

OPEN CONJECTURES ON CONGRUENCES

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ABSTRACT. We collect here various conjectures on congruences made by the author in a series of papers, some of which involve binary quadratic forms and other advanced theories. Part A consists of 100 unsolved conjectures of the author while conjectures in Part B have been recently confirmed. We hope that this material will interest number theorists and stimulate further research. Number theorists are welcome to work on those open conjectures.

INTRODUCTION

Congruences modulo primes have been widely investigated since the time of Fermat. However, we find that there are still lots of new challenging congruences that cannot be easily solved. They appeal for new powerful tools or advanced theory.

Here we collect various conjectures of the author on congruences, most of which can be found in the author's papers available from `arxiv` or his homepage. We use two sections to state conjectures and related remarks. Part A contains 100 unsolved conjectures of the author while Part B consists of conjectures that have been recently confirmed. Most of the congruences here are *super* congruences in the sense that they happen to hold modulo some higher power of p . The topic of super congruences is related to the p -adic Γ -function, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi-Yau manifolds, and some sophisticated combinatorial identities involving harmonic numbers (cf. K. Ono's book [O]). The recent theory of super congruences also involves Bernoulli and Euler numbers (see [S10]) and various series related to π (cf. [Zu]). Many

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congruences collected here are about $\sum_{k=0}^{p-1} a_k/m^k$ modulo powers of a prime p , where m is an integer not divisible by p and the quantity a_k is a sum or a product of some binomial coefficients which usually arises from enumerative combinatorics.

For clarity, we often state the prime version of a conjecture instead of the general version. The reader should consult the original papers for more detailed information and related results.

Now we introduce some basic notations in this paper.

As usual, we set

$$\mathbb{N} = \{0, 1, 2, \dots\} \quad \text{and} \quad \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

The Kronecker symbol $\delta_{m,n}$ takes 1 or 0 according as $m = n$ or not. The rising factorial $(x)_n$ is defined by $(x)_n = \prod_{k=0}^{n-1} (x+k)$, and $(x)_0$ is regarded as 1. For an integer m and a positive odd number n , the notation $\left(\frac{m}{n}\right)$ stands for the Jacobi symbol. For an odd prime p , we use $q_p(2)$ to denote the Fermat quotient $(2^{p-1} - 1)/p$. For a prime p and a rational number x , the p -adic valuation of x is given by

$$\nu_p(x) = \sup\{a \in \mathbb{N} : x \equiv 0 \pmod{p^a}\}.$$

For a polynomial or a power series $P(x)$, we write $[x^n]P(x)$ for the coefficient of x^n in the expansion of $P(x)$. For $k_1, \dots, k_n \in \mathbb{N}$, we define the multinomial coefficient

$$\binom{k_1 + \dots + k_n}{k_1, \dots, k_n} := \frac{(k_1 + \dots + k_n)!}{k_1! \dots k_n!}.$$

Harmonic numbers are given by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^n \frac{1}{k} \quad (n = 1, 2, 3, \dots).$$

For $n \in \mathbb{N}$, C_n denotes the *Catalan number* $\frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$ and $C_n^{(2)}$ stands for the (first kind) *second-order Catalan number* $\frac{1}{2n+1} \binom{3n}{n} = \binom{3n}{n} - 2 \binom{3n}{n-1}$. Note that if p is an odd prime then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \quad \text{for every } k = \frac{p+1}{2}, \dots, p-1.$$

Bernoulli numbers B_0, B_1, B_2, \dots are rational numbers given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that $B_{2n+1} = 0$ for all $n \in \mathbb{Z}^+$ and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Euler numbers E_0, E_1, E_2, \dots are integers defined by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that $E_{2n+1} = 0$ for all $n \in \mathbb{N}$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left(|x| < \frac{\pi}{2}\right).$$

For $A, B \in \mathbb{Z}$ we define the Lucas sequences $u_n = u_n(A, B)$ ($n \in \mathbb{N}$) and $v_n = v_n(A, B)$ ($n \in \mathbb{N}$) as follows:

$$u_0 = 0, \quad u_1 = 1, \quad \text{and } u_{n+1} = Au_n - Bu_{n-1} \quad (n = 1, 2, 3, \dots);$$

$$v_0 = 0, \quad v_1 = 1, \quad \text{and } v_{n+1} = Av_n - Bv_{n-1} \quad (n = 1, 2, 3, \dots).$$

PART A. CONJECTURES THAT REMAIN UNSOLVED

Conjecture A1 ([S09e]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Also,

$$3 \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{7}(3p - 4x^2) = 32y^2 - \frac{8}{7}p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = 1 \text{ \& } p = x^2 + 7y^2 \text{ } (x, y \in \mathbb{Z}), \\ \frac{8}{7}p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = -1. \end{cases}$$

Remark. (a) The first congruence modulo p can be easily deduced from (6) in Ahlgren [A, Theorem 5]. M. Jameson and Ken Ono (at Wisconsin

Univ.) are working on the the first congruence in Conj. A1 but they have not yet proved it fully.

(b) Let p be an odd prime with $(\frac{p}{7}) = 1$. It is well known that $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$. We ever wrote that the author was unable to guess $\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \pmod{p}$ in the case $(\frac{p}{7}) = 1$ though we conjectured the first congruence in Conj. A1 on November 13, 2009. After reading this remark, on Nov. 28, 2009 Bilgin Ali and Bruno Mishutka guessed that

$$\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} 11y^2/3 - x^2 \pmod{p} & \text{if } 3 \mid y, \\ 4(y^2 - x^2)/3 \pmod{p} & \text{if } 3 \nmid y. \end{cases}$$

Since $x^2 \equiv -7y^2 \pmod{p}$, we can simplify the congruence as follows:

$$\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv -\frac{32}{21}x^2 \equiv \frac{32}{3}y^2 \pmod{p}.$$

Note that the second congruence in Conj. A1 is now a congruence mod p^2 . By [S10, Theorem 1.3], this congruence follows from the first congruence in Conj. A1.

Conjecture A2 ([S10]). *For any prime $p > 3$ we have*

$$\sum_{k=0}^{(p-1)/2} (21k+8) \binom{2k}{k}^3 \equiv 8p + (-1)^{(p-1)/2} 32p^3 E_{p-3} \pmod{p^4}.$$

If p is a prime and a is a positive integer with $p^a \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{\lfloor \frac{2}{3}p^a \rfloor} (21k+8) \binom{2k}{k}^3 \equiv 8p^a \pmod{p^{a+5+(-1)^p}}.$$

Also, for each prime $p > 5$ we have

$$\sum_{k=1}^{p-1} \frac{21k-8}{k^3 \binom{2k}{k}^3} + \frac{p-1}{p^3} \equiv \frac{H_{p-1}}{p^2} (15p-6) + \frac{12}{5} p^2 B_{p-5} \pmod{p^3}.$$

Remark. (a) By [S10] the first congruence in Conj. A2 has the following equivalent form:

$$\sum_{k=1}^{(p-1)/2} \frac{21k-8}{k^3 \binom{2k}{k}^3} \equiv (-1)^{(p+1)/2} 4E_{p-3} \pmod{p} \quad (\text{for any prime } p > 3).$$

Note that $\sum_{k=1}^{\infty} (21k-8)/(k^3 \binom{2k}{k}^3) = \zeta(2) = \pi^2/6$ by [Z] or [PP, (7)].

(b) The author [S10] proved that for any prime p and positive integer a we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} (21k+8) \binom{2k}{k}^3 \equiv 8 + 16p^3 B_{p-3} \pmod{p^4},$$

where B_{-1} is regarded as zero, and B_0, B_1, B_2, \dots are Bernoulli numbers. (When $p > 5$, the congruence even holds mod p^5 if we replace $16p^3 B_{p-3}$ by $-48p H_{p-1}$.) See Conj. A1 for our guess on $\sum_{k=0}^{p-1} \binom{2k}{k}^3$ and $\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \pmod{p^2}$. We also conjecture that any integer $n > 1$ satisfying the congruence $\sum_{k=0}^{n-1} (21k+8) \binom{2k}{k}^3 \equiv 8n \pmod{n^4}$ must be a prime; this has been verified for $n \leq 10^4$.

Conjecture A3 ([S09e, S10]). (i) *Set*

$$a_n := \frac{1}{4n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (35k+8) \binom{4k}{k, k, k, k} 81^{n-1-k} \quad \text{for } n \in \mathbb{Z}^+.$$

(Note that $a_1 = 1/3$ and $(4n+6)a_{n+1} = 81na_n + (35n+8) \binom{2n-1}{n} \frac{C_{2n}}{2}$ for $n = 1, 2, 3, \dots$) Then $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) *Let p be a prime. If $p > 3$ then*

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{35k+8}{81^k} \binom{4k}{k, k, k, k} \equiv 8 \pmod{p^3} \quad \text{for all } a \in \mathbb{Z}^+.$$

If $\left(\frac{p}{7}\right) = 1$, i.e., $p \equiv 1, 2, 4 \pmod{7}$, then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{81^k} \pmod{p^3}$$

and

$$3 \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv 5 \sum_{k=0}^{p-1} \frac{k \binom{4k}{k, k, k, k}}{81^k} \pmod{p^3}.$$

If $\left(\frac{p}{7}\right) = -1$ and $p \neq 3$, then

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{81^k} \equiv 0 \pmod{p^2}.$$

(iii) *We have*

$$\sum_{k=1}^{\infty} \frac{(35k-8)81^k}{k^3 \binom{4k}{k, k, k, k}} = 12\pi^2.$$

Conjecture A4 ([S09e, S10]). (i) Set

$$a_n := \frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (11k+3) \binom{2k}{k}^2 \binom{3k}{k} 64^{n-1-k} \quad \text{for } n \in \mathbb{Z}^+.$$

(Note that $a_1 = 1/2$ and $(2n+3)a_{n+1} = 32na_n + (11n+3)\binom{2n-1}{n}C_n^{(2)}$ for $n = 1, 2, 3, \dots$.) Then $a_n \in \mathbb{Z}$ for every $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ \& } 4p = x^2 + 11y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases} \end{aligned}$$

Furthermore,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{11k+3}{64^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 3 + \frac{7}{2}p^3 B_{p-3} \pmod{p^4} \quad \text{for all } a \in \mathbb{Z}^+.$$

(iii) We have

$$\sum_{k=1}^{\infty} \frac{(11k-3)64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = 8\pi^2.$$

Also, if $p > 3$ is a prime then

$$p \sum_{k=1}^{(p-1)/2} \frac{(11k-3)64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \equiv 32q_p(2) - \frac{64}{3}p^2 B_{p-3} \pmod{p^3},$$

where $q_p(2) = (2^{p-1} - 1)/p$.

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $\left(\frac{p}{11}\right) = 1$ we can write $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$. The only known result about the parameters in the representation $4p = x^2 + 11y^2$ is the following one due to Jacobi (see, e.g., [HW]): If $p = 11f + 1$ is a prime and $4p = x^2 + 11y^2$ with $x \equiv 2 \pmod{11}$, then $x \equiv \binom{6f}{3f} \binom{3f}{f} / \binom{4f}{2f} \pmod{p}$.

