p -adic Γ -function, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi-Yau manifolds, and some sophisticated combinatorial identities involving harmonic numbers (cf. Ken Ono's book [O]).

2010 *Mathematics Subject Classification*. Primary 11B65, 11A07; Secondary 05A10, 11A41, 11B37, 11B68, 11E25, 11S99.

Keywords. Binomial coefficients, Catalan numbers, Bernoulli numbers, Euler numbers, binary quadratic forms, congruences modulo prime powers.

Copyright is owned by the author Zhi-Wei Sun. The material on the author's homepage has been linked to Number Theory Web since Nov. 27, 2009.

For clarity, we often state the prime version of a conjecture instead of the general version. The reader should consult the original papers for more detailed information and related results.

Now we introduce some basic notations in this paper. As usual, we set

$$\mathbb{N} = \{0, 1, 2, \dots\} \quad \text{and} \quad \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

The Kronecker symbol $\delta_{m,n}$ takes 1 or 0 according as $m = n$ or not. The rising factorial $(x)_n$ is defined by $(x)_n = \prod_{k=0}^{n-1} (x+k)$, and $(x)_0$ is regarded as 1. For an integer m and a positive odd number n , the notation $\left(\frac{m}{n}\right)$ stands for the Jacobi symbol. For an odd prime p , we use $q_p(2)$ to denote the Fermat quotient $(2^{p-1} - 1)/p$. For a prime p and a rational number x , the p -adic valuation of x is given by

$$\nu_p(x) = \sup\{a \in \mathbb{N} : x \equiv 0 \pmod{p^a}\}.$$

For $k_1, \dots, k_n \in \mathbb{N}$, we define the multinomial coefficient

$$\binom{k_1 + \dots + k_n}{k_1, \dots, k_n} := \frac{(k_1 + \dots + k_n)!}{k_1! \dots k_n!}.$$

Harmonic numbers are given by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^n \frac{1}{k} \quad (n = 1, 2, 3, \dots).$$

For $n \in \mathbb{N}$ by C_n we mean the *Catalan number* $\frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$. Note that if p is an odd prime then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \quad \text{for every } k = \frac{p+1}{2}, \dots, p-1.$$

Bernoulli numbers B_0, B_1, B_2, \dots are rational numbers given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that $B_{2n+1} = 0$ for all $n \in \mathbb{Z}^+$ and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Euler numbers E_0, E_1, E_2, \dots are integers defined by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that $E_{2n+1} = 0$ for all $n \in \mathbb{N}$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left(|x| < \frac{\pi}{2}\right).$$

PART A. CONJECTURES THAT REMAIN UNSOLVED

Conjecture A1 ([S09e]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Also,

$$3 \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{7}(3p - 4x^2) = 32y^2 - \frac{8}{7}p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ } (x, y \in \mathbb{Z}), \\ \frac{8}{7}p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = -1. \end{cases}$$

Remark. (a) Marie Jameson and Ken Ono (at Wisconsin Univ.) are working on this conjecture but they have not yet proved it fully.

(b) Let p be an odd prime with $\left(\frac{p}{7}\right) = 1$. It is well known that $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$. We ever wrote that the author was unable to guess $\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \pmod{p}$ in the case $\left(\frac{p}{7}\right) = 1$ though we conjectured the first congruence in Conj. A1 on November 13, 2009. After reading this remark, on Nov. 28, 2009 Bilgin Ali and Bruno Mishutka guessed that

$$\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} 11y^2/3 - x^2 \pmod{p} & \text{if } 3 \mid y, \\ 4(y^2 - x^2)/3 \pmod{p} & \text{if } 3 \nmid y. \end{cases}$$

Since $x^2 \equiv -7y^2 \pmod{p}$, we can simplify the congruence as follows:

$$\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv -\frac{32}{21}x^2 \equiv \frac{32}{3}y^2 \pmod{p}.$$

Note that the second congruence in Conj. A1 is now a congruence mod p^2 . By [S10, Theorem 1.3], this congruence follows from the first congruence in Conj. A1.

Conjecture A2 ([S10]). *For any prime $p > 3$ we have*

$$\sum_{k=0}^{(p-1)/2} (21k + 8) \binom{2k}{k}^3 \equiv 8p + (-1)^{(p-1)/2} 32p^3 E_{p-3} \pmod{p^4},$$

where E_0, E_1, E_2, \dots are Euler numbers. If p is a prime and a is a positive integer with $p^a \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{\lfloor \frac{2}{3}p^a \rfloor} (21k+8) \binom{2k}{k}^3 \equiv 8p^a \pmod{p^{a+5+(-1)^p}}.$$

Also, for each prime $p > 5$ we have

$$\sum_{k=1}^{p-1} \frac{21k-8}{k^3 \binom{2k}{k}^3} + \frac{p-1}{p^3} \equiv \frac{H_{p-1}}{p^2} (15p-6) + \frac{12}{5} p^2 B_{p-5} \pmod{p^3}.$$

Remark. (a) By [S10] the first congruence in Conj. A2 has the following equivalent form:

$$\sum_{k=1}^{(p-1)/2} \frac{21k-8}{k^3 \binom{2k}{k}^3} \equiv (-1)^{(p+1)/2} 4E_{p-3} \pmod{p} \quad (\text{for any prime } p > 3).$$

Note that $\sum_{k=1}^{\infty} (21k-8)/(k^3 \binom{2k}{k}^3) = \zeta(2) = \pi^2/6$ by [Z] or [PP, (7)].

(b) The author [S10] proved that for any prime p and positive integer a we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} (21k+8) \binom{2k}{k}^3 \equiv 8 + 16p^3 B_{p-3} \pmod{p^4},$$

where B_{-1} is regarded as zero, and B_0, B_1, B_2, \dots are Bernoulli numbers. (When $p > 5$, the congruence even holds mod p^5 if we replace $16p^3 B_{p-3}$ by $-48p H_{p-1}$.) See Conj. A1 for our guess on $\sum_{k=0}^{p-1} \binom{2k}{k}^3$ and $\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \pmod{p^2}$. We also conjecture that any integer $n > 1$ satisfying the congruence $\sum_{k=0}^{n-1} (21k+8) \binom{2k}{k}^3 \equiv 8n \pmod{n^4}$ must be a prime; this has been verified for $n \leq 10^4$.

Conjecture A3 ([S09e, S10]). (i) *Set*

$$a_n := \frac{1}{4n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (35k+8) \binom{4k}{k, k, k, k} 81^{n-1-k} \quad \text{for } n \in \mathbb{Z}^+.$$

(Note that $a_1 = 1/3$ and $(4n+6)a_{n+1} = 81na_n + (35n+8) \frac{\binom{2n-1}{n} C_{2n}}{2}$ for $n = 1, 2, 3, \dots$.) Then $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let p be a prime. If $p > 3$ then

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{35k+8}{81^k} \binom{4k}{k, k, k, k} \equiv 8 \pmod{p^3} \quad \text{for all } a \in \mathbb{Z}^+.$$

If $\left(\frac{p}{7}\right) = 1$, i.e., $p \equiv 1, 2, 4 \pmod{7}$, then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{81^k} \pmod{p^3}$$

and

$$3 \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv 5 \sum_{k=0}^{p-1} \frac{k \binom{4k}{k, k, k, k}}{81^k} \pmod{p^3}.$$

If $\left(\frac{p}{7}\right) = -1$ and $p \neq 3$, then

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{81^k} \equiv 0 \pmod{p^2}.$$

(iii) We have

$$\sum_{k=1}^{\infty} \frac{(35k-8)81^k}{k^3 \binom{4k}{k, k, k, k}} = 12\pi^2.$$

Conjecture A4 ([S09e, S10]). (i) Set

$$a_n := \frac{1}{n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (11k+3) \binom{2k}{k}^2 \binom{3k}{k} 64^{n-1-k} \quad \text{for } n \in \mathbb{Z}^+.$$

(Note that $a_1 = 1/2$ and $(2n+3)a_{n+1} = 32na_n + (11n+3) \binom{2n-1}{n} C_n^{(2)}$ for $n = 1, 2, 3, \dots$, where $C_n^{(2)}$ denotes the second-order Catalan number $\binom{3n}{n} / (2n+1) = \binom{3n}{n} - 2 \binom{3n}{n-1}$). Then $a_n \in \mathbb{Z}$ for every $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases} \end{aligned}$$

Furthermore,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{11k+3}{64^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 3 + \frac{7}{2} p^3 B_{p-3} \pmod{p^4} \quad \text{for all } a \in \mathbb{Z}^+.$$

(iii) We have

$$\sum_{k=1}^{\infty} \frac{(11k-3)64^k}{\binom{2k}{k}^2 \binom{3k}{k}} = 8\pi^2.$$

Also, if $p > 3$ is a prime then

$$p \sum_{k=1}^{(p-1)/2} \frac{(11k-3)64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \equiv 32q_p(2) - \frac{64}{3}p^2 B_{p-3} \pmod{p^3},$$

where $q_p(2) = (2^{p-1} - 1)/p$.