Conjecture A5 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (10k+3) \binom{2k}{k}^2 \binom{3k}{k} 8^{n-1-k}.$$

(Note that $a_1 = 1/2$ and $(2n+3)a_{n+1} = 4na_n + (10n+3)\binom{2n-1}{n}C_n^{(2)}$ for $n = 1, 2, 3, \dots$) Then $a_n \in \mathbb{Z}$ for all $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = 1 \text{ \& } p = x^2 + 2y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = -1. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{10k+3}{8^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 3 + \frac{49}{8} p^3 B_{p-3} \pmod{p^4}.$$

(iii) We have

$$\sum_{k=1}^{\infty} \frac{(10k-3)8^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = \frac{\pi^2}{2}.$$

Conjecture A6 ([S09e, S10]). Let $p > 3$ be a prime. If $p \equiv 1 \pmod{6}$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv 4x^2 - 2p \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^3}{16^k} \equiv p - \frac{4x^2}{3} \pmod{p^2}.$$

If $p \equiv 5 \pmod{6}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv 0 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^3}{16^k} \equiv \frac{p}{3} \pmod{p^2}.$$

Furthermore,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv 1 + \frac{7}{6} p^3 B_{p-3} \pmod{p^4} \text{ for all } a \in \mathbb{Z}^+,$$

and

$$\sum_{k=0}^{(p-1)/2} (3k+1) \frac{\binom{2k}{k}^3}{16^k} \equiv p + 2 \left(\frac{-1}{p}\right) p^3 E_{p-3} \pmod{p^4}.$$

Also,

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 16^{n-1-k} \in \mathbb{Z} \text{ for all } n = 2, 3, 4, \dots$$

Remark. The author [S09e] determined $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / 16^k \pmod{p}$. Note that $a_1 = 1/4$ and $(2n+1)a_{n+1} = 8na_n + (3n+1)\binom{2n-1}{n}^2$ for $n = 1, 2, 3, \dots$. Also, the first identity of J. Guillera [G3] in the case $a = 1/2$ gives $\sum_{k=1}^{\infty} (3k-1)16^k / (k^3 \binom{2k}{k}^3) = \pi^2/2$.

Conjecture A7 ([S09e, S10]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 4 \mid p-1 \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} \equiv p \left(\frac{-1}{p} \right) + p^3 E_{p-3} \pmod{p^4}$$

and furthermore

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 (-8)^{n-1-k} \in \mathbb{Z}^+ \quad \text{for all } n = 2, 3, 4, \dots$$

Remark. Note that $a_1 = 1/4$ and $(2n+1)a_{n+1} + 4na_n = (3n+1) \binom{2n-1}{n}^2$ for $n = 1, 2, 3, \dots$. Also, the third identity of Guillera [G3] with $a = 1/2$ gives $\sum_{k=1}^{\infty} (3k-1)(-8)^k / (k^3 \binom{2k}{k}^3) = -2G$, where G is the Catalan constant defined by $G = \sum_{k=0}^{\infty} (-1)^k / (2k+1)^2 = 0.915965594 \dots$

Conjecture A8 ([S10]). (i) *For $n \in \mathbb{Z}^+$ set*

$$\frac{1}{n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (5k+1) \binom{2k}{k}^2 \binom{3k}{k} (-192)^{n-1-k}.$$

(Note that $a_1 = 1/6$ and $(2n+3)a_{n+1} + 96na_n = (5n+1) \binom{2n-1}{n} C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Then $a_n \in \mathbb{Z}$ for $n = 2, 3, 4, \dots$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) *Let $p > 3$ be a prime. Then*

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{5k+1}{(-192)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv \left(\frac{p^a}{3} \right) \pmod{p^2} \quad \text{for any } a \in \mathbb{Z}^+.$$

We also have

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-192)^k} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } 4p = x^2 + 27y^2 \text{ (} x, y \in \mathbb{Z} \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Remark. It is well known that for any prime $p \equiv 1 \pmod{3}$ there are unique $x, y \in \mathbb{Z}^+$ such that $4p = x^2 + 27y^2$ (see, e.g., [C]). Also, Ramanujan [R] found that

$$\sum_{k=0}^{\infty} (5k+1) \left(-\frac{9}{16} \right)^k \frac{(1/2)_k (1/3)_k (2/3)_k}{(1)_k^3} = \sum_{k=0}^{\infty} \frac{5k+1}{(-192)^k} \binom{2k}{k}^2 \binom{3k}{k} = \frac{4\sqrt{3}}{\pi}.$$

Conjecture A9 (Discovered on March 23, 2010). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-96)^{3k}} \equiv \begin{cases} \left(\frac{-6}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1 \text{ \& } 4p = x^2 + 19y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{342k + 25}{(-96)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} \equiv 25p \left(\frac{-6}{p}\right) \pmod{p^3}.$$

Furthermore, for any $n = 2, 3, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (342k + 25)(-96^3)^{n-1-k} \binom{6k}{3k} \binom{3k}{k,k,k} \in \mathbb{Z}$$

unless $2n + 1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. It is well known that $\mathbb{Q}(\sqrt{-19})$ has class number one and hence for any odd prime p with $\left(\frac{p}{19}\right) = 1$ there are unique positive integers x and y such that $4p = x^2 + 19y^2$. D. V. Chudnovsky and G. V. Chunovsky [CC] obtained that

$$\sum_{k=0}^{\infty} \frac{342k + 25}{(-96)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} = \frac{32\sqrt{6}}{\pi}.$$

See also [BB] and [G5] for more Ramanujan-type series involving $1/\pi$.

Conjecture A10 (Discovered on March 23, 2010). *If $p > 5$ is a prime, then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-960)^{3k}} \equiv \begin{cases} \left(\frac{p}{15}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{43}\right) = 1 \text{ \& } 4p = x^2 + 43y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{43}\right) = -1. \end{cases}$$

Also, for any $n = 2, 3, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (5418k + 263)(-960^3)^{n-1-k} \binom{6k}{3k} \binom{3k}{k,k,k} \in \mathbb{Z}$$

unless $2n + 1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. It is well known that $\mathbb{Q}(\sqrt{-43})$ has class number one and hence for any odd prime p with $(\frac{p}{43}) = 1$ there are unique positive integers x and y such that $4p = x^2 + 43y^2$. D. V. Chudnovsky and G. V. Chunovsky [CC] showed that

$$\sum_{k=0}^{\infty} \frac{5418k + 263}{(-960)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{640\sqrt{15}}{3\pi},$$

and Zudilin [Zu] suggested that for any prime $p > 5$ we should have

$$\sum_{k=0}^{p-1} \frac{5418k + 263}{(-960)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 263p \left(\frac{-15}{p} \right) \pmod{p^3}.$$

Conjecture A11 (Discovered on March 24, 2010). *Let $p > 5$ be a prime with $p \neq 11$. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k, k, k}}{(-5280)^{3k}} \\ \equiv & \begin{cases} \left(\frac{-330}{p} \right) (x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{67} \right) = 1 \text{ \& } 4p = x^2 + 67y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{67} \right) = -1. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} \frac{261702k + 10177}{(-5280)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 10177p \left(\frac{-330}{p} \right) \pmod{p^3}.$$

Furthermore, for any $n = 2, 3, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (261702k + 10177) (-5280^3)^{n-1-k} \binom{6k}{3k} \binom{3k}{k, k, k} \in \mathbb{Z}$$

unless $2n + 1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. It is well known that $\mathbb{Q}(\sqrt{-67})$ has class number one and hence for any odd prime p with $(\frac{p}{67}) = 1$ there are unique positive integers x and y such that $4p = x^2 + 67y^2$. It is known that (cf. [CC] and [G5])

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(261702k + 10177) (-1)^k (1/2)_k (1/6)_k (5/6)_k}{440^{3k}} \\ & = \sum_{k=0}^{\infty} \frac{261702k + 10177}{(-5280)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{3 \times 440^2}{\pi \sqrt{330}}. \end{aligned}$$

Conjecture A12 (Discovered on March 24, 2010). *Let $p > 5$ be a prime with $p \neq 23, 29$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k,k,k}}{(-640320)^{3k}} \equiv \begin{cases} \left(\frac{-10005}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{163}\right) = 1 \text{ \& } 4p = x^2 + 163y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{163}\right) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} \equiv 13591409p \left(\frac{-10005}{p}\right) \pmod{p^3}.$$

Furthermore, for $n = 2, 3, \dots$, if we denote by a_n the number

$$\frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (545140134k + 13591409)(-640320^3)^{n-1-k} \binom{6k}{3k} \binom{3k}{k,k,k},$$

then $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For any odd prime p with $\left(\frac{p}{163}\right) = 1$, there are unique positive integers x and y such that $4p = x^2 + 163y^2$. D. V. Chudnovsky and G. V. Chudnovsky [CC] got the formula

$$\sum_{k=0}^{\infty} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k,k,k} = \frac{3 \times 53360^2}{2\pi\sqrt{10005}},$$

which enabled them to hold the record for the calculation of π during 1989-1994.

Conjecture A13 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$\frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (15k+4) \binom{2k}{k}^2 \binom{3k}{k} (-27)^{n-1-k}.$$

(Note that $a_1 = 1/3$ and $(4n+6)a_{n+1} + 27na_n = (15n+4)\binom{2n-1}{n}C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Then $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ \& } p = x^2 + 15y^2 \text{ } (x, y \in \mathbb{Z}), \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ \& } p = 5x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{15k+4}{(-27)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4 \left(\frac{p^a}{3}\right) \pmod{p^2}.$$

(iii) (Discovered on April 1, 2010) We have

$$\sum_{k=1}^{\infty} \frac{(15k-4)(-27)^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = -27K,$$

where

$$K := L\left(2, \left(\frac{-3}{\cdot}\right)\right) = \sum_{k=1}^{\infty} \frac{\binom{k}{3}}{k^2} = 0.781302412896486296867187429624\dots$$

Remark. Let $p > 5$ be a prime. By the theory of binary quadratic forms (cf. [C]), if $p \equiv 1, 4 \pmod{15}$ then $p = x^2 + 15y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 2, 8 \pmod{15}$ then $p = 5x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. On April 1, 2010, the author also conjectured that

$$\sum_{k=0}^{p-1} \frac{5k+1}{(-144)^k} \binom{4k}{k, k, k, k} \equiv p \left(\frac{p}{3}\right) \pmod{p^3}$$

for any prime $p > 3$, and that

$$\sum_{k=1}^{\infty} \frac{(5k-1)(-144)^k}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} = -\frac{45}{2}K,$$

which converges more slowly than the series in Conj. A13(iii).

Conjecture A14 ([S09e, S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^2 \binom{3k}{k} 6^{3(n-1-k)}$$

and

$$b_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (8k+1) \binom{4k}{k, k, k, k} 48^{2(n-1-k)}.$$

(Note that $a_1 = 1/6$ and $b_1 = 1/12$. Also, $(2n+3)a_{n+1} = 108na_n + (6n+1)\binom{2n-1}{n}C_n^{(2)}$ and $(2n+3)b_{n+1} = 1152nb_n + (8n+1)\binom{2n-1}{n}\frac{C_{2n}}{2}$ for all $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $a_n, b_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n, 3b_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{6^{3k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1 \text{ i.e., } p \equiv 13, 17, 19, 23 \pmod{24}; \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{48^{2k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ 2p - 4x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1 \text{ i.e., } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{6k+1}{6^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}$$

and

$$\begin{aligned} & \frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} \\ & \equiv \frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}. \end{aligned}$$

Remark. (a) Let $p > 3$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1, 7 \pmod{24}$ then $p = x^2 + 6y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 5, 11 \pmod{24}$ then $p = 2x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$.

(b) Ramanujan [R] found that

$$\sum_{k=0}^{\infty} (6k+1) \frac{(1/2)_k (1/3)_k (2/3)_k}{2^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{6k+1}{6^{3k}} \binom{2k}{k}^2 \binom{3k}{k} = \frac{3\sqrt{3}}{\pi}$$

and

$$\sum_{k=0}^{\infty} (8k+1) \frac{(1/2)_k (1/4)_k (3/4)_k}{9^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{8k+1}{48^{2k}} \binom{4k}{k, k, k, k} = \frac{2\sqrt{3}}{\pi}.$$

Conjecture A15 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (20k+3) \binom{4k}{k, k, k, k} (-2^{10})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(2n+3)a_{n+1} + 512na_n = (20n+3) \binom{2n-1}{n} \frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Then $(-1)^{n-1} a_n \in \mathbb{Z}^+$ for all $n = 2, 3, 4, \dots$

(ii) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} \equiv 3p \left(\frac{-1}{p} \right) + 3p^3 E_{p-3} \pmod{p^4},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} \equiv p \left(\frac{-1}{p} \right) (2^{p-1} + 2 - (2^{p-1} - 1)^2) \pmod{p^4}$$

provided $p > 3$. We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{10})^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ \& } p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2(p - x^2) \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ \& } 2p = x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-5}{p} \right) = -1, \text{ i.e., } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

Remark. Let $p \neq 2, 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1, 9 \pmod{20}$ then $p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 3, 7 \pmod{20}$ then $2p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$. Note also that the second congruence mod p^3 in part (ii) has been obtained by W. Zudilin [Zu] as a p -adic analogue of the Ramanujan series

$$\sum_{k=0}^{\infty} \frac{20k+3}{(-2^{10})^k} \binom{4k}{k, k, k, k} = \frac{8}{\pi}.$$

Conjecture A16 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (10k+1) \binom{4k}{k, k, k, k} 12^{4(n-1-k)}.$$

(Note that $a_1 = 1/12$ and $(2n+3)a_{n+1} = 10368na_n + (10n+1)\binom{2n-1}{n}\frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $a_n \in \mathbb{Z}$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{10k+1}{12^{4k}} \binom{4k}{k, k, k, k} \equiv p \left(\frac{-2}{p} \right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{12^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ \& } p = x^2 + 10y^2 \text{ } (x, y \in \mathbb{Z}), \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 7, 13, 23, 37 \pmod{40} \text{ \& } p = 2x^2 + 5y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-10}{p} \right) = -1, \text{ i.e., } p \equiv 3, 17, 21, 27, 29, 31, 33, 39 \pmod{40}. \end{cases}$$

Remark. (a) Let $p > 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $\left(\frac{-2}{p} \right) = \left(\frac{p}{5} \right) = 1$ then $p = x^2 + 10y^2$ for some $x, y \in \mathbb{Z}$; if $\left(\frac{-2}{p} \right) = \left(\frac{p}{5} \right) = -1$ then $p = 2x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$.

(b) Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (10k+1) \frac{(1/2)_k (1/4)_k (3/4)_k}{81^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{10k+1}{12^{4k}} \binom{4k}{k, k, k, k} = \frac{9\sqrt{2}}{4\pi}.$$

Conjecture A17 (Discovered on March 23, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{10}3^4)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{13}{p} \right) = \left(\frac{-1}{p} \right) = 1 \text{ \& } p = x^2 + 13y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left(\frac{13}{p} \right) = \left(\frac{-1}{p} \right) = -1 \text{ \& } 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{13}{p} \right) = -\left(\frac{-1}{p} \right). \end{cases}$$

We also have

$$\sum_{k=0}^{p-1} \frac{260k+23}{(-82944)^k} \binom{4k}{k, k, k, k} \equiv 23p \left(\frac{-1}{p} \right) + \frac{5}{3} p^3 E_{p-3} \pmod{p^4}.$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (260k+23) \binom{4k}{k, k, k, k} (-82944)^{n-1-k} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. Ramanujan (cf. [Be, p. 353]) found that

$$\sum_{k=0}^{\infty} \frac{(260k+23)(1/2)_k(1/4)_k(3/4)_k}{k!^3 18^{2k}} = \sum_{k=0}^{\infty} \frac{260k+23}{(-82944)^k} \binom{4k}{k, k, k, k} = \frac{72}{\pi}.$$

Conjecture A18 (Discovered on March 23, 2010). *Let $p > 3$ be a prime with $p \neq 11$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{1584^{2k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-11}{p} \right) = \left(\frac{2}{p} \right) = 1 \text{ \& } p = x^2 + 22y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } \left(\frac{-11}{p} \right) = \left(\frac{2}{p} \right) = -1 \text{ \& } p = 2x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-11}{p} \right) = -\left(\frac{2}{p} \right). \end{cases}$$

We also have

$$\sum_{k=0}^{p-1} \frac{280k+19}{1584^{2k}} \binom{4k}{k, k, k, k} \equiv 19p \left(\frac{p}{11} \right) \pmod{p^3}.$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (280k+19) \binom{4k}{k, k, k, k} 1584^{2(n-1-k)} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. Ramanujan (cf. [Be, p. 354]) found that

$$\sum_{k=0}^{\infty} \frac{(280k+19)(1/2)_k(1/4)_k(3/4)_k}{k!^3 99^{2k}} = \sum_{k=0}^{\infty} \frac{280k+19}{1584^{2k}} \binom{4k}{k, k, k, k} = \frac{2 \times 99^2}{\pi \sqrt{11}}.$$

Conjecture A19 (Discovered on March 23, 2010). *Let $p > 3$ be a prime with $p \neq 7$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{10}21^4)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = 1 \text{ \& } p = x^2 + 37y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = -1 \text{ \& } 2p = x^2 + 37y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-37}{p}\right) = -1. \end{cases}$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (21460k + 1123) \binom{4k}{k, k, k, k} (-2^{10}21^4)^{n-1-k} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. Ramanujan (cf. [Be, p. 353]) found that

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(21460k + 1123)(-1)^k (1/2)_k (1/4)_k (3/4)_k}{k!^3 882^k} \\ &= \sum_{k=0}^{\infty} \frac{21460k + 1123}{(-2^{10}21^4)^k} \binom{4k}{k, k, k, k} = \frac{2^3 21^2}{\pi}. \end{aligned}$$

Conjecture A20 ([S10]). *For $n \in \mathbb{Z}^+$ set*

$$a_n := \frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (51k + 7) \binom{2k}{k}^2 \binom{3k}{k} (-12^3)^{n-1-k}.$$

(Note that $a_1 = 7/6$ and $(2n+3)a_{n+1} + 864na_n = (51n+7)\binom{2n-1}{n}C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) *Given an integer $n > 1$, we have $(-1)^{n-1}a_n \in \mathbb{Z}^+$ unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.*

(ii) *Let $p > 3$ be a prime. Then*

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{51k + 7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 7 \left(\frac{p^a}{3}\right) \pmod{p^2} \text{ for any } a \in \mathbb{Z}^+.$$

We also have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-12^3)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = 1 \text{ \& } 4p = x^2 + 51y^2 \ (x, y \in \mathbb{Z}), \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{17}\right) = -1 \text{ \& } 4p = 3x^2 + 17y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = -\left(\frac{p}{17}\right). \end{cases}$$

Remark. (a) Let $p > 3$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $(\frac{p}{3}) = (\frac{p}{17}) = 1$ then $4p = x^2 + 51y^2$ for some $x, y \in \mathbb{Z}$; if $(\frac{p}{3}) = (\frac{p}{17}) = -1$ then $4p = 3x^2 + 17y^2$ for some $x, y \in \mathbb{Z}$. (b) Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (51k + 7) \frac{(1/2)_k (1/3)_k (2/3)_k}{(-16)^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{51k + 7}{(-12^3)^k} \binom{2k}{k}^2 \binom{3k}{k} = \frac{12\sqrt{3}}{\pi}.$$

Conjecture A21 (Discovered on March 23, 2010). *Let $p > 3$ be a prime with $p \neq 11$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{396^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{29}{p}) = (\frac{-2}{p}) = 1 \text{ \& } p = x^2 + 58y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } (\frac{29}{p}) = (\frac{-2}{p}) = -1 \text{ \& } p = 2x^2 + 29y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-58}{p}) = -1. \end{cases}$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1) \binom{2n}{n}} \sum_{k=0}^{n-1} (26390k + 1103) \binom{4k}{k, k, k, k} 396^{4(n-1-k)} \in \mathbb{Z}$$

unless $2n + 1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. Ramanujan (cf. [Be, p. 354]) found that

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(26390k + 1103)(1/2)_k (1/4)_k (3/4)_k}{k! 399^{4k}} \\ &= \sum_{k=0}^{\infty} \frac{26390k + 1103}{396^{4k}} \binom{4k}{k, k, k, k} = \frac{99^2}{2\pi\sqrt{2}}. \end{aligned}$$

Conjecture A22 (Discovered on March 26, 2010). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-48)^{3k}} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = (\frac{p}{41}) = 1 \text{ \& } 4p = x^2 + 123y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } (\frac{p}{3}) = (\frac{p}{41}) = -1 \text{ \& } 4p = 3x^2 + 41y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{123}) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{615k + 53}{(-48)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv 53p \left(\frac{p}{3}\right) \pmod{p^3}.$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (615k + 53) \binom{2k}{k}^2 \binom{3k}{k} (-48)^{3(n-1-k)} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. It is known (cf. [G5]) that

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(615k + 53)(-1)^k (1/2)_k (1/3)_k (2/3)_k}{k!^3 2^{10k}} \\ &= \sum_{k=0}^{\infty} \frac{615k + 53}{(-48)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} = \frac{96\sqrt{3}}{\pi}. \end{aligned}$$

Conjecture A23 (Discovered on March 26, 2010). *Let $p > 5$ be a prime.*

Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-300)^{3k}} \\ & \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = 1 \text{ \& } 4p = x^2 + 267y^2, \\ 2p - 3x^2 \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = \left(\frac{p}{89}\right) = -1 \text{ \& } 4p = 3x^2 + 89y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{267}\right) = -1. \end{cases} \end{aligned}$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n\binom{2n}{n}} \sum_{k=0}^{n-1} (14151k + 827) \binom{2k}{k}^2 \binom{4k}{2k} (-300)^{3(n-1-k)} \in \mathbb{Z}$$

unless $n-1$ is a power of 2 in which case $2a_n$ is an odd integer.