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11}) = 1$ we can write $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$. The only known result about the parameters in the representation $4p = x^2 + 11y^2$ is the following one due to Jacobi (see, e.g., [HW]): If $p = 11f + 1$ is a prime and $4p = x^2 + 11y^2$ with $x \equiv 2 \pmod{11}$, then $x \equiv \binom{6f}{3f} \binom{3f}{f} / \binom{4f}{2f} \pmod{p}$.

Conjecture A5 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (10k+3) \binom{2k}{k}^2 \binom{3k}{k} 8^{n-1-k}.$$

(Note that $a_1 = 1/2$ and $(2n+3)a_{n+1} = 4na_n + (10n+3) \binom{2n-1}{n} C_n^{(2)}$ for $n = 1, 2, 3, \dots$) Then $a_n \in \mathbb{Z}$ for all $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-2}{p}) = 1 \text{ \& } p = x^2 + 2y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p}) = -1. \end{cases} \end{aligned}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{10k+3}{8^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 3 + \frac{49}{8}p^3 B_{p-3} \pmod{p^4}.$$

(iii) We have

$$\sum_{k=1}^{\infty} \frac{(10k-3)8^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = \frac{\pi^2}{2}.$$

Conjecture A6 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$\frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (5k+1) \binom{2k}{k}^2 \binom{3k}{k} (-192)^{n-1-k}.$$

(Note that $a_1 = 1/6$ and $(2n+3)a_{n+1} + 96na_n = (5n+1)\binom{2n-1}{n}C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Then $a_n \in \mathbb{Z}$ for $n = 2, 3, 4, \dots$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{5k+1}{(-192)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2} \text{ for any } a \in \mathbb{Z}^+.$$

We also have

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-192)^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } 4p = x^2 + 27y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Remark. It is well known that for any prime $p \equiv 1 \pmod{3}$ there are unique $x, y \in \mathbb{Z}^+$ such that $4p = x^2 + 27y^2$ (see, e.g., [C]). Also, Ramanujan [R] found that

$$\sum_{k=0}^{\infty} (5k+1) \left(-\frac{9}{16}\right)^k \frac{(1/2)_k (1/3)_k (2/3)_k}{(1)_k^3} = \sum_{k=0}^{\infty} \frac{5k+1}{(-192)^k} \binom{2k}{k}^2 \binom{3k}{k} = \frac{4\sqrt{3}}{\pi}.$$

Conjecture A7 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$\frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (15k+4) \binom{2k}{k}^2 \binom{3k}{k} (-27)^{n-1-k}.$$

(Note that $a_1 = 1/3$ and $(4n+6)a_{n+1} + 27na_n = (15n+4)\binom{2n-1}{n}C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Then $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2 \text{ } (x, y \in \mathbb{Z}), \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 5x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{15k+4}{(-27)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4 \left(\frac{p^a}{3} \right) \pmod{p^2}.$$

Remark. Let $p > 5$ be a prime. By the theory of binary quadratic forms (cf. [C]), if $p \equiv 1, 4 \pmod{15}$ then $p = x^2 + 15y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 2, 8 \pmod{15}$ then $p = 5x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$.

Conjecture A8 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^2 \binom{3k}{k} 6^{3(n-1-k)}$$

and

$$b_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (8k+1) \binom{4k}{k, k, k, k} 48^{2(n-1-k)}.$$

(Note that $a_1 = 1/6$ and $b_1 = 1/12$. Also, $(2n+3)a_{n+1} = 108na_n + (6n+1) \binom{2n-1}{n} C_n^{(2)}$ and $(2n+3)b_{n+1} = 1152nb_n + (8n+1) \binom{2n-1}{n} \frac{C_{2n}}{2}$ for all $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $a_n, b_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n, 3b_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{6^{3k}} \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \text{ } (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1 \text{ i.e., } p \equiv 13, 17, 19, 23 \pmod{24}; \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{48^{2k}} \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 4x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1 \text{ i.e., } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases} \end{aligned}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{6k+1}{6^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv \left(\frac{p^a}{3} \right) \pmod{p^2}$$

and

$$\begin{aligned} & \frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} \\ & \equiv \frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}. \end{aligned}$$

Remark. (a) Let $p > 3$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1, 7 \pmod{24}$ then $p = x^2 + 6y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 5, 11 \pmod{24}$ then $p = 2x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$.

(b) Ramanujan [R] found that

$$\sum_{k=0}^{\infty} (6k+1) \frac{(1/2)_k (1/3)_k (2/3)_k}{2^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{6k+1}{6^{3k}} \binom{2k}{k}^2 \binom{3k}{k} = \frac{3\sqrt{3}}{\pi}$$

and

$$\sum_{k=0}^{\infty} (8k+1) \frac{(1/2)_k (1/4)_k (3/4)_k}{9^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} = \frac{2\sqrt{3}}{\pi}.$$

Conjecture A9 ([S10]). For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (51k+7) \binom{2k}{k}^2 \binom{3k}{k} (-12^3)^{n-1-k}.$$

(Note that $a_1 = 7/6$ and $(2n+3)a_{n+1} + 864na_n = (51n+7) \binom{2n-1}{n} C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $(-1)^{n-1} a_n \in \mathbb{Z}^+$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{51k+7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 7 \left(\frac{p^a}{3}\right) \pmod{p^2} \text{ for any } a \in \mathbb{Z}^+.$$

We also have

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-12^3)^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } 4p = x^2 + 51y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 4p = 3x^2 + 17y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = -\left(\frac{p}{17}\right). \end{cases} \end{aligned}$$

Remark. (a) Let $p > 3$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $\left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1$ then $4p = x^2 + 51y^2$ for some $x, y \in \mathbb{Z}$; if $\left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1$ then $4p = 3x^2 + 17y^2$ for some $x, y \in \mathbb{Z}$. (b) Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (51k + 7) \frac{(1/2)_k (1/3)_k (2/3)_k}{(-16)^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{51k + 7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} = \frac{12\sqrt{3}}{\pi}.$$

Conjecture A10 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (20k+3) \binom{4k}{k, k, k, k} (-2^{10})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(2n+3)a_{n+1} + 512na_n = (20n+3)\binom{2n-1}{n}\frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Then $(-1)^{n-1}a_n \in \mathbb{Z}^+$ for all $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} \equiv 3p \left(\frac{-1}{p}\right) + 3p^3 E_{p-3} \pmod{p^4},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} \equiv p \left(\frac{-1}{p}\right) (2^{p-1} + 2 - (2^{p-1} - 1)^2) \pmod{p^4}$$

provided $p > 3$. We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{10})^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2(p - x^2) \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-5}{p}\right) = -1, \text{ i.e., } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

Remark. Let $p \neq 2, 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1, 9 \pmod{20}$ then $p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 3, 7 \pmod{20}$ then $2p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$. Note also that the second congruence mod p^3 in part (ii) has been obtained by W. Zudilin [Zu] as a p -adic analogue of the Ramanujan series

$$\sum_{k=0}^{\infty} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} = \frac{8}{\pi}.$$

Conjecture A11 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (10k+1) \binom{4k}{k, k, k, k} 12^{4(n-1-k)}.$$

(Note that $a_1 = 1/12$ and $(2n+3)a_{n+1} = 10368na_n + (10n+1)\binom{2n-1}{n}\frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{10k+1}{12^{4k}} \binom{4k}{k, k, k, k} \equiv p \left(\frac{-2}{p} \right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{12^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ \& } p = x^2 + 10y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 7, 13, 23, 37 \pmod{40} \text{ \& } p = 2x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-10}{p} \right) = -1, \text{ i.e., } p \equiv 3, 17, 21, 27, 29, 31, 33, 39 \pmod{40}. \end{cases}$$

Remark. (a) Let $p > 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $\left(\frac{-2}{p} \right) = \left(\frac{p}{5} \right) = 1$ then $p = x^2 + 10y^2$ for some $x, y \in \mathbb{Z}$; if $\left(\frac{-2}{p} \right) = \left(\frac{p}{5} \right) = -1$ then $p = 2x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$.