Remark. It is known (cf. [G5]) that

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(14151k + 827)(-1)^k (1/2)_k (1/3)_k (2/3)_k}{k!^3 500^{2k}} \\ &= \sum_{k=0}^{\infty} \frac{14151k + 827}{(-300)^{3k}} \binom{2k}{k}^2 \binom{4k}{2k} = \frac{1500\sqrt{3}}{\pi} \end{aligned}$$

and W. Zudilin [Zu] suggested that for any prime $p > 5$ we have

$$\sum_{k=0}^{p-1} \frac{14151k + 827}{(-300)^{3k}} \binom{2k}{k}^2 \binom{3k}{k} \equiv 827p \left(\frac{p}{3}\right) \pmod{p^3}.$$

Conjecture A24 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (28k+3) \binom{4k}{k, k, k, k} (-3 \times 2^{12})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(2n+3)a_{n+1} + 6144na_n = (28n+3)\binom{2n-1}{n} \frac{C_{2n}}{2}$ for $n \in \mathbb{Z}^+$.) Then we have $(-1)^{n-1}a_n \in \mathbb{Z}^+$ for all $n = 2, 3, 4, \dots$

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{28k+3}{(-3 \times 2^{12})^k} \binom{4k}{k, k, k, k} \equiv 3p \left(\frac{p}{3}\right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-3 \times 2^{12})^k} \equiv \begin{cases} (-1)^{\lfloor x/6 \rfloor} (4x^2 - 2p) \pmod{p^2} & \text{if } 12 \mid p-1 \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x-1 \text{ \& } 2 \mid y), \\ -4\left(\frac{xy}{3}\right)xy \pmod{p^2} & \text{if } 12 \mid p-5 \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x-1 \text{ \& } 2 \mid y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Remark. Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (28k+3) \frac{(1/2)_k (1/4)_k (3/4)_k}{(-48)^k (1)_k^3} = \sum_{k=0}^{\infty} \frac{28k+3}{(-3 \times 2^{12})^k} \binom{4k}{k, k, k, k} = \frac{16\sqrt{3}}{3\pi}.$$

Conjecture A25 (Discovered on March 23, 2010). Let $p > 5$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{(-2^{14}3^45)^k} \equiv \begin{cases} \varepsilon(x)(4x^2 - 2p) & \text{if } 4 \mid p-1, \left(\frac{p}{5}\right) = 1, p = x^2 + y^2 \text{ \& } 2 \nmid x, \\ 4xy \pmod{p^2} & \text{if } 4 \mid p-1, \left(\frac{p}{5}\right) = -1, p = x^2 + y^2 \text{ \& } 5 \nmid x-y, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where $\varepsilon(x)$ takes -1 or 1 according as $5 \mid x$ or not. We also have

$$\sum_{k=0}^{p-1} \frac{644k+41}{(-2^{14}3^45)^k} \binom{4k}{k, k, k, k} \equiv 41p \left(\frac{-5}{p}\right) \pmod{p^3}.$$

Furthermore, for $n = 2, 3, 4, \dots$ we have

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (644k+41) \binom{4k}{k, k, k, k} (-2^{14}3^45)^{n-1-k} \in \mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3a_n \in \mathbb{Z} \setminus 3\mathbb{Z}$.

Remark. Ramanujan (cf. [Be, p. 353]) found that

$$\sum_{k=0}^{\infty} (644k+41) \frac{(1/2)_k (1/4)_k (3/4)_k}{k!^3 (-5)^k 72^k} = \sum_{k=0}^{\infty} \frac{644k+41}{(-2^{14}3^45)^k} \binom{4k}{k, k, k, k} = \frac{288}{\pi\sqrt{5}}.$$

Conjecture A26 ([S10]). (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{10n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (154k+15) \binom{6k}{3k} \binom{3k}{k, k, k} (-2^{15})^{n-1-k}.$$

(Note that $a_1 = 1/4$ and $(4n+6)a_{n+1} + 32768na_n = (\frac{154}{5}n+3)\binom{6n-1}{3n}C_n^{(2)}$ for $n \in \mathbb{Z}^+$.) Given an integer $n > 1$, we have $(-1)^{n-1}a_n \in \mathbb{Z}^+$ unless $2n+1$ is a power of 5 in which case $5a_n \in \mathbb{Z} \setminus 5\mathbb{Z}$.

(ii) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{154k+15}{(-2^{15})^k} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 15p \left(\frac{-2}{p} \right) \pmod{p^3}.$$

We also have

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k, k, k}}{(-2^{15})^k} \\ & \equiv \begin{cases} \left(\frac{-2}{p} \right) (x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{11} \right) = 1 \text{ \& } 4p = x^2 + 11y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11} \right) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases} \end{aligned}$$

Remark. Ramanujan [R] obtained that

$$\sum_{k=0}^{\infty} (28k+3) \frac{(-27)^k}{29^k} \cdot \frac{(1/2)_k (1/6)_k (5/6)_k}{(1)_k^3} = \sum_{k=0}^{\infty} \frac{154k+15}{(-2^{15})^k} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{32\sqrt{2}}{\pi}.$$

Conjecture A27 ([S09e, S10]). *Let $p \equiv 1 \pmod{4}$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{64^k} \binom{2k}{k}^3 \equiv 0 \pmod{p^2}.$$

When $p > 5$ we also have

$$\sum_{k=0}^{p^a-1} \frac{k^3 \binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for all } a = 1, 2, 3, \dots$$

Remark. Let p be an odd prime. Rodriguez-Villegas [RV] conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv a(p) \pmod{p^2},$$

where the sequence $\{a(n)\}_{n \geq 1}$ is defined by

$$\sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{4n})^6.$$

This was proved by many authors, see, e.g., E. Mortenson [M2]. Ishikawa [I] pointed out that if $p = x^2 + y^2$ with x odd and y even then $a(p) = 4x^2 - 2p$ by the Jacobi-Macdonald formula. The author [S10] showed that $\sum_{p/2 < k < p} (4k+1) \binom{2k}{k}^3 / 64^k \equiv 0 \pmod{p^4}$ if $p \equiv 3 \pmod{4}$. He also determined $\sum_{k=0}^{p-1} k^3 \binom{2k}{k}^3 / 64^k$ modulo p .

Conjecture A28 ([S09e, S10]). *Let p be an odd prime. If $\left(\frac{-2}{p}\right) = 1$ (i.e., $p \equiv 1, 3 \pmod{8}$) and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right) (4x^2 - 2p) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{28^{4k}} \equiv 4x^2 - 2p \pmod{p^2}.$$

If $\left(\frac{-2}{p}\right) = -1$ (i.e., $p \equiv 5, 7 \pmod{8}$), then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{28^{4k}} \equiv 0 \pmod{p^2} \quad \text{provided } p \neq 7.$$

Also,

$$\sum_{k=0}^{(p-1)/2} (4k+1) \frac{\binom{2k}{k}^3}{(-64)^k} \equiv p \left(\frac{-1}{p} \right) + p^3 E_{p-3} \pmod{p^4},$$

and

$$\sum_{k=0}^{p-1} \frac{40k+3}{28^{4k}} \binom{4k}{k, k, k, k} \equiv 3p \left(\frac{p}{3} \right) \pmod{p^2} \quad \text{provided } p \neq 7.$$

Moreover,

$$a_n := \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (40k+3) \binom{4k}{k, k, k, k} 28^{4(n-1-k)}$$

are integers for all $n = 2, 3, 4, \dots$

Remark. (a) E. Mortenson [M4] proved the following conjecture of van Hamme [vH]:

$$\sum_{k=0}^{(p-1)/2} (4k+1) \binom{-1/2}{k}^3 \equiv (-1)^{(p-1)/2} p \pmod{p^3} \quad \text{for any odd prime } p.$$

Note that $\binom{-1/2}{k}^3 = \binom{2k}{k}^3 / (-64)^k$ for $k = 0, 1, 2, \dots$. In [S10b] the author proved that

$$2n \binom{2n}{n} \mid \sum_{k=0}^{n-1} (4k+1) \binom{2k}{k}^3 (-64)^{n-1-k}$$

for all $n = 2, 3, \dots$. On March 5, 2010 the author showed that the fifth congruence in Conj. A28 is equivalent to the second one in Conj. A31. By [S10], for any odd prime p we have $\sum_{p/2 < k < p} (4k+1) \binom{2k}{k}^3 / (-64)^k \equiv 0 \pmod{p^4}$.

(b) It is known (cf. [G5] and [G2]) that

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3 \binom{2k}{k}^3} = -16G,$$

where $G := \sum_{k=0}^{\infty} (-1)^k / (2k+1)^2$ is the Catalan constant. Ramanujan (cf. [Be, p. 354]) found that

$$\sum_{k=0}^{\infty} (40k+3) \frac{(1/2)_k (1/4)_k (3/4)_k}{k! 3^7 4^k} = \sum_{k=0}^{\infty} \frac{40k+3}{28^{4k}} \binom{4k}{k, k, k, k} = \frac{49}{3\pi\sqrt{3}}.$$

Conjecture A29 ([S10]). *Let p be an odd prime. If $p \equiv 1, 3 \pmod{8}$, then*

$$\sum_{k=0}^{p-1} \frac{16k+3}{256^k} \binom{4k}{k, k, k, k} \equiv 0 \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{36k+5}{12^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} \equiv 0 \pmod{p^2}.$$

Remark. A related conjecture of Rodriguez-Villegas [RV] partially confirmed by Mortenson [M2] states that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k, k, k, k}}{256^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = 1 \text{ \& } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-2}{p}\right) = -1, \text{ i.e., } p \equiv 5, 7 \pmod{8}, \end{cases}$$

and

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k, k, k}}{12^{3k}} &= \sum_{k=0}^{p-1} \frac{(6k)!}{(3k)!(k!)^3} 1728^{-k} \\ &\equiv \begin{cases} \left(\frac{2}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } 4 \mid p-1 \text{ \& } p = x^2 + y^2 \ (2 \nmid x, 2 \mid y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Conjecture A30 (The first and the second congruences were discovered during March 13-15, 2010 while all the others were discovered during Feb. 24-26, 2010). *Let $p > 5$ be a prime. If $p > 7$ then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} H_{p-1} - \frac{13}{27} \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^4}.$$

We also have

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - \frac{H_{p-1}}{p^3} \equiv \frac{7}{54p} \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -\frac{7}{45} p B_{p-5} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -2B_{p-3} \pmod{p},$$

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv \frac{56}{15} p B_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k 16^k} \equiv -2H_{(p-1)/2} - \frac{7}{2} p^2 B_{p-3} \pmod{p^3},$$

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}^2}{k 16^k} \equiv -\frac{21}{2} H_{p-1} \pmod{p^4}.$$

Remark. It is known that $H_{p-1}/p^2 \equiv -B_{p-3}/3 \pmod{p}$ for any prime $p > 3$ and $\sum_{k=1}^{p-1} 1/k^3 \equiv -\frac{6}{5}p^2 B_{p-5} \pmod{p^3}$ for each prime $p > 5$, and that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

Also, [T, Theorem 4.2] implies that $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv -\frac{4}{15}p B_{p-3} \pmod{p^2}$ for any prime $p > 5$. Tauraso conjectured that $\sum_{k=1}^{p-1} \binom{2k}{k}^2 / (k16^k) \equiv -2H_{(p-1)/2} \pmod{p^3}$ for each prime $p > 3$. During March 6-7, 2010 the author proved that the third, the fourth and the fifth congruences in Conj. A30 are pairwise equivalent and that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p}$$

for any prime $p > 3$. *Mathematica* (version 7) yields

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2}{k16^k} = 4 \log 2 - \frac{8G}{\pi}$$

where $G = \sum_{k=0}^{\infty} (-1)^k / (2k+1)^2$ is the Catalan constant.