(b) Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (10k+1) \frac{(1/2)_k (1/4)_k (3/4)_k}{81^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{10k+1}{12^{4k}} \binom{4k}{k, k, k, k} = \frac{9\sqrt{2}}{4\pi}.$$

Conjecture A12 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (28k+3) \binom{4k}{k, k, k, k} (-3 \times 2^{12})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(2n+3)a_{n+1} + 6144na_n = (28n+3)\binom{2n-1}{n}\frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Then we have $(-1)^{n-1}a_n \in \mathbb{Z}^+$ for all $n = 2, 3, 4, \dots$

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{28k+3}{(-3 \times 2^{12})^k} \binom{4k}{k, k, k, k} \equiv 3p \left(\frac{p}{3} \right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(3 \times 2^{12})^k} \equiv \begin{cases} (-1)^{\lfloor x/6 \rfloor} (4x^2 - 2p) \pmod{p^2} & \text{if } 12 \mid p-1 \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x-1 \text{ \& } 2 \mid y\text{),} \\ -4\left(\frac{xy}{3}\right)xy \pmod{p^2} & \text{if } 12 \mid p-5 \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x-1 \text{ \& } 2 \mid y\text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Remark. Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (28k+3) \frac{(1/2)_k (1/4)_k (3/4)_k}{(-48)^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{28k+3}{(-3 \times 2^{12})^k} \binom{4k}{k,k,k,k} = \frac{16\sqrt{3}}{3\pi}.$$

Conjecture A13 ([S10]). *For any prime $p > 3$ we have*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) - \frac{25}{9} p^2 E_{p-3} \pmod{p^3}.$$

Remark. A related conjecture of Rodriguez-Villega [RV] proved by Mortenson [M2] states that if $p > 3$ is a prime then

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} = \sum_{k=0}^{p-1} \frac{(6k)!}{k!(2k)!(3k)!} (2^4 3^3)^{-k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Using the Gosper algorithm we find the identity

$$\sum_{k=0}^n (36k+5) \binom{6k}{3k} \binom{3k}{k} 432^{n-k} = (6n+1)(6n+5) \binom{6n}{3n} \binom{3n}{n} \quad (n \in \mathbb{N})$$

which implies that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^n \frac{36k+5}{432^k} \binom{6k}{3k} \binom{3k}{k} = \frac{18}{\pi}$$

and

$$\sum_{k=0}^{p-1} \frac{36k+5}{432^k} \binom{6k}{3k} \binom{3k}{k} \equiv 5p^2 \pmod{p^3} \quad \text{for any prime } p > 3.$$

Conjecture A14 ([S10]). *Let $p > 3$ be a prime. When $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \begin{cases} (-1)^{\lfloor x/6 \rfloor} (2x - p/(2x)) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ \left(\frac{xy}{3}\right) (2y - p/(2y)) \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p^a-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2}.$$

Provided that $p \equiv 1 \pmod{4}$, we have

$$\sum_{k=0}^{p^a-1} \frac{k \binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^a} \quad \text{for all } a \in \mathbb{Z}^+.$$

Conjecture A15 ([S10]). *For any prime $p \equiv 1 \pmod{4}$ we have*

$$\sum_{k=0}^{p-1} \frac{36k + 5}{12^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 0 \pmod{p^2}.$$

Remark. A related conjecture of Rodriguez-Villega [RV] partially confirmed by Mortenson [M2] states that if $p > 3$ is a prime then

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k, k, k}}{12^{3k}} &= \sum_{k=0}^{p-1} \frac{(6k)!}{(3k)!(k!)^3} 1728^{-k} \\ &\equiv \begin{cases} \left(\frac{p}{3}\right) (4x^2 - 2p) \pmod{p^2} & \text{if } 4 \mid p-1 \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x, 2 \mid y \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Conjecture A16 ([S10]). (i) *For $n \in \mathbb{Z}^+$ set*

$$a_n := \frac{1}{10n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (154k + 15) \binom{6k}{3k} \binom{3k}{k, k, k} (-2^{15})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(4n+6)a_{n+1} + 32768na_n = \left(\frac{154}{5}n + 3\right) \binom{6n-1}{3n} C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $(-1)^{n-1}a_n \in \mathbb{Z}^+$ unless $2n+1$ is a power of 5 in which case $5a_n \in \mathbb{Z} \setminus 5\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{154k+15}{(-2^{15})^k} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 15p \left(\frac{-2}{p} \right) \pmod{p^3}.$$

We also have

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k, k, k}}{(-2^{15})^k} \\ & \equiv \begin{cases} \left(\frac{-2}{p} \right) (x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{11} \right) = 1 \text{ \& } 4p = x^2 + 11y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11} \right) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases} \end{aligned}$$

Remark. Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (28k+3) \frac{(-27)^k}{2^{9k}} \cdot \frac{(1/2)_k (1/6)_k (5/6)_k}{(1)_k^3} = \sum_{k=0}^{\infty} \frac{154k+15}{(-2^{15})^k} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{32\sqrt{2}}{\pi}.$$

Conjecture A17 (The first and the second congruences were discovered during March 13-15, 2010 while all the others were discovered during Feb. 24-26, 2010). Let $p > 5$ be a prime. If $p > 7$ then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} H_{p-1} - \frac{13}{27} \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^4}.$$

We also have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - \frac{H_{p-1}}{p^3} & \equiv \frac{7}{54p} \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -\frac{7}{45} p B_{p-5} \pmod{p^2}, \\ \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} & \equiv -2B_{p-3} \pmod{p}, \\ \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \binom{2k}{k} & \equiv \frac{56}{15} p B_{p-3} \pmod{p^2}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k 16^k} & \equiv -2H_{(p-1)/2} - \frac{7}{2} p^2 B_{p-3} \pmod{p^3}, \\ \sum_{p/2 < k < p} \frac{\binom{2k}{k}^2}{k 16^k} & \equiv -\frac{21}{2} H_{p-1} \pmod{p^4}. \end{aligned}$$

Remark. It is known that $H_{p-1}/p^2 \equiv -B_{p-3}/3 \pmod{p}$ for any prime $p > 3$ and $\sum_{k=1}^{p-1} 1/k^3 \equiv -\frac{6}{5}p^2 B_{p-5}$ for each prime $p > 5$, and that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

Also, [T, Theorem 4.2] implies that $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv -\frac{4}{15}pB_{p-3} \pmod{p^2}$ for any prime $p > 5$. Tauraso conjectured that $\sum_{k=1}^{p-1} \binom{2k}{k}^2 / (k16^k) \equiv -2H_{(p-1)/2} \pmod{p^3}$ for each prime $p > 3$. During March 6-7, 2010 the author proved that the third, the fourth and the fifth congruences in Conj. A17 are pairwise equivalent and that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3}E_{p-3} \pmod{p}$$

for any prime $p > 3$.

Conjecture A18 (Discovered during March 5-15, 2010, see [S10]). *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{(2k-1)\binom{2k}{k}} \equiv E_{p-3} + (-1)^{(p-1)/2} - 1 \pmod{p},$$

which is actually equivalent to

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)4^k} \equiv pE_{p-3} \pmod{p^2}.$$

If $p > 3$, then

$$\sum_{k=1}^{p-1} \frac{4^k}{k^2 \binom{2k}{k}} + \frac{4q_p(2)}{p} \equiv -2q_p^2(2) + pB_{p-3} \pmod{p^2}$$

and

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{p}{8} \sum_{k=1}^{(p-1)/2} \frac{16^k}{k(2k-1)\binom{2k}{k}} \equiv \frac{p}{3}E_{p-3} \pmod{p^2}.$$

Remark. R. Sprugnoli [Sp] proved that $\sum_{k=1}^{\infty} 4^k / (k^2 \binom{2k}{k}) = \pi^2/2$ and $\sum_{k=2}^{\infty} 4^k / ((k-1)^2 \binom{2k}{k}) = \pi^2 - 4$. Let $p > 3$ be a prime. During March 6-7, 2010 the author showed that

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}$$

(which is equivalent to $\sum_{p/2 < k < p} \binom{2k}{k} / (k4^k) \equiv (-1)^{(p-1)/2} 2pE_{p-3} \pmod{p}$) and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)4^k} \equiv (-1)^{(p+1)/2} q_p(2) \pmod{p^2},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$. On March 9, 2010 he proved that

$$\sum_{k=2}^{(p-1)/2} \frac{4^k}{(k-1)^2 \binom{2k}{k}} \equiv 8E_{p-3} - 4 - 12 \left(\frac{-1}{p} \right) \pmod{p}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{4^k}{(k+1) \binom{2k}{k}} \equiv \left(\frac{-1}{p} \right) (4 - 2E_{p-3}) - 2 \pmod{p}.$$

Conjecture A19 (The first, the second and the third congruences were discovered on March 14, 2010 while the other two were discovered on March 6, 2010). *Let $p > 5$ be a prime and let $H_{p-1} = \sum_{k=1}^{p-1} 1/k$. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv (-1)^{(p-1)/2} \left(\frac{H_{p-1}}{12} + \frac{3p^4}{160} B_{p-5} \right) \pmod{p^5}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} \equiv (-1)^{(p-1)/2} \left(\frac{H_{p-1}}{4p^2} + \frac{p^2}{36} B_{p-5} \right) \pmod{p^3}.$$

We also have

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 (-16)^k} &\equiv \frac{H_{p-1}}{5p} \pmod{p^3}, \\ \sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)^2 (-16)^k} &\equiv -\frac{p}{4} B_{p-3} \pmod{p^2}, \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 (-32)^k} &\equiv -\left(\frac{2}{p} \right) \frac{q_p^2(2)}{2} \pmod{p}. \end{aligned}$$

Remark. On March 6, 2010 the author proved the first congruence in Conj. A19 mod p^2 and the second congruence mod p , and used some new identities to establish the following congruences (for any prime $p > 3$):

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 4^k} \equiv (-1)^{(p+1)/2} \frac{q_p^2(2)}{2} \pmod{p},$$

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \equiv - \left(\frac{-2}{p}\right) \frac{q_p(2)}{2} + \left(\frac{-2}{p}\right) \frac{p}{8} q_p^2(2) \pmod{p^2}.$$

Conjecture A20 ([S10]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{4}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p^a-1)/2} \pmod{p^3}.$$

If $p > 3$, and $p \equiv 1, 3 \pmod{8}$ or $a > 1$, then

$$\sum_{k=0}^{\lfloor \frac{r}{8}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p^a-1)/2} \pmod{p^3} \quad \text{for } r = 5, 7.$$

Remark. The author [S10] showed that $\sum_{k=0}^{\lfloor p/2 \rfloor} \binom{2k}{k}^2 / 16^k \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}$ for any odd prime p .

Conjecture A21 ([S09e, S10]). *Let $p > 3$ be a prime. If $p \equiv 1 \pmod{6}$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv 4x^2 - 2p \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^3}{16^k} \equiv p - \frac{4x^2}{3} \pmod{p^2}.$$

If $p \equiv 5 \pmod{6}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^3}{16^k} \equiv \frac{p}{3} \pmod{p^2}.$$

Furthermore,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv 1 + \frac{7}{6} p^3 B_{p-3} \pmod{p^4} \quad \text{for all } a \in \mathbb{Z}^+,$$

and

$$\sum_{k=0}^{(p-1)/2} (3k+1) \frac{\binom{2k}{k}^3}{16^k} \equiv p + 2 \left(\frac{-1}{p} \right) p^3 E_{p-3} \pmod{p^4}.$$

Also,

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 16^{n-1-k} \in \mathbb{Z} \text{ for all } n = 2, 3, 4, \dots,$$

and

$$\sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3 \binom{2k}{k}^3} = \frac{\pi^2}{2}.$$

Remark. The author [S09e] determined $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / 16^k \pmod{p}$. Note that $a_1 = 1/4$ and $(2n+1)a_{n+1} = 8na_n + (3n+1)\binom{2n-1}{n}^2$ for $n = 1, 2, 3, \dots$

Conjecture A22 ([S10]). *Let p be an odd prime. If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{64^k} \binom{2k}{k}^3 \equiv 0 \pmod{p^2}.$$

Remark. Let p be an odd prime. Rodriguez-Villegas [RV] conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv a(p) \pmod{p^2},$$

where the sequence $\{a(n)\}_{n \geq 1}$ is defined by

$$\sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{4n})^6.$$

This was proved by many authors, see, e.g., E. Mortenson [M2]. Ishikawa [I] pointed out that if $p = x^2 + y^2$ with x odd and y even then $a(p) = 4x^2 - 2p$ by the Jacobi-Macdonald formula. The author [S10] showed that $\sum_{p/2 < k < p} (4k+1) \binom{2k}{k}^3 / 64^k \equiv 0 \pmod{p^4}$ for any prime $p \equiv 3 \pmod{4}$.

Conjecture A23 ([S09e, S10]). *Let p be an odd prime. If $\left(\frac{-2}{p}\right) = 1$ (i.e., $p \equiv 1, 3 \pmod{8}$) and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv (-1)^{(p-1)/2} (4x^2 - 2p) \pmod{p^2}.$$

If $\left(\frac{-2}{p}\right) = -1$ (i.e., $p \equiv 5, 7 \pmod{8}$), then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2}.$$

Also,

$$\sum_{k=0}^{(p-1)/2} (4k+1) \frac{\binom{2k}{k}^3}{(-64)^k} \equiv p \left(\frac{-1}{p}\right) + p^3 E_{p-3} \pmod{p^4}.$$

Moreover,

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (4k+1) \binom{2k}{k}^3 (-64)^{n-1-k} \in \mathbb{Z} \text{ for all } n = 2, 3, 4, \dots$$

Remark. E. Mortenson [M4] proved the following conjecture of van Hamme [vH]:

$$\sum_{k=0}^{(p-1)/2} (4k+1) \binom{-1/2}{k}^3 \equiv (-1)^{(p-1)/2} p \pmod{p^3} \text{ for any odd prime } p.$$

Note that $\binom{-1/2}{k}^3 = \binom{2k}{k}^3 / (-64)^k$ for $k = 0, 1, 2, \dots$. Also, $a_1 = 1/4$ and $(2n+1)a_{n+1} + 32na_n = (4n+1)\binom{2n-1}{n}^2$ for $n = 1, 2, 3, \dots$. On March 5, 2010 the author showed that the third congruence in Conj. A23 is equivalent to the first one in Conjecture A18. By [S10], for any odd prime p we have $\sum_{p/2 < k < p} (4k+1) \binom{2k}{k}^3 / (-64)^k \equiv 0 \pmod{p^4}$.

Conjecture A24 ([S10]). (i) For each $n = 2, 3, \dots$ we have

$$\begin{aligned} \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^3 256^{n-1-k} &\in \mathbb{Z}, \\ \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^3 (-512)^{n-1-k} &\in \mathbb{Z}, \\ \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (42k+5) \binom{2k}{k}^3 4096^{n-1-k} &\in \mathbb{Z}. \end{aligned}$$

(ii) Let p be an odd prime. We have

$$\frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{42k+5}{4096^k} \binom{2k}{k}^3 \equiv \left(\frac{-1}{p^a}\right) \left(5 + \frac{p^3}{4} B_{p-3}\right) \pmod{p^4}$$

for all $a \in \mathbb{Z}^+$. When $p > 5$ the congruence even holds mod p^5 if we replace $\frac{p^3}{4}B_{p-3}$ by $-\frac{3}{4}pH_{p-1}$. If $p > 3$, then

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{6k+1}{256^k} \binom{2k}{k}^3 &\equiv p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{(p-1)/2} \frac{6k+1}{(-512)^k} \binom{2k}{k}^3 &\equiv p \left(\frac{-2}{p} \right) + \frac{p^3}{4} \left(\frac{2}{p} \right) E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} \frac{42k+5}{4096^k} \binom{2k}{k}^3 &\equiv 5p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}. \end{aligned}$$

Remark. Those congruences in part (ii) mod p^3 are van Hamme's conjectures (cf. [vH]) which are p -adic analogues of corresponding Ramanujan series.

Conjecture A25 ([S09e]). *If $p > 5$ is a prime with $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p^a-1} \frac{k^3 \binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for all } a = 1, 2, 3, \dots$$

Remark. For any prime p the author has determined $\sum_{k=0}^{p-1} k^3 \binom{2k}{k}^3 / 64^k$ modulo p .

Conjecture A26 ([S09e, S10]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} \equiv p \left(\frac{-1}{p} \right) + p^3 E_{p-3} \pmod{p^4}$$

and furthermore

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 (-8)^{n-1-k} \in \mathbb{Z}^+ \quad \text{for all } n = 2, 3, 4, \dots$$

(Note that $a_1 = 1/4$ and $(2n+1)a_{n+1} + 4na_n = (3n+1)\binom{2n-1}{n}^2$ for $n = 1, 2, 3, \dots$) If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \left(1 - \frac{1}{(-8)^k} \right) \equiv 0 \pmod{p^3}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \left(1 + \frac{1}{(-2)^k}\right) \equiv 0 \pmod{p^3}.$$

Conjecture A27 ([S10]). *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \pmod{p^2}. \end{aligned}$$

Moreover, if $p \equiv 1 \pmod{4}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \pmod{p^3};$$

if $p \equiv 1 \pmod{3}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \pmod{p^3}.$$

Conjecture A28 (Discovered on March 2, 2010). *Let p be an odd prime.*

(i) *If $p \equiv 1 \pmod{4}$ then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv \frac{1}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} \\ &\equiv \frac{1}{3} \left(\frac{2}{p}\right) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}; \end{aligned}$$

when $p \equiv 3 \pmod{4}$ we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv -\frac{7}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv -\left(\frac{2}{p}\right) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}, \end{aligned}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p} \text{ for } m = -8, 64, -512 \text{ if } p > 3.$$

(ii) If $p \equiv 1 \pmod{3}$ then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{1}{2} \left(\frac{-1}{p} \right) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}.$$

If $p \equiv 2 \pmod{3}$ then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p^2}.$$

(iii) If $p \equiv 5, 7 \pmod{8}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-64)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p}.$$

Remark. Partially motivated by an observation of M. Jameson and K. Ono which occurred during their attempt to prove Conjecture A1, we discovered the current Conjecture A28.