Conjecture A31 (Discovered during March 5-16, 2010, see [S10]). *Let p be an odd prime. Then*

$$p \sum_{k=1}^{p-1} \frac{2^k}{k \binom{2k}{k}} \equiv \left(\frac{-1}{p} \right) - 1 - p q_p(2) + p^2 E_{p-3} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{(2k-1) \binom{2k}{k}} \equiv E_{p-3} + (-1)^{(p-1)/2} - 1 \pmod{p},$$

which is actually equivalent to

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)4^k} \equiv p E_{p-3} \pmod{p^2}.$$

If $p > 3$, then

$$\sum_{k=1}^{p-1} \frac{4^k}{k^2 \binom{2k}{k}} + \frac{4q_p(2)}{p} \equiv -2q_p^2(2) + pB_{p-3} \pmod{p^2}$$

and

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{p}{8} \sum_{k=1}^{(p-1)/2} \frac{16^k}{k(2k-1)\binom{2k}{k}} \equiv \frac{p}{3} E_{p-3} \pmod{p^2}.$$

Remark. It is known that $\sum_{k=1}^{\infty} 2^k / (k \binom{2k}{k}) = \pi/2$. Also, the first congruence is known to hold mod p^2 (cf. [T]). On April 20, 2010 the author noted that **Mathematica** yields $\sum_{k=1}^{\infty} 4^k / (k(2k-1) \binom{2k}{k}) = 4G$ where G denotes the Catalan constant $\sum_{k=0}^{\infty} (-1)^k / (2k+1)^2$. Glaisher (cf. [Ma]) got the formulae $\sum_{k=0}^{\infty} \binom{2k}{k} / ((2k+1)4^k) = \pi/2$ and $\sum_{k=1}^{\infty} 4^k / (k^2 \binom{2k}{k}) = \pi^2/2$. R. Sprugnoli [Sp] showed that $\sum_{k=2}^{\infty} 4^k / ((k-1)^2 \binom{2k}{k}) = \pi^2 - 4$. Let $p > 3$ be a prime. During March 6-7, 2010 the author showed that

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}$$

(which is equivalent to $\sum_{p/2 < k < p} \binom{2k}{k} / (k4^k) \equiv (-1)^{(p-1)/2} 2pE_{p-3} \pmod{p}$) and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)4^k} \equiv (-1)^{(p+1)/2} q_p(2) \pmod{p^2},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$. On March 9, 2010 he proved that

$$\sum_{k=2}^{(p-1)/2} \frac{4^k}{(k-1)^2 \binom{2k}{k}} \equiv 8E_{p-3} - 4 - 12 \left(\frac{-1}{p} \right) \pmod{p}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{4^k}{(k+1) \binom{2k}{k}} \equiv \left(\frac{-1}{p} \right) (4 - 2E_{p-3}) - 2 \pmod{p}.$$

Conjecture A32 (Discovered in March 2010). *Let $p > 5$ be a prime and let $H_{p-1} = \sum_{k=1}^{p-1} 1/k$. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv (-1)^{(p-1)/2} \left(\frac{H_{p-1}}{12} + \frac{3p^4}{160} B_{p-5} \right) \pmod{p^5},$$

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} \equiv (-1)^{(p-1)/2} \left(\frac{H_{p-1}}{4p^2} + \frac{p^2}{36} B_{p-5} \right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 (-16)^k} \equiv \frac{H_{p-1}}{5p} \pmod{p^3},$$

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k}}{(2k+1)^2 (-16)^k} \equiv -\frac{p}{4} B_{p-3} \pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 (-32)^k} \equiv -\binom{2}{p} \frac{q_p^2(2)}{2} \pmod{p}.$$

Remark. On March 6, 2010 the author proved the first congruence in Conj. A32 mod p^2 and the second congruence mod p , and used some new identities to establish the following congruences (for any prime $p > 3$):

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 4^k} \equiv (-1)^{(p+1)/2} \frac{q_p^2(2)}{2} \pmod{p},$$

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \equiv -\binom{-2}{p} \frac{q_p(2)}{2} + \binom{-2}{p} \frac{p}{8} q_p^2(2) \pmod{p^2}.$$

It is known (cf. [Ma]) that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2 (-16)^k} = \frac{\pi^2}{10}$$

which can be easily proved by using $1/(2k+1) = \int_0^1 x^{2k} dx$. In March 2010 the author suggested that $\sum_{k=0}^{\infty} \binom{2k}{k} / ((2k+1)^3 16^k) = 7\pi^3/216$ via a public message to Number Theory List, and then Olivier Gerard pointed out there is a computer proof via certain math. softwares like **Mathematica** (Version 7). **Mathematica** also yields the following identities:

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)8^k} = \frac{\pi}{2\sqrt{2}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2 (-32)^k} = \frac{\pi^2 - 3 \log^2 2}{6\sqrt{2}}.$$

Conjecture A33 ([S10]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{4}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p^a-1)/2} \pmod{p^3}.$$

If $p > 3$, and $p \equiv 1, 3 \pmod{8}$ or $a > 1$, then

$$\sum_{k=0}^{\lfloor \frac{r}{8}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p^a-1)/2} \pmod{p^3} \quad \text{for } r = 5, 7.$$

Remark. The author [S10] showed that $\sum_{k=0}^{\lfloor p/2 \rfloor} \binom{2k}{k}^2 / 16^k \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}$ for any odd prime p .

Conjecture A34 ([S10]). (i) *For each $n = 2, 3, \dots$ we have*

$$\begin{aligned} \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^3 256^{n-1-k} &\in \mathbb{Z}, \\ \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^3 (-512)^{n-1-k} &\in \mathbb{Z}, \\ \frac{1}{2n \binom{2n}{n}} \sum_{k=0}^{n-1} (42k+5) \binom{2k}{k}^3 4096^{n-1-k} &\in \mathbb{Z}. \end{aligned}$$

(ii) *Let p be an odd prime. We have*

$$\frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{42k+5}{4096^k} \binom{2k}{k}^3 \equiv \left(\frac{-1}{p^a} \right) \left(5 + \frac{p^3}{4} B_{p-3} \right) \pmod{p^4}$$

for all $a \in \mathbb{Z}^+$. When $p > 5$ the congruence even holds mod p^5 if we replace $\frac{p^3}{4} B_{p-3}$ by $-\frac{3}{4} p H_{p-1}$. If $p > 3$, then

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{6k+1}{256^k} \binom{2k}{k}^3 &\equiv p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{(p-1)/2} \frac{6k+1}{(-512)^k} \binom{2k}{k}^3 &\equiv p \left(\frac{-2}{p} \right) + \frac{p^3}{4} \left(\frac{2}{p} \right) E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} \frac{42k+5}{4096^k} \binom{2k}{k}^3 &\equiv 5p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}. \end{aligned}$$

Remark. Those congruences in part (ii) mod p^3 are van Hamme's conjectures (cf. [vH]) which are p -adic analogues of corresponding Ramanujan series.

Conjecture A35 ([S09e, S10]). *Let p be an odd prime. If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \left(1 - \frac{1}{(-8)^k}\right) \equiv 0 \pmod{p^3}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \left(1 + \frac{1}{(-2)^k}\right) \equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^2}{(-8)^k} \equiv 0 \pmod{p^2}.$$

Remark. When $p > 3$ is a prime congruent to 3 mod 4, the author proved on March 5, 2010 that

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2},$$

which is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} H_k \equiv 0 \pmod{p}$$

since $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / 64^k \equiv 0 \pmod{p^2}$ and $\binom{p-1}{k} (-1)^k \equiv 1 - pH_k \pmod{p^2}$.

Conjecture A36 ([S10]). *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \pmod{p^2}. \end{aligned}$$

Moreover, if $p \equiv 1 \pmod{4}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \pmod{p^3};$$

if $p \equiv 1 \pmod{3}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \pmod{p^3}.$$

Conjecture A37 (Discovered on March 2, 2010). *Let p be an odd prime.*

(i) *If $p \equiv 1 \pmod{4}$ then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv \frac{1}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} \\ &\equiv \frac{1}{3} \binom{2}{p} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}; \end{aligned}$$

when $p \equiv 3 \pmod{4}$ we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv -\frac{7}{2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \sum_{k < j \leq 2k} \frac{1}{j} &\equiv -\binom{2}{p} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}, \end{aligned}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p} \text{ for } m = -8, 64, -512 \text{ if } p > 3.$$

(ii) *If $p \equiv 1 \pmod{3}$ then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{1}{2} \binom{-1}{p} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \pmod{p^2}.$$

If $p \equiv 2 \pmod{3}$ then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p^2}.$$

(iii) *If $p \equiv 5, 7 \pmod{8}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-64)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv 0 \pmod{p}.$$

Remark. Partially motivated by an observation of M. Jameson and K. Ono which occurred during their attempt to prove Conjecture A1, we discovered the current Conj. A37.

Conjecture A38 (Discovered on March 3, 2010). *Let $p > 3$ be a prime.*

(i) *If $p \equiv 2 \pmod{3}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{16^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{256^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{1}{6} \pmod{p}.$$

(ii) *If $p \equiv 3 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-8)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-512)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv -\frac{1}{6} \pmod{p}.$$

(iii) *If $p \equiv 5, 7 \pmod{8}$ then*

$$\sum_{k=0}^{(p-1)/2} \frac{k \binom{2k}{k}^3}{(-64)^k} \sum_{k < j \leq 2k} \frac{1}{j} \equiv \frac{(-1)^{(p-1)/2}}{6} \pmod{p}.$$

Remark. For certain reasons we omit other similar observations.

Conjecture A39 (Discovered on April 4, 2010). (i) *For any $n \in \mathbb{Z}^+$ we have*

$$a_n := \frac{1}{8n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} (205k^2 + 160k + 32) (-1)^{n-1-k} \binom{2k}{k}^5 \in \mathbb{Z}^+.$$

(ii) *Let p be an odd prime. If $p \neq 3$ then*

$$\sum_{k=0}^{(p-1)/2} (205k^2 + 160k + 32) (-1)^k \binom{2k}{k}^5 \equiv 32p^2 + \frac{896}{3} p^5 B_{p-3} \pmod{p^6}.$$

If $p \neq 5$ then

$$\sum_{k=0}^{p-1} (205k^2 + 160k + 32) (-1)^k \binom{2k}{k}^5 \equiv 32p^2 + 64p^3 H_{p-1} \pmod{p^7}.$$

Remark. Note that $a_1 = 1$ and

$$4(2n+1)^2 a_{n+1} + n^2 a_n = (205n^2 + 160n + 32) \binom{2n-1}{n}^3 \quad \text{for } n = 1, 2, \dots$$

The author created the sequence $\{a_n\}_{n>0}$ at OEIS as A176285 (cf. [S10a]). In 1997 T. Amdeberhan and D. Zeilberger [AZ] used the WZ method to obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^k (205k^2 - 160k + 32)}{k^5 \binom{2k}{k}^5} = -2\zeta(3).$$

Conjecture A40 (Discovered on April 5, 2010). (i) *For any odd prime p we have*

$$\sum_{k=0}^{p-1} \frac{28k^2 + 18k + 3}{(-64)^k} \binom{2k}{k}^4 \binom{3k}{k} \equiv 3p^2 - \frac{7}{2}p^5 B_{p-3} \pmod{p^6}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{28k^2 + 18k + 3}{(-64)^k} \binom{2k}{k}^4 \binom{3k}{k} \equiv 3p^2 + 6 \left(\frac{-1}{p}\right) p^4 E_{p-3} \pmod{p^5}.$$

(ii) *For any integer $n > 1$, we have*

$$\sum_{k=0}^{n-1} (28k^2 + 18k + 3) \binom{2k}{k}^4 \binom{3k}{k} (-64)^{n-1-k} \equiv 0 \pmod{(2n+1)n^2 \binom{2n}{n}^2}.$$