Conjecture A29 (Discovered on March 3, 2010). *Let $p > 3$ be a prime.*

(i) If $p \equiv 2 \pmod{3}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \left(\frac{-1}{p} \right) \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{1}{6} \pmod{p}.$$

(ii) If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \left(\frac{2}{p} \right) \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv -\frac{1}{6} \pmod{p}.$$

(iii) If $p \equiv 5, 7 \pmod{8}$ then

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-64)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{(-1)^{(p-1)/2}}{6} \pmod{p}.$$

Remark. For certain reasons we omit other similar observations.

Conjecture A30 ([S09e, S10]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{24^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{(-216)^k} \equiv \begin{cases} \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod{6}, \\ 0 \pmod{p} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$

Remark. In [M2] Mortenson proved the following conjecture of Rodriguez-Villegas [RV]: For any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{27^k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Conjecture A31 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k+1}}{48^k} \equiv 0 \pmod{p^2}.$$

If $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv \frac{p}{2x} - x \pmod{p^2}.$$

If $p \equiv 2 \pmod{3}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv 0 \pmod{p}.$$

Conjecture A32 ([S09e]). *Let $p > 3$ be a prime. If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv (-1)^{(p-1)/4} \left(\frac{p}{3}\right) \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv (-1)^{(p-1)/4} \left(\frac{p}{3}\right) \left(x - \frac{p}{2x}\right) \pmod{p^2}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv 0 \pmod{p}.$$

Conjecture A33 ([S09e]). *Let p be an odd prime. If $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$, then*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv \begin{cases} (-1)^{(p-1)/8+(x-1)/2} (2x - p/(2x)) \pmod{p^2} & \text{if } 8 \mid p-1, \\ p/(2x) - 2x \pmod{p^2} & \text{if } 8 \mid p-3. \end{cases}$$

If $p \equiv 5, 7 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2}.$$

Conjecture A34 ([S09e]). *Let $p > 3$ be a prime. If $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ with $(\frac{x}{7}) = 1$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv \left(\frac{p}{3}\right) \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv 8 \left(\frac{p}{3}\right) \left(\frac{p}{2x} - x\right) \pmod{p^2}.$$

If $(\frac{p}{7}) = -1$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}^2}{63^k} \equiv 0 \pmod{p}.$$

Conjecture A35 ([S09e]). *For any prime $p \equiv 1 \pmod{4}$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{2k}{k+1}^2}{(-8)^k} \equiv -2p \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+1}}{64^k} \equiv 0 \pmod{p^2}.$$

Conjecture A36 ([S09e]). *For any prime $p \equiv 3 \pmod{4}$, we have*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^2}{(-8)^k} \equiv 0 \pmod{p^2}.$$

Remark. When $p > 3$ is a prime congruent to 3 mod 4, the author proved on March 5, 2010 that

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2},$$

which is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} H_k \equiv 0 \pmod{p} \quad \text{with} \quad H_k = \sum_{0 < j \leq k} \frac{1}{j},$$

since $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / 64^k \equiv 0 \pmod{p^2}$ and $\binom{p-1}{k} (-1)^k \equiv 1 - pH_k \pmod{p^2}$.

Conjecture A37 (Discovered on March 11, 2010). *For any $k, n \in \mathbb{Z}^+$ we have*

$$(2n+1) \binom{2n}{n} C_{n+k} \binom{n+k+1}{2k} \equiv 0 \pmod{\binom{2k}{k}}.$$

Conjecture A38 ([S09e]). *Let p be an odd prime. If $p \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{k C_k^3}{16^k} \equiv 2p - 2 \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^3}{64^k} \equiv 8 \pmod{p^2}.$$

Remark. The author [S09e] determined $\sum_{k=0}^{p-1} C_k^3 / 64^k$ modulo any odd prime p .

Conjecture A39 ([S10]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv (-1)^{(p-1)/2} - 3p^2 E_{p-3} \pmod{p^3}.$$

Remark. Mortenson [M2] proved the following conjecture of Rodriguez-Villegas [RV]: For any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p} \right) \pmod{p^2}.$$

Conjecture A40 ([S09e]). *Let p be a prime with $p \equiv 1, 3 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{128^k} \equiv p \pmod{p^2}.$$

Conjecture A41 ([S09e]). *Let p be a prime with $p \equiv 1 \pmod{3}$ and hence $p = x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. Then we have*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{3k}{k}}{54^k} \equiv 0 \pmod{p} \text{ and } \sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{54^k} \equiv p \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k+1}}{108^k} \equiv 0 \pmod{p} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \frac{4}{9}(p - 2x^2) \pmod{p^2}.$$

Remark. In [S09a] the author determined $\sum_{k=0}^{p-1} \binom{3k}{k}/m^k \pmod{p}$ for any prime $p > 3$ and any $m \in \mathbb{Z}$ with $p \nmid m$. A conjecture of Rodriguez-Villegas [RV] has the following equivalent form:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1 \text{ \& } p = x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

See [M3] for related result.

Conjecture A42 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_k^{(2)}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k} C_k^{(2)}}{27^k} \equiv 0 \pmod{p^2},$$

where

$$C_k^{(2)} = \frac{\binom{3k}{k}}{2k+1} = \binom{3k}{k} - 2 \binom{3k}{k-1}$$

is a second-order Catalan number (of the first kind). Furthermore,

$$\sum_{k=0}^{p-1} (4k+1) \frac{\binom{2k}{k} C_k^{(2)}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^4}.$$

Conjecture A43 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{C_k C_k^{(2)}}{27^k} \equiv 2 \left(\frac{p}{3} \right) - p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{C_k \bar{C}_k^{(2)}}{27^k} \equiv -7 \pmod{p},$$

where

$$\bar{C}_k^{(2)} = \frac{2}{k+1} \binom{3k}{k} = 2 \binom{3k}{k} - \binom{3k}{k+1}$$

is a second-order Catalan number of the second kind. Hence

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k-1} \binom{3k}{k-1}}{27^k} \equiv \left(\frac{p}{3} \right) - p \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k+1} \binom{3k}{k+1}}{27^k} \equiv 2 \left(\frac{p}{3} \right) - 7 \pmod{p}.$$

Remark. Note that

$$\binom{2k}{k-1} \binom{3k}{k-1} = \left(\binom{2k}{k} - C_k \right) \frac{\binom{3k}{k} - C_k^{(2)}}{2}$$

and

$$\binom{2k}{k+1} \binom{3k}{k+1} = \left(\binom{2k}{k} - C_k \right) \left(2 \binom{3k}{k} - \bar{C}_k^{(2)} \right).$$

Conjecture A44 ([S09h]). *Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \binom{k}{3} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-3)/4} \left(4y - \frac{p}{3y} \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \binom{k}{3} \frac{k \binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p+1)/4} y \pmod{p^2}.$$

If $p \equiv 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{k}{3} \frac{\binom{2k}{k}^2}{16^k} \equiv 0 \pmod{p^2}.$$

Recall that the Pell sequence $\{P_n\}_{n \geq 0}$ and its companion $\{Q_n\}_{n \geq 0}$ are defined as follows:

$$\begin{aligned} P_0 &= 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad (n = 1, 2, 3, \dots); \\ Q_0 &= 2, \quad Q_1 = 2, \quad \text{and} \quad Q_{n+1} = 2Q_n + Q_{n-1} \quad (n = 1, 2, 3, \dots). \end{aligned}$$

Conjecture A45 ([S09h]). (i) *Let p be a prime with $p \equiv 1, 3 \pmod{8}$. Write $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(p-3)/8} (p/(2x) - 2x) \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{kP_k}{(-8)^k} \binom{2k}{k}^2 \equiv \frac{(-1)^{(x+1)/2}}{2} \left(x + \frac{p}{2x}\right) \pmod{p^2}.$$

(ii) *If $p \equiv 5 \pmod{8}$ is a prime, then*

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $p \equiv 7 \pmod{8}$ is a prime, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{P_k}{8^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

Conjecture A46 ([S09h]). *Let p be an odd prime.*

(i) *If $p \equiv 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $y \equiv 1, 3 \pmod{p}$, then*

$$\sum_{k=0}^{p-1} \frac{P_k}{32^k} \binom{2k}{k}^2 \equiv (-1)^{(y-1)/2} \left(2y - \frac{p}{4y}\right) \pmod{p^2}.$$

If $p \equiv 7 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{P_k}{32^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

(ii) *Suppose that $p \equiv 1, 3 \pmod{8}$, $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$ and also $y \equiv 1, 3 \pmod{8}$ when $p \equiv 3 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{kP_k}{32^k} \binom{2k}{k}^2 \equiv \begin{cases} (-1)^{(p-1)/8} (p/(4x) - x/2) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(y+1)/2} y \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Conjecture A47 ([S09h]). *Let p be an odd prime.*

(i) *When $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{(-8)^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(4x - \frac{p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kQ_k}{(-8)^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(p-3)/8} 2(x + p/x) \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

(ii) *When $p \equiv 5, 7 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{(-8)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Conjecture A48 ([S09h]). *Let p be an odd prime.*

(i) *When $p \equiv 1 \pmod{8}$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{32^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/8} \left(4x - \frac{p}{x}\right) \pmod{p^2}.$$

If $p \equiv 5 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{Q_k}{32^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

(ii) *If $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$ and also $y \equiv 1, 3 \pmod{8}$ when $p \equiv 3 \pmod{8}$, then*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2}{32^k} Q_k \equiv \begin{cases} (-1)^{(p-1)/8} (p/x - 2x) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(y+1)/2} 2y \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

We define the sequence $\{S_n\}_{n \geq 0}$ and its companion $\{T_n\}_{n \geq 0}$ as follows:

$$S_0 = 0, \quad S_1 = 1, \quad \text{and } S_{n+1} = 4S_n - S_{n-1} \quad (n = 1, 2, 3, \dots);$$

$$T_0 = 2, \quad T_1 = 4, \quad \text{and } T_{n+1} = 4T_n - T_{n-1} \quad (n = 1, 2, 3, \dots).$$

Conjecture A49 ([S09h]). *Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{S_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p+1)/4} \left(4y - \frac{p}{3y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kS_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-3)/4} \left(6y - \frac{7p}{3y}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{S_k}{4^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture A50 ([S09h]). *Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{S_k}{64^k} \binom{2k}{k}^2 \equiv 2y - \frac{p}{6y} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kS_k}{64^k} \binom{2k}{k}^2 \equiv y \pmod{p^2}.$$

If $p \equiv 11 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \frac{S_k}{64^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Conjecture A51 ([S09h]). *Let $p \equiv 1 \pmod{3}$ be a prime.*

(i) *If $p \equiv 1 \pmod{12}$ and $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{p-1} \frac{T_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/4+(x-1)/2} \left(4x - \frac{p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{T_k}{64^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(4x - \frac{p}{x}\right) \pmod{p^2};$$

also

$$\sum_{k=0}^{p-1} \frac{kT_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/4+(x+1)/2} \left(4x - \frac{2p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kT_k}{64^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(2x - \frac{p}{x}\right) \pmod{p^2}.$$

(ii) If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{T_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-3)/4} \left(12y - \frac{p}{y}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{kT_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p+1)/4} \left(20y - \frac{8p}{y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kT_k}{64^k} \binom{2k}{k}^2 \equiv 4y \pmod{p^2}.$$

Conjecture A52 ([S09h]). Let $p \equiv 2 \pmod{3}$ be a prime. If $p \equiv 5 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \frac{T_k}{4^k} \binom{2k}{k}^2 \equiv \sum_{k=0}^{p-1} \frac{T_k}{64^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $p \equiv 11 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{T_k}{(-4)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

Conjecture A53 ([S09i]). Let p be a prime with $p \equiv \pm 1 \pmod{12}$. Then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} (-1)^k S_k \equiv (-1)^{(p-1)/2} S_{p-1} \pmod{p^3}.$$

Remark. The author has proved the congruence mod p^2 .

For $A, B \in \mathbb{Z}$ we define the Lucas sequences $u_n = u_n(A, B)$ ($n \in \mathbb{N}$) and $v_n = v_n(A, B)$ ($n \in \mathbb{N}$) as follows:

$$u_0 = 0, \quad u_1 = 1, \quad \text{and } u_{n+1} = Au_n - Bu_{n-1} \quad (n = 1, 2, 3, \dots);$$

$$v_0 = 0, \quad v_1 = 1, \quad \text{and } v_{n+1} = Av_n - Bv_{n-1} \quad (n = 1, 2, 3, \dots).$$

Conjecture A54 ([S09i]). *Let p be a prime with $p \equiv \pm 1 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} \frac{u_k(4,2)}{(-2)^k} \equiv (-1)^{(p-1)/2} u_{p-1}(4,2) \pmod{p^3}.$$

Remark. The author has proved the congruence mod p^2 . Note that

$$u_k(4,2) = \begin{cases} 2^{k/2} P_k & \text{if } k \text{ is even,} \\ 2^{(k-3)/2} Q_k & \text{if } k \text{ is odd.} \end{cases}$$

The author [S09h] showed that

$$\sum_{k=0}^{(p-1)/2} \frac{u_k(4,2)}{16^k} \binom{2k}{k} \equiv \frac{(-1)^{\lfloor (p-4)/8 \rfloor}}{2} \left(1 - \left(\frac{2}{p} \right) \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{v_k(4,2)}{16^k} \binom{2k}{k} \equiv 2(-1)^{\lfloor p/8 \rfloor} \left(\frac{-1}{p} \right) \pmod{p^2}.$$

Conjecture A55 ([S09i]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} ((-1)^k - (-3)^{-k}) \equiv \left(\frac{p}{3} \right) (3^{p-1} - 1) \pmod{p^3}.$$

Remark. The congruence mod p^2 follows from [S09b].

Recall that the Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined as follows:

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+1} = F_n + F_{n-1} \text{ (} n = 1, 2, 3, \dots \text{)}.$$

Conjecture A56 ([S09i]). *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{(-1)^{\lfloor n/5 \rfloor - 1}}{(2n+1)n^2 \binom{2n}{n}} \sum_{k=0}^{n-1} F_{2k+1} \binom{2k}{k} \equiv \begin{cases} 6 \pmod{25} & \text{if } n \equiv 0 \pmod{5}, \\ 4 \pmod{25} & \text{if } n \equiv 1 \pmod{5}, \\ 1 \pmod{25} & \text{if } n \equiv 2, 4 \pmod{5}, \\ 9 \pmod{25} & \text{if } n \equiv 3 \pmod{5}. \end{cases}$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geq b$ then the sum

$$\frac{1}{5^{2a}} \sum_{k=0}^{5^a-1} F_{2k+1} \binom{2k}{k}$$

modulo 5^b only depends on b .

Remark. In [S09i] the author proved that if $p \neq 2, 5$ is a prime then

$$\sum_{k=0}^{p-1} F_{2k} \binom{2k}{k} \equiv (-1)^{\lfloor p/5 \rfloor} \left(1 - \left(\frac{p}{5}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} F_{2k+1} \binom{2k}{k} \equiv (-1)^{\lfloor p/5 \rfloor} \left(\frac{p}{5}\right) \pmod{p^2}.$$

Recall that the usual q -analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{0 \leq k < n} q^k$$

which tends to n as $q \rightarrow 1$. For any $n, k \in \mathbb{N}$ with $n \geq k$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{0 < r \leq n} [r]_q}{(\prod_{0 < s \leq k} [s]_q)(\prod_{0 < t \leq n-k} [t]_q)}$$

is a natural extension of the usual binomial coefficient $\binom{n}{k}$. A q -analogue of Fibonacci numbers introduced by I. Schur [Sc] is defined as follows:

$$F_0(q) = 0, F_1(q) = 1, \text{ and } F_{n+1}(q) = F_n(q) + q^n F_{n-1}(q) \quad (n = 1, 2, 3, \dots).$$

Conjecture A57 ([S09i]). *Let a and m be positive integers. Then, in the ring $\mathbb{Z}[q]$, we have the following congruence*

$$\sum_{k=0}^{5^a m - 1} q^{-2k(k+1)} \begin{bmatrix} 2k \\ k \end{bmatrix}_q F_{2k+1}(q) \equiv 0 \pmod{[5^a]_q^2}.$$

Conjecture A58 ([S09i]). *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{(-1)^{n-1}}{n^2(n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} S_{k+1} \binom{2k}{k} \equiv \begin{cases} 1 \pmod{9} & \text{if } n \equiv 0, 2 \pmod{9}, \\ 4 \pmod{9} & \text{if } n \equiv 5, 6 \pmod{9}, \\ -2 \pmod{9} & \text{otherwise.} \end{cases}$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geq b - 1$ then the sum

$$\frac{1}{3^{2a}} \sum_{k=0}^{3^a - 1} S_{k+1} \binom{2k}{k}$$

modulo 3^b only depends on b .

Remark. In [S09i] the author proved that if $p > 3$ is a prime then

$$\sum_{k=0}^{p-1} S_k \binom{2k}{k} \equiv 2 \left(\binom{p}{3} - \binom{-1}{p} \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} S_{k+1} \binom{2k}{k} \equiv \binom{p}{3} \pmod{p^2}.$$

Conjecture A59 ([S09f]). *For any nonnegative integer n we have*

$$\frac{1}{(2n+1)^2 \binom{2n}{n}} \sum_{k=0}^n \frac{\binom{2k}{k}}{16^k} \equiv \begin{cases} 1 \pmod{9} & \text{if } 3 \mid n, \\ 4 \pmod{9} & \text{if } 3 \nmid n. \end{cases}$$

. Also,

$$\frac{1}{3^{2a}} \sum_{k=0}^{(3^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv (-1)^a 10 \pmod{27}$$

for every $a = 1, 2, 3, \dots$.