Also,

$$\sum_{k=1}^{\infty} \frac{(28k^2 - 18k + 3)(-64)^k}{k^5 \binom{2k}{k}^4 \binom{3k}{k}} = -14\zeta(3).$$

Conjecture A41 (Discovered on April 5, 2010). *Let p be an odd prime.*

(i) *If $p \neq 3$, then*

$$\sum_{k=0}^{p-1} \frac{10k^2 + 6k + 1}{(-256)^k} \binom{2k}{k}^5 \equiv p^2 - \frac{7}{6}p^5 B_{p-3} \pmod{p^6}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{10k^2 + 6k + 1}{(-256)^k} \binom{2k}{k}^5 \equiv p^2 + \frac{7}{3}p^5 B_{p-3} \pmod{p^6}.$$

(ii) *If $p \neq 5$, then*

$$\sum_{k=0}^{p-1} \frac{74k^2 + 27k + 3}{4096^k} \binom{2k}{k}^4 \binom{3k}{k} \equiv 3p^2 + 7p^5 B_{p-3} \pmod{p^6}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{74k^2 + 27k + 3}{4096^k} \binom{2k}{k}^4 \binom{3k}{k} \equiv 3p^2 - \frac{9}{4}p^3 H_{p-1} \pmod{p^7}.$$

Remark. By [G3, Identity 8] and [G4], we have

$$\sum_{k=1}^{\infty} \frac{(10k^2 - 6k + 1)(-256)^k}{k^5 \binom{2k}{k}^5} = -28\zeta(3)$$

and

$$\sum_{k=0}^{\infty} \frac{74k^2 + 27k + 3}{4096^k} \binom{2k}{k}^4 \binom{3k}{k} = \frac{48}{\pi^2}.$$

Conjecture A42 (Discovered on April 6, 2010). (i) For any prime $p \neq 2, 5$ we have

$$\sum_{k=0}^{p-1} \frac{21k^3 + 22k^2 + 8k + 1}{256^k} \binom{2k}{k}^7 \equiv p^3 \pmod{p^8}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{168k^3 + 76k^2 + 14k + 1}{2^{20k}} \binom{2k}{k}^7 \equiv \left(\frac{-1}{p}\right) p^3 \pmod{p^8}.$$

(ii) For any integer $n > 1$, we have

$$\sum_{k=0}^{n-1} (21k^3 + 22k^2 + 8k + 1) \binom{2k}{k}^7 256^{n-1-k} \equiv 0 \pmod{2n^3 \binom{2n}{n}^3}$$

and

$$\sum_{k=0}^{n-1} (168k^3 + 76k^2 + 14k + 1) \binom{2k}{k}^7 2^{20(n-1-k)} \equiv 0 \pmod{2n^3 \binom{2n}{n}^3}.$$

Remark. (a) B. Gourevich and Guillera (see [G1, Section 4]) conjectured

$$\sum_{k=0}^{\infty} \frac{168k^3 + 76k^2 + 14k + 1}{2^{20k}} \binom{2k}{k}^7 = \frac{32}{\pi^3}$$

and

$$\sum_{k=1}^{\infty} \frac{(21k^3 - 22k^2 + 8k - 1)256^k}{k^7 \binom{2k}{k}^7} = \frac{\pi^4}{8}$$

respectively. Zudilin [Zu] suggested that for any odd prime p we might have

$$\sum_{k=0}^{p-1} \frac{168k^3 + 76k^2 + 14k + 1}{2^{20k}} \binom{2k}{k}^7 \equiv \left(\frac{-1}{p}\right) p^3 \pmod{p^7},$$

which is weaker than the second congruence in Conj. A42(i).

(b) Let $a_1 = 2$ and

$$(2n+1)^3 a_n = 32n^3 a_{n-1} + (21n^3 + 22n^2 + 8n + 1) \binom{2n-1}{n}^4 \text{ for } n = 2, 3, \dots$$

Then for any $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} a_n &= \frac{1}{16(2n+1)^3 \binom{2n}{n}^3} \sum_{k=0}^n (21k^3 + 22k^2 + 8k + 1) 256^{n-k} \binom{2k}{k}^7 \\ &= \frac{1}{2(n+1)^3 \binom{2(n+1)}{n+1}^3} \sum_{k=0}^n (21k^3 + 22k^2 + 8k + 1) 256^{n-k} \binom{2k}{k}^7. \end{aligned}$$

The author created the sequence $\{a_n\}_{n>0}$ at OEIS as A176477 (cf. [S10a]). We not only conjectured that $a_n \in \mathbb{Z}^+$ for all $n = 1, 2, 3, \dots$ but also guessed that a_n is odd if and only if $n = 2^k$ for some $k \in \mathbb{Z}^+$. We have a similar conjecture related to the last congruence in Conj. A42.

Conjecture A43 ([S10]). *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p} \right) - \frac{25}{9} p^2 E_{p-3} \pmod{p^3}.$$

Remark. A related conjecture of Rodriguez-Villegas [RV] proved by Mortenson [M2] states that if $p > 3$ is a prime then

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} = \sum_{k=0}^{p-1} \frac{(6k)!}{k!(2k)!(3k)!} (2^4 3^3)^{-k} \equiv \left(\frac{-1}{p} \right) \pmod{p^2}.$$

Using the Gosper algorithm we find the identity

$$\sum_{k=0}^n (36k + 5) \binom{6k}{3k} \binom{3k}{k} 432^{n-k} = (6n + 1)(6n + 5) \binom{6n}{3n} \binom{3n}{n} \quad (n \in \mathbb{N})$$

which implies that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^n \frac{36k + 5}{432^k} \binom{6k}{3k} \binom{3k}{k} = \frac{18}{\pi}$$

and

$$\sum_{k=0}^{p-1} \frac{36k + 5}{432^k} \binom{6k}{3k} \binom{3k}{k} \equiv 5p^2 \pmod{p^3} \quad \text{for any prime } p > 3.$$

Conjecture A44 ([S10]). *Let $p > 3$ be a prime. When $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \begin{cases} (-1)^{\lfloor x/6 \rfloor} (2x - p/(2x)) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ \left(\frac{xy}{3}\right) (2y - p/(2y)) \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p^a-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2}.$$

Provided that $p \equiv 1 \pmod{4}$, we have

$$\sum_{k=0}^{p^a-1} k \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv 0 \pmod{p^a} \quad \text{for all } a \in \mathbb{Z}^+.$$

Conjecture A45 ([S09e, S10]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{24^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{(-216)^k} \equiv \begin{cases} \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod{6}, \\ 0 \pmod{p} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$

Remark. In [M2] Mortenson proved the following conjecture of Rodriguez-Villegas [RV]: For any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{27^k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Conjecture A46 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k+1}}{48^k} \equiv 0 \pmod{p^2}.$$

If $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} k \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv \frac{p}{2x} - x \pmod{p^2}.$$

If $p \equiv 2 \pmod{3}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv 0 \pmod{p}.$$

Conjecture A47 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-192)^k} \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{(-192)^k} \equiv \frac{1}{4} \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{48^k} \pmod{p^2}.$$

Conjecture A48 ([S09e]). *Let $p > 3$ be a prime. If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv (-1)^{(p-1)/4} \left(\frac{p}{3}\right) \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv (-1)^{(p-1)/4} \left(\frac{p}{3}\right) \left(x - \frac{p}{2x}\right) \pmod{p^2}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv 0 \pmod{p}.$$

Conjecture A49 ([S09e]). *Let p be an odd prime. If $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$, then*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv \begin{cases} (-1)^{(p-1)/8+(x-1)/2} (2x - p/(2x)) \pmod{p^2} & \text{if } 8 \mid p-1, \\ p/(2x) - 2x \pmod{p^2} & \text{if } 8 \mid p-3. \end{cases}$$

If $p \equiv 5, 7 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2}.$$

Conjecture A50 ([S09e]). *Let $p > 3$ be a prime. If $\left(\frac{p}{7}\right) = 1$ and $p = x^2 + 7y^2$ with $\left(\frac{x}{7}\right) = 1$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv \left(\frac{p}{3}\right) \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv 8 \left(\frac{p}{3}\right) \left(\frac{p}{2x} - x\right) \pmod{p^2}.$$

If $\left(\frac{p}{7}\right) = -1$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}^2}{63^k} \equiv 0 \pmod{p}.$$

Conjecture A51 ([S09e]). *For any prime $p \equiv 1 \pmod{4}$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{2k}{k+1}^2}{(-8)^k} \equiv -2p \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+1}}{64^k} \equiv 0 \pmod{p^2}.$$

Conjecture A52 ([S09e]). *Let p be an odd prime. If $p \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{k C_k^3}{16^k} \equiv 2p - 2 \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^3}{64^k} \equiv 8 \pmod{p^2}.$$

Remark. The author [S09e] determined $\sum_{k=0}^{p-1} C_k^3/64^k$ modulo any odd prime p .

Conjecture A53 ([S10]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv (-1)^{(p-1)/2} - 3p^2 E_{p-3} \pmod{p^3}$$

and

$$p \sum_{k=1}^{(p-1)/2} \frac{64^k}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{2k}} \equiv 16 \left(p E_{p-3} - \left(\frac{-1}{p}\right) q_p(2) \right) \pmod{p^2}.$$

Remark. Mortenson [M2] proved the following conjecture of Rodriguez-Villegas [RV]: For any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}.$$

Conjecture A54 ([S09e]). *Let p be a prime with $p \equiv 1, 3 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{128^k} \equiv p \pmod{p^2}.$$

Remark. **Mathematica** yields that $\sum_{k=0}^{\infty} \binom{4k}{2k} C_k / 128^k = 4\sqrt{\pi} / (\Gamma(\frac{1}{8})\Gamma(\frac{11}{8}))$.

Conjecture A55 ([S09e]). *Let p be a prime with $p \equiv 1 \pmod{3}$ and hence $p = x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. Then we have*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{3k}{k}}{54^k} \equiv 0 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{54^k} \equiv p \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \frac{4}{9}(p - 2x^2) \pmod{p^2}.$$

Remark. In [S09a] the author determined $\sum_{k=0}^{p-1} \binom{3k}{k} / m^k \pmod{p}$ for any prime $p > 3$ and any $m \in \mathbb{Z}$ with $p \nmid m$. A conjecture of Rodriguez-Villegas [RV] has the following equivalent form:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{3}\right) = 1 \text{ \& } p = x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

See [M3] for related result. **Mathematica** yields that

$$\sum_{k=0}^{\infty} \frac{k \binom{2k}{k} \binom{3k}{k}}{54^k} = \frac{\sqrt{\pi}}{9\Gamma(\frac{4}{3})\Gamma(\frac{7}{6})} \text{ and } \sum_{k=0}^{\infty} \frac{\binom{3k}{k} C_k}{54^k} = \frac{3\sqrt{\pi}}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{6})}.$$

Conjecture A56 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_k^{(2)}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} \frac{k \binom{2k}{k} C_k^{(2)}}{27^k} \equiv 0 \pmod{p^2},$$

where

$$C_k^{(2)} = \frac{\binom{3k}{k}}{2k+1} = \binom{3k}{k} - 2 \binom{3k}{k-1}$$

is a second-order Catalan number (of the first kind). Furthermore,

$$\sum_{k=0}^{p-1} (4k+1) \frac{\binom{2k}{k} C_k^{(2)}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^4}.$$

Conjecture A57 ([S09e]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{C_k C_k^{(2)}}{27^k} \equiv 2 \binom{p}{3} - p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{C_k \bar{C}_k^{(2)}}{27^k} \equiv -7 \pmod{p},$$

where

$$\bar{C}_k^{(2)} = \frac{2}{k+1} \binom{3k}{k} = 2 \binom{3k}{k} - \binom{3k}{k+1}$$

is a second-order Catalan number of the second kind. Hence

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k-1} \binom{3k}{k-1}}{27^k} \equiv \binom{p}{3} - p \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k+1} \binom{3k}{k+1}}{27^k} \equiv 2 \binom{p}{3} - 7 \pmod{p}.$$

Remark. Note that

$$\binom{2k}{k-1} \binom{3k}{k-1} = \left(\binom{2k}{k} - C_k \right) \frac{\binom{3k}{k} - C_k^{(2)}}{2}$$

and

$$\binom{2k}{k+1} \binom{3k}{k+1} = \left(\binom{2k}{k} - C_k \right) \left(2 \binom{3k}{k} - \bar{C}_k^{(2)} \right).$$

Mathematica yields that

$$\sum_{k=0}^{\infty} \frac{C_k \bar{C}_k^{(2)}}{27^k} = \frac{81\sqrt{3}}{4\pi} - 9.$$

Conjecture A58 ([S09h]). *Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \binom{k}{3} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-3)/4} \left(4y - \frac{p}{3y} \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \left(\frac{k}{3}\right) \frac{k \binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p+1)/4} y \pmod{p^2}.$$

If $p \equiv 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \left(\frac{k}{3}\right) \frac{\binom{2k}{k}^2}{16^k} \equiv 0 \pmod{p^2}.$$

Recall that the Pell sequence $\{P_n\}_{n \geq 0}$ and its companion $\{Q_n\}_{n \geq 0}$ are defined as follows:

$$P_0 = 0, P_1 = 1, \text{ and } P_{n+1} = 2P_n + P_{n-1} \quad (n = 1, 2, 3, \dots);$$

$$Q_0 = 2, Q_1 = 2, \text{ and } Q_{n+1} = 2Q_n + Q_{n-1} \quad (n = 1, 2, 3, \dots).$$

Conjecture A59 ([S09h]). (i) Let p be a prime with $p \equiv 1, 3 \pmod{8}$. Write $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$. Then

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(p-3)/8} (p/(2x) - 2x) \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{k P_k}{(-8)^k} \binom{2k}{k}^2 \equiv \frac{(-1)^{(x+1)/2}}{2} \left(x + \frac{p}{2x}\right) \pmod{p^2}.$$

(ii) If $p \equiv 5 \pmod{8}$ is a prime, then

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $p \equiv 7 \pmod{8}$ is a prime, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{P_k}{8^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

Conjecture A60 ([S09h]). *Let p be an odd prime.*

(i) *If $p \equiv 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $y \equiv 1, 3 \pmod{p}$, then*

$$\sum_{k=0}^{p-1} \frac{P_k}{32^k} \binom{2k}{k}^2 \equiv (-1)^{(y-1)/2} \left(2y - \frac{p}{4y} \right) \pmod{p^2}.$$

If $p \equiv 7 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{P_k}{32^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

(ii) *Suppose that $p \equiv 1, 3 \pmod{8}$, $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$ and also $y \equiv 1, 3 \pmod{8}$ when $p \equiv 3 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{kP_k}{32^k} \binom{2k}{k}^2 \equiv \begin{cases} (-1)^{(p-1)/8} (p/(4x) - x/2) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(y+1)/2} y \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Conjecture A61 ([S09h]). *Let p be an odd prime.*

(i) *When $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{(-8)^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(4x - \frac{p}{x} \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kQ_k}{(-8)^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(p-3)/8} 2(x + p/x) \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

(ii) *When $p \equiv 5, 7 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{(-8)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Conjecture A62 ([S09h]). *Let p be an odd prime.*

(i) *When $p \equiv 1 \pmod{8}$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod{8}$, we have*

$$\sum_{k=0}^{p-1} \frac{Q_k}{32^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/8} \left(4x - \frac{p}{x} \right) \pmod{p^2}.$$

If $p \equiv 5 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{Q_k}{32^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

(ii) If $p \equiv 1, 3 \pmod{8}$ and $p = x^2 + 2y^2$ with $x \equiv 1, 3 \pmod{8}$ and also $y \equiv 1, 3 \pmod{8}$ when $p \equiv 3 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2}{32^k} Q_k \equiv \begin{cases} (-1)^{(p-1)/8} (p/x - 2x) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(y+1)/2} 2y \pmod{p^2} & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

Conjecture A63 ([S09h]). Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{u_k(4, 1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p+1)/4} \left(4y - \frac{p}{3y} \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k u_k(4, 1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-3)/4} \left(6y - \frac{7p}{3y} \right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{u_k(4, 1)}{4^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture A64 ([S09h]). Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{u_k(4, 1)}{64^k} \binom{2k}{k}^2 \equiv 2y - \frac{p}{6y} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k u_k(4, 1)}{64^k} \binom{2k}{k}^2 \equiv y \pmod{p^2}.$$

If $p \equiv 11 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \frac{u_k(4, 1)}{64^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Conjecture A65 ([S09h]). *Let $p \equiv 1 \pmod{3}$ be a prime.*

(i) *If $p \equiv 1 \pmod{12}$ and $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{p-1} \frac{v_k(4,1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/4+(x-1)/2} \left(4x - \frac{p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{v_k(4,1)}{64^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(4x - \frac{p}{x}\right) \pmod{p^2};$$

also

$$\sum_{k=0}^{p-1} \frac{kv_k(4,1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/4+(x+1)/2} \left(4x - \frac{2p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kv_k(4,1)}{64^k} \binom{2k}{k}^2 \equiv (-1)^{(x-1)/2} \left(2x - \frac{p}{x}\right) \pmod{p^2}.$$

(ii) *If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{v_k(4,1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-3)/4} \left(12y - \frac{p}{y}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{kv_k(4,1)}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p+1)/4} \left(20y - \frac{8p}{y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kv_k(4,1)}{64^k} \binom{2k}{k}^2 \equiv 4y \pmod{p^2}.$$

Conjecture A66 ([S09h]). *Let $p \equiv 2 \pmod{3}$ be a prime. If $p \equiv 5 \pmod{12}$, then*

$$\sum_{k=0}^{p-1} \frac{v_k(4,1)}{4^k} \binom{2k}{k}^2 \equiv \sum_{k=0}^{p-1} \frac{v_k(4,1)}{64^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $p \equiv 11 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{v_k(4,1)}{(-4)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

Recall Apéry numbers are those integers

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n \in \mathbb{N})$$

which play a central role in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$.

Conjecture A67 ([S10d]). *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} A_k \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}; \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k A_k \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k A_k(-2) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \ \& \ p = x^2 + y^2 \ (2 \nmid x, 2 \mid y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \end{aligned}$$

where

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k.$$

Conjecture A68. (i) ([S10d]) *For any $\varepsilon \in \{\pm 1\}$, $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n-1} (2k+1) \varepsilon^k A_k(x)^m \equiv 0 \pmod{n}.$$

(ii) ([S10d]) *If p is an odd prime, then*

$$\sum_{k=0}^{p-1} (2k+1) (-1)^k A_k(x) \equiv p \left(\frac{1-4x}{p} \right) \pmod{p^2}.$$

Also, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} (2k+1) A_k(-3) \equiv \sum_{k=0}^{p-1} (2k+1) (-1)^k A_k \equiv p \left(\frac{p}{3} \right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k(-2) \equiv p - \frac{4}{3}p^2 q_p(2) \pmod{p^3}.$$

(iii) (Discovered on August 18, 2010) *For any prime $p > 3$ we have*

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)^3 A_k &\equiv p^3 + 4p^4 H_{p-1} - p^6 \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^9} \\ &\equiv p^3 + 4p^4 H_{p-1} + \frac{6}{5}p^8 B_{p-5} \pmod{p^9} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} (2k+1)^3 (-1)^k A_k \equiv -\frac{p}{3} \left(\frac{p}{3} \right) \pmod{p^3}.$$

Remark. The author [S10d] proved that $n \mid \sum_{k=0}^{n-1} (2k+1)A_k(x)$ for all $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ and that $\sum_{k=0}^{p-1} (2k+1)A_k \equiv p \pmod{p^4}$ for any prime $p > 3$. The reader may consult sequences A178790 and A178791 in [S10a] for the sequences $\frac{1}{n} \sum_{k=0}^n (2k+1)A_k$ ($n = 1, 2, 3, \dots$) and $\frac{1}{n} \sum_{k=0}^n (2k+1)(-1)^k A_k$ ($n = 1, 2, 3, \dots$). Motivated by the author's work in [S10d], Guo and Zeng [GZ] showed that $n^3 \mid \sum_{k=0}^{n-1} (2k+1)^3 A_k$ for all $n \in \mathbb{Z}^+$ and that $\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 \pmod{p^6}$ for any prime $p > 3$.

Conjecture A69 ([S10e]). *Let p be an odd prime.*

(i) *We have*

$$\begin{aligned} &\sum_{k=0}^{p-1} (-1)^k A_k \left(\frac{1}{4} \right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} &\sum_{k=0}^{p-1} (-1)^k A_k(16) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod{3} \text{ and } p = x^2 + 7y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7} \text{ (i.e., } (\frac{p}{7}) = -1). \end{cases} \end{aligned}$$

(ii) *If $p \neq 5$, then*

$$\begin{aligned} &\sum_{k=0}^{p-1} A_k(-4) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ and } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ and } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-5}{p}) = -1, \text{ i.e., } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases} \end{aligned}$$

If $p \neq 3$, then

$$\sum_{k=0}^{p-1} A_k(9) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases}$$

Those integers

$$D_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \quad (n \in \mathbb{N})$$

are called central Delannoy numbers; they arise naturally in many enumeration problems in combinatorics.

Conjecture A70 ([S10d]). *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2 \left(\frac{-1}{p} \right) E_{p-3} \pmod{p}.$$

When $p > 3$ we have

$$\sum_{k=0}^{p-1} (2k+1)D_k^2 \equiv p^2 - 4p^3q_p(2) - 2p^4q_p(2)^2 \pmod{p^5},$$

$$\sum_{n=0}^{p-1} D_n S_n \equiv 1 + 4pq_p(2) - 2p^2q_p(2)^2 \pmod{p^3},$$

and

$$\sum_{n=1}^{(p-1)/2} D_n S_n \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{)}, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where

$$S_n = \sum_{k=0}^n \binom{n+k}{2k} C_k = \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} \binom{n+k}{k}$$

is the n th Schröder number.

Remark. The author [S10d] proved that

$$\sum_{k=0}^{p-1} D_k \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}$$

for any odd prime p .

Just like $A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$ we define

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k.$$

Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n .

Conjecture A71 ([S10d]). (i) *For any $n \in \mathbb{Z}$ the numbers*

$$s(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k \left(\frac{1}{4} \right)$$

and

$$t(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k \left(-\frac{1}{4} \right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ and $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ respectively. Moreover, the numerators of $s(1), s(3), s(5), \dots$ are congruent to 1 modulo 12 and the numerators of $s(2), s(4), s(6), \dots$ are congruent to 7 modulo 12. If p is an odd prime and $a \in \mathbb{Z}^+$, then

$$s(p^a) \equiv t(p^a) \equiv 1 \pmod{p}.$$

For $p = 3$ and $a \in \mathbb{Z}^+$ we have

$$s(3^a) \equiv 4 \pmod{3^2} \quad \text{and} \quad t(3^a) \equiv -8 \pmod{3^5}.$$

(ii) *Let p be a prime. For any positive integer n and p -adic integer x , we have*

$$\nu_p \left(\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k(x) \right) \geq \min\{\nu_p(n), \nu_p(4x-1)\}$$

and

$$\nu_p \left(\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x+1)\}.$$

Conjecture A72 ([S10e]). *Let p be an odd prime.*

(i) *We have*

$$\begin{aligned} \sum_{k=0}^{p-1} D_k \left(-\frac{1}{2}\right)^3 &= \sum_{k=0}^{p-1} (-1)^k D_k \left(-\frac{1}{2}\right)^3 \equiv \sum_{k=0}^{p-1} (-1)^k A_k \left(-\frac{1}{4}\right) \\ &\equiv \begin{cases} \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

If $p \neq 3$, then

$$\begin{aligned} \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} (-1)^k D_k \left(\frac{1}{2}\right)^3 \\ \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

(ii) *Suppose that $p > 3$. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} D_k (-3)^3 &= \sum_{k=0}^{p-1} (-1)^k D_k (2)^3 \\ &\equiv \begin{cases} \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } p = x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

We also have

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k D_k (2)^3 &\equiv \sum_{k=0}^{p-1} (-1)^k D_k \left(-\frac{1}{4}\right)^3 \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} (-1)^k D_k \left(\frac{1}{8}\right)^3 \\ &\equiv \sum_{k=0}^{p-1} (-1)^k A_k \left(\frac{1}{16}\right) \pmod{p^2}. \end{aligned}$$

(iii) *Assume that $p > 5$. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} D_k (3)^3 &= \sum_{k=0}^{p-1} (-1)^k D_k (-4)^3 \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and } p = x^2 + 15y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and } p = 5x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} (-1)^k D_k \left(-\frac{1}{16}\right)^3 \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} (-1)^k D_k (-4)^3 \pmod{p^2}.$$

Remark. It is easy to show that $(-1)^n D_n(x) = D_n(-x-1)$.

Recall that

$$T_n := [x^n](1+x+x^2)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}$$

is called a central trinomial coefficient. And those numbers

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k \quad (n = 0, 1, 2, \dots)$$

are called Motzkin numbers. In general, for $b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define

$$T_n(b, c) := [x^n](x^2 + bx + c)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k$$

and

$$M_n(b, c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k} c^k.$$

Conjecture A73. (i) ([S10e]) *Let b and c be integers. For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1} T_k(b, c) M_k(b, c) (b^2 - 4c)^{n-1-k} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $c(b^2 - 4c)$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b, c) M_k(b, c)}{(b^2 - 4c)^k} \equiv \frac{pb^2}{2c} \left(\left(\frac{b^2 - 4c}{p} \right) - 1 \right) \pmod{p^2}.$$

(ii) ([S10e]) *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{T_k(3, 3) M_k(3, 3)}{(-3)^k} \equiv \begin{cases} 2p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod{3}, \\ p^3 - p^2 - 3p \pmod{p^4} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

(iii) ([S10d]) For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2 - 6p) \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p - 1) \left(\frac{p}{3}\right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} T_k M_k \equiv \frac{4}{3} \left(\frac{p}{3}\right) + \frac{p}{6} \left(1 - 9 \left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Remark. The author [S10e] proved that if p is an odd prime not dividing $c(b^2 - 4c)$ then

$$\sum_{k=0}^{p-1} \frac{T_k(b, c)M_k(b, c)}{(b^2 - 4c)^k} \equiv 0 \pmod{p}.$$

He also showed that if p is an odd prime not dividing $c(b^2 - 4c^2)$ then

$$\sum_{k=0}^{p-1} \frac{T_k(b, c^2)M_k(b, c^2)}{(b - 2c)^{2k}} \equiv \frac{4b}{b + 2c} \left(\frac{b^2 - 4c^2}{p}\right) \pmod{p}.$$

Conjecture A74 ([S10e]). Let b and c be integers. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (8ck + 4c + b)T_k(b, c^2)^2(b - 2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b - 2c)$, then

$$\sum_{k=0}^{p-1} (8ck + 4c + b) \frac{T_k(b, c^2)^2}{(b - 2c)^{2k}} \equiv p(b + 2c) \left(\frac{b^2 - 4c^2}{p}\right) \pmod{p^2}.$$

Remark. In [S10e] the author proved that for any $b, c \in \mathbb{Z}$ and odd prime $p \nmid b - 2c$ we have

$$\sum_{k=0}^{p-1} \frac{T_k(b, c^2)^2}{(b - 2c)^{2k}} \equiv \left(\frac{-c^2}{p}\right) \pmod{p}.$$

The author (cf. [S10a]) added the sequence

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k + 5)T_k^2 \quad (n = 1, 2, 3, \dots)$$

as A179100 at Sloane's OEIS. H. Q. Cao and the author [CS] showed that

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2} \quad \text{for any prime } p > 3.$$

Conjecture A75 ([S10d]). *Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1} (2k+1)T_k(b, c)^2 (b^2 - 4c)^{n-1-k} \equiv 0 \pmod{n^2}.$$

If c is nonzero and p is an odd prime not dividing $b^2 - 4c$, then

$$\frac{1}{p^2} \sum_{k=0}^{p-1} (2k+1) \frac{T_k(b, c)^2}{(b^2 - 4c)^k} \equiv 1 + \frac{b^2}{c} \cdot \frac{\left(\frac{b^2-4c}{p}\right) - 1}{2} \pmod{p}.$$

Remark. Note that $D_n = T_n(3, 2)$ with $3^2 - 4 \times 2 = 1$. Thus the conjecture implies that

$$\sum_{k=0}^{n-1} (2k+1)D_k^2 \equiv 0 \pmod{n^2}$$

for all $n \in \mathbb{Z}^+$. (The author has added the sequence $a(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)D_k^2$ ($n = 1, 2, 3, \dots$) as A178808 in OEIS (cf. [S10a].) Given $b, c \in \mathbb{Z}$, the author proved that

$$\sum_{k=0}^{n-1} (2k+1)T_k(b, c)^2 (4c - b^2)^{n-1-k} \equiv 0 \pmod{n} \quad \text{for } n = 1, 2, 3, \dots$$

(cf. [S10d]) and that

$$\sum_{k=0}^{p-1} \frac{T_k(b, c)^2}{(b^2 - 4c)^k} \equiv \left(\frac{c(b^2 - 4c)}{p} \right) \pmod{p}$$

for any odd prime $p \nmid b^2 - 4c$ (see [S10e]).

Conjecture A76 ([S10d]). *Let $b, c \in \mathbb{Z}$ with $b^2 - 4c = 1$.*

(i) *For any $m, n \in \mathbb{Z}^+$ We have*

$$\sum_{k=0}^{n-1} (2k+1)T_k(b, c)^m \equiv 0 \pmod{n}.$$

If p is a prime not dividing c , then

$$\sum_{k=0}^{p-1} (2k+1)T_k(b, c)^3 \equiv p \left(\frac{-2b-1}{p} \right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (2k+1)T_k(b, c)^4 \equiv p \pmod{p^2}.$$

(ii) For any odd prime p we have

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)^2 T_k(7, 12) &\equiv \left(\frac{p}{3}\right) - 4p \pmod{p^2}, \\ \sum_{k=0}^{p-1} (2k+1)^2 D_k &\equiv \left(\frac{-1}{p}\right) - 2p + (2 - E_{p-3})p^2 \pmod{p^3}, \\ \sum_{k=0}^{p-1} (2k+1)^4 D_k &\equiv 13 \left(\frac{-1}{p}\right) - 20p + (24 - 13E_{p-3})p^2 \pmod{p^3}. \end{aligned}$$

Remark. The author [S10d] proved the first congruence for $m = 1$ and noted that

$$\sum_{k=0}^{n-1} (2k+1) T_k 3^{n-1-k} = n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} (k+1) \binom{2k}{k}$$

for all $n = 1, 2, 3, \dots$ and that if $b, c \in \mathbb{Z}$ and $b^2 - 4c = 1$ then

$$\sum_{k=0}^{p-1} (2k+1) T_k(b, c) \equiv p \pmod{p^2}$$

for any prime $p \nmid c$. Recall that $D_k = T_k(3, 2)$ and $(-1)^k D_k = T_k(-3, 2)$. Guo and Zeng [GZ] extended the author's result by showing that $n \mid \sum_{k=0}^{n-1} (2k+1)^m \varepsilon^k D_k$ for all $\varepsilon \in \{\pm 1\}$, $n \in \mathbb{Z}^+$ and $m = 1, 3, 5, \dots$ (the case $m = 1$ is due to the author [S10d]). It is easy to show that for any odd prime p we have

$$\sum_{k=0}^{p-1} (2k+1)^3 D_k \equiv 5p \pmod{p^2}, \quad \sum_{k=0}^{p-1} (2k+1)^3 (-1)^k D_k \equiv -p - 2p^2 \pmod{p^3}.$$

Conjecture A77 ([S10d, S10e]). *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1} \frac{T_k(2, -1)^2}{8^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{T_k(2, -3)^2}{16^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2}.$$

If $p > 3$, then

$$\sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{4^k} \equiv \sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{36^k} \equiv \sum_{k=0}^{p-1} \frac{T_k(6, -3)^2}{48^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

(ii) *We have*

$$\begin{aligned} \sum_{k=0}^{p-1} T_k(1, 2)^2 &\equiv \sum_{k=0}^{p-1} \frac{T_k(2, -2)^2}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{T_k(2, -1)^2}{(-8)^k} \\ &\equiv \begin{cases} 2x \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x - 1 \text{),} \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} \sum_{k=0}^{p-1} T_k(1, -1)^2 &\equiv \sum_{k=0}^{p-1} \frac{T_k(2, 2)^2}{4^k} \\ &\equiv \begin{cases} \left(\frac{2}{p}\right)2x \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 4 \mid x - 1 \text{),} \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{T_k(2, 2)^2}{4^k} - \sum_{k=0}^{p-1} \frac{T_k(2, 1)^2}{8^k} \equiv \begin{cases} 0 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Remark. The author [S10e] proved that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{4^k} \equiv \sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{36^k} \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

Conjecture A78 ([S10e]). *Let $p > 3$ be a prime. Then*

$$\begin{aligned} &\sum_{k=0}^{p-1} \left(\frac{T_k(2, 9)}{(-4)^k}\right)^3 \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} &\sum_{k=0}^{p-1} \left(\frac{T_k(2, 9)}{8^k}\right)^3 \\ &\equiv \begin{cases} \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2, \\ \left(\frac{-1}{p}\right)(8x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

When $\left(\frac{-6}{p}\right) = 1$ we have

$$\sum_{k=0}^{p-1} (72k + 47) \frac{T_k(2, 9)^3}{(-64)^k} \equiv 42p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} (72k + 25) \frac{T_k(2, 9)^3}{512^k} \equiv 12p \left(\frac{3}{p}\right) \pmod{p^2}.$$

Motivated by central trinomial coefficients