Conjecture A60 ([S09f]). *Let p be an odd prime and let $m \equiv 4 \pmod{p}$. Then*

$$\nu_p \left(\sum_{k=0}^n \frac{\binom{2k}{k}}{m^k} \right) \geq \nu_p \left((2n+1) \binom{2n}{n} \right) \quad \text{for any } n \in \mathbb{Z}^+.$$

Moreover, if $p > 3$ then

$$\frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv (-1)^{(p^a-1)/2} \pmod{p}.$$

Conjecture A61 ([S09d]). *Let p be an odd prime and let $h \in \mathbb{Z}$ with $2h-1 \equiv 0 \pmod{p}$. If $a \in \mathbb{Z}^+$ and $p^a > 3$, then*

$$\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(-\frac{h}{2} \right)^k \equiv 0 \pmod{p^{a+1}}.$$

Also, for any $n \in \mathbb{Z}^+$ we have

$$\nu_p \left(\sum_{k=0}^{n-1} \binom{hn-1}{k} \binom{2k}{k} \left(-\frac{h}{2} \right)^k \right) \geq \nu_p(n).$$

Conjecture A62 ([S09d]). *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod{3}$. Then*

$$\nu_3\left(\frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k}\right) \geq \min\{\nu_3(n), \nu_3(m-1)\} - 1$$

for every $n \in \mathbb{Z}^+$. Furthermore,

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \binom{3^a-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k} \equiv -\frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for each integer $a > \nu_3(m-1)$. Also,

$$\sum_{k=0}^{3^a-1} \binom{3^a-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k} \equiv -3^{2a-1} \pmod{3^{2a}} \text{ for every } a = 2, 3, \dots$$

Conjecture A63 ([S09c]). *For any prime p and positive integer n we have*

$$\nu_p\left(\sum_{k=0}^{n-1} \binom{(p-1)k}{k, \dots, k}\right) \geq \nu_p(n)$$

and

$$\nu_p\left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \binom{(p-1)k}{k, \dots, k}\right) \geq \nu_p(n).$$

Remark. The author [S09c] proved that an integer $p > 1$ is a prime if and only if

$$\sum_{k=0}^{p-1} \binom{(p-1)k}{k, \dots, k} \equiv 0 \pmod{p}.$$

He also showed that if $n \in \mathbb{Z}^+$ is a multiple of a prime p then

$$\sum_{k=0}^{n-1} \binom{(p-1)k}{k, \dots, k} \equiv 0 \pmod{p}.$$

Conjecture A64 (Sun and Tauraso [ST]). *Let $p \neq 2, 5$ be a prime and let $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \left(1 - 2F_{p^a - (\frac{p^a}{5})}\right) \pmod{p^3},$$

where $\{F_n\}_{n \geq 0}$ is the Fibonacci sequence.

Remark. The congruence mod p^2 was proved by the author in [S09b].

Conjecture A65 ([S09f]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{3}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{5}{8}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a} \right) \pmod{p^2}.$$

Remark. The author [S09f] proved that $\sum_{k=0}^{\lfloor p^a/2 \rfloor} \binom{2k}{k}/16^k \equiv (\frac{3}{p^a}) \pmod{p^2}$ for any odd prime p and $a \in \mathbb{Z}^+$.

Conjecture A66 ([S09f]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p^a \equiv 1, 2 \pmod{5}$, or $a > 1$ and $p \not\equiv 3 \pmod{5}$,*

$$\sum_{k=0}^{\lfloor \frac{4}{5}p^a \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a} \right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod{5}$, or $a > 1$ and $p \not\equiv 2 \pmod{5}$, then

$$\sum_{k=0}^{\lfloor \frac{3}{5}p^a \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a} \right) \pmod{p^2}.$$

Thus, if $p^a \equiv 1 \pmod{5}$ then

$$\sum_{\frac{3}{5}p^a < k < \frac{4}{5}p^a} (-1)^k \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

Conjecture A67 ([S09a, S10]). *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv -3p q_p^2(2) \pmod{p^2},$$

and

$$p \sum_{k=1}^{p-1} \frac{1}{k 2^k \binom{3k}{k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ -3/5 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

We also have

$$p \sum_{k=0}^{(p-1)/2} \frac{25k-3}{2^k \binom{3k}{k}} \equiv \left(\frac{-1}{p} \right) - \left(\frac{2}{p} \right) \frac{5p}{2} \pmod{p^2}$$

and

$$2p \sum_{k=0}^{p-1} \frac{25k-3}{2^k \binom{3k}{k}} \equiv 3 \left(\frac{-1}{p} \right) + (E_{p-3} - 9)p^2 \pmod{p^4}.$$

Remark. Gosper announced in 1974 that $\sum_{k=0}^{\infty} (25k-3)/(2^k \binom{3k}{k}) = \pi/2$. In [ZPS] Zhao, Pan and Sun proved that $\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv 0 \pmod{p}$ for any odd prime p .

Conjecture A68 (raised on Nov. 2, 2009 via a message to Number Theory List). *If $n > 1$ is an odd integer satisfying the Morley congruence*

$$\binom{n-1}{(n-1)/2} \equiv (-1)^{(n-1)/2} 4^{n-1} \pmod{n^3},$$

then n must be a prime.

Remark. In 1895 Morley [Mo] showed that $\binom{p-1}{(p-1)/2} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \pmod{p^3}$ for any prime $p > 3$. The author has verified the conjecture for $n < 10^4$.

Conjecture A69 ([S09b]). *If an odd integer $n > 1$ satisfies the congruence*

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(n-1)/2} \pmod{n^2},$$

then n must be a prime.

Remark. The author [S09b] proved that if p is an odd prime then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

And he verified the conjecture for $n < 10^4$ via Mathematica. On the author's request, Qing-Hu Hou at Nankai Univ. finished the verification for $n < 10^5$.

Conjecture A70 (Discovered in 2007). *Let p be a prime and let $l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. If n or r is not divisible by p then we have*

$$\begin{aligned} & \nu_p \left(\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \binom{(k-r)/p}{l} \right) \\ & \geq \left\lfloor \frac{n-lp-1}{p-1} \right\rfloor + \nu_p \left(\binom{\lfloor (n-l-1)/(p-1) \rfloor}{l} \right). \end{aligned}$$

Remark. D. Wan [W] proved that the inequality holds if the last term on the right-hand side is omitted (see also Sun and Wan [SW]).

Conjecture A71 ([S09b]). *Let p be any prime and let r be an integer. For $a \in \mathbb{N}$ define*

$$S_r(p^a) = \sum_{\substack{0 < k < p^a \\ k \equiv r \pmod{p-1}}} C_k.$$

Then, for any $a \in \mathbb{N}$ we have

$$S_r(p^{a+2}) \equiv S_r(p^a) \pmod{p^{(1+\delta_{p,2})(a+1)}}.$$

Furthermore,

$$\frac{S_r(p^{a+2}) - S_r(p^a)}{p^{(1+\delta_{p,2})(a+1)}} + p(\delta_{p^a,2} + \delta_{p^a,3}) \pmod{p^2}$$

does not depend on $a \in \mathbb{Z}^+$.

Conjecture A72 ([S09b]). Let p be a prime, and let $d \in \{0, \dots, p\}$ and $r \in \mathbb{Z}$. For $a \in \mathbb{N}$ define

$$T_r^{(d)}(p^a) = \sum_{\substack{0 < k < p^a \\ k \equiv r \pmod{p-1}}} \binom{2k}{k+d}.$$

Then, for any $a \in \mathbb{N}$ we have

$$T_r^{(d)}(p^{a+2}) \equiv T_r^{(d)}(p^a) \pmod{p^a};$$

furthermore

$$\frac{T_r^{(d)}(p^{a+2}) - T_r^{(d)}(p^a)}{p^a} \pmod{p}$$

does not depend on $a \in \mathbb{Z}^+$. If $a \in \mathbb{N}$ and $d < p = 2$, then

$$T_r^{(d)}(2^{a+2}) \equiv T_r^{(d)}(2^a) \pmod{2^{2a+2+\delta_{d,0}(1-\delta_{a,0})}}.$$

If $a \in \mathbb{Z}^+$, $d \in \{0, 1\}$ and $p = 3$, then

$$T_r^{(d)}(3^{a+2}) \equiv T_r^{(d)}(3^a) \pmod{3^{a+1+\delta_{d,1}(1-\delta_{a,1})}}.$$

PART B. CONJECTURES THAT HAVE BEEN CONFIRMED

Conjecture B1 (raised in an early version of [S10], and confirmed by Kasper Andersen). For any positive integer n , the arithmetic mean

$$s_n := \frac{1}{n} \sum_{k=0}^{n-1} (21k + 8) \binom{2k}{k}^3$$

is always an integer divisible by $4 \binom{2n}{n}$.

Remark. On Feb. 11, 2010, Andersen proved this conjecture by noting that $t_n := s_n / (4 \binom{2n}{n})$ coincides with

$$r_n := \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2.$$

Conjecture B2 (raised in [S09e], and confirmed by Zhi-Hong Sun). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} ((-2)^{-k} - 4^{-k}) \binom{2k}{k}^2 \equiv 0 \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2}{16^k} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 3 \pmod{4}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 0 \pmod{p^2}.$$

Remark. In [S09e] the author proved those congruences modulo p except the first one.

Conjecture B3 (raised in [S09e], and confirmed by Roberto Tauraso). *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^2}{16^k} \equiv 12p^2 - 4 \pmod{p^3} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{kC_k^2}{16^k} \equiv 4 - 10p^2 \pmod{p^3}.$$

Also,

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{64^k} \equiv p \pmod{p^2},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} C_k}{64^k} \equiv (-1)^{(p-1)/2} \frac{2}{3} p \pmod{p^2} \text{ provided } p > 3.$$

Remark. The author [S09e] showed that $\sum_{k=0}^{p-1} C_k^2/16^k \equiv -3 \pmod{p}$ for any odd prime p , and his PhD student Yong Zhang proved the first and the second congruences mod p^2 .

Conjecture B4 (raised in [S09e], and confirmed by Roberto Tauraso). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{27^k} \equiv p \pmod{p^2} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{\binom{3k}{k} C_k}{27^k} \equiv \frac{p}{2} \binom{p}{3} \pmod{p^2}.$$

Conjecture B5 (raised in [S09e], and confirmed by the author's PhD student Yong Zhang). *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k+1}^2}{16^k} \equiv (-1)^{(p-1)/2} - 4 + p^2(8 + E_{p-3}) \pmod{p^3}.$$

If $p > 3$, then

$$\sum_{k=0}^{(p-1)/2} \frac{C_k C_{k+1}}{16^k} \equiv 8 \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{2k}{k+1}}{8^k} \equiv 0 \pmod{p}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{C_k C_{k+1}}{(-16)^k} \equiv -10 \pmod{p}.$$

Remark. As for the first congruence in Conjecture B5, the author [S09e] proved the congruence mod p and then his PhD student Yong Zhang showed the congruence mod p^2 . Following the author's recent method in [S10], Zhang confirmed the congruence with the help of the software **Sigma**.

Conjecture B6 (raised in [S09c], and confirmed by the author's PhD student Yong Zhang). *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod{3}$. Then*

$$\nu_3 \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{m^k} \right) \geq \min\{\nu_3(n), \nu_3(m-1) - 1\}$$

for every $n \in \mathbb{Z}^+$. Furthermore,

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for any integer $a \geq \nu_3(m-1)$.

Conjecture B7 (raised in [S09f], and confirmed by Hao Pan and the author). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{4}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}}{(-4)^k} \equiv \left(\frac{2}{p^a} \right) \pmod{p^2}.$$

Conjecture B8 (raised in [S09h], and confirmed by the author's former student Hui-Qin Cao). *If p is a prime with $p \equiv 11 \pmod{12}$, then*

$$\sum_{k=0}^{p-1} \left(\frac{k}{3} \right) \frac{\binom{2k}{k}^2}{(-16)^k} \equiv 0 \pmod{p}.$$

Conjecture B9 (raised in [S09g], and confirmed by the author's former student Li-Lu Zhao (Hong Kong University)). *Let m be any positive even integer. If p is a prime with $p - 1 \nmid 3m$, then*

$$\sum_{k=1}^{p-1} \frac{H_{k,m}^2}{k^m} \equiv 0 \pmod{p},$$

where $H_{k,m} := \sum_{j=1}^k 1/j^m$.

Remark. The author [S09g] proved the congruence in the case $2p/3 < m < p$. For a prime $p > 3$ the author [S09g] established the following basic congruences for harmonic numbers:

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}, \quad \sum_{k=1}^{p-1} H_k^2 \equiv 2p - 2 \pmod{p^2}, \quad \sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p} \quad \text{provided } p > 5,$$

where H_k denotes the harmonic number $\sum_{j=1}^k 1/j$.

REFERENCES

- [C] D. A. Cox, *Primes of the Form $x^2 + ny^2$* , John Wiley & Sons, 1989.
- [HW] R. H. Hudson and K. S. Williams, *Binomial coefficients and Jacobi sums*, Trans. Amer. Math. Soc. **281** (1984), 431–505.
- [I] T. Ishikawa, *Super congruence for the Apéry numbers*, Nagoya Math. J. **118** (1990), 195–202.
- [Mo] F. Morley, *Note on the congruence $2^{4n} \equiv (-1)^n (2n)!/(n!)^2$, where $2n + 1$ is a prime*, Ann. Math. **9** (1895), 168–170.

- [M1] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory **99** (2003), 139–147.
- [M2] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ by geometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. **355** (2003), 987–1007.
- [M3] E. Mortenson, *Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), 321–330.
- [M4] E. Mortenson, *A p -adic supercongruence conjecture of van Hamme*, Proc. Amer. Math. Soc. **136** (2008), 4321–4328.
- [O] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -series*, Amer. Math. Soc., Providence, R.I., 2003.
- [PP] K. H. Pilehrood and T. H. Pilehrood, *Generating function identities for $\zeta(2n+2), \zeta(2n+3)$ via the WZ method*, Electron. J. Combin. **15** (2008), #R35, 9 pp.
- [R] S. Ramanujan, *Modular equations and approximations to π* , Quart. J. Math. (Oxford) (2) **45** (1914), 350–372.
- [RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), pp. 223–231, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.
- [Sc] I. Schur, *Gesammelte Abhandlungen*, Vol. 2, Springer, Berlin, 1973, pp. 117–136.
- [Sp] R. Sprugnoli, *Sums of reciprocals of the central binomial coefficients*, Integers **6** (2006), #A27, 18pp (electronic).
- [S09a] Z. W. Sun, *Various congruences involving binomial coefficients and higher-order Catalan numbers*, arXiv:0909.3808. <http://arxiv.org/abs/0909.3808>.
- [S09b] Z. W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, preprint, arXiv:0909.5648. <http://arxiv.org/abs/0909.5648>.
- [S09c] Z. W. Sun, *p -adic valuations of some sums of multinomial coefficients*, preprint, arXiv:0910.3892. <http://arxiv.org/abs/0910.3892>.
- [S09d] Z. W. Sun, *On sums of binomial coefficients modulo p^2* , preprint, arXiv:0910.5667. <http://arxiv.org/abs/0910.5667>.
- [S09e] Z. W. Sun, *On congruences related to central binomial coefficients*, preprint, arXiv:0911.2415. <http://arxiv.org/abs/0911.2415>.
- [S09f] Z. W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients (II)*, preprint, arXiv:0911.3060. <http://arxiv.org/abs/0911.3060>.
- [S09g] Z. W. Sun, *Arithmetic theory of harmonic numbers*, preprint, arXiv:0911.4433. <http://arxiv.org/abs/0911.4433>.
- [S09h] Z. W. Sun, *Congruences involving binomial coefficients and Lucas sequences*, preprint, arXiv:0912.1280. <http://arxiv.org/abs/0912.1280>.
- [S09i] Z. W. Sun, *Curious congruences for Fibonacci numbers*, preprint, arXiv:0912.2671. <http://arxiv.org/abs/0912.2671>.
- [S10] Z. W. Sun, *Super congruences and Euler numbers*, preprint, arXiv:1001.4453. <http://arxiv.org/abs/1001.4453>.
- [ST] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math., in press. <http://dx.doi.org/10.1016/j.aam.2010.01.001>.
- [SW] Z. W. Sun and D. Wan, *Lucas-type congruences for cyclotomic ψ -coefficients*, Int. J. Number Theory **4** (2008), 155–170.
- [T] R. Tauraso, *Congruences involving the reciprocals of central binomial coefficients*, preprint, arXiv:0906.5150. <http://arxiv.org/abs/0906.5150>.
- [vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p -adic Functional Analysis (Nijmegen, 1996), pp. 223–236, Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997.
- [W] D. Wan, *Combinatorial congruences and ψ -operators*, Finite Fields Appl. **12** (2006), 693–703.

- [ZPS] L. Zhao, H. Pan and Z. W. Sun, *Some congruences for the second-order Catalan numbers*, Proc. Amer. Math. Soc. **138** (2010), 37–46.
- [Z] D. Zeilberger, *Closed form (pun intended!)*, Contemporary Math. **143** (1993), 579–607.
- [Zu] W. Zudilin, *Ramanujan-type supercongruences*, J. Number Theory **129** (2009), 1848–1857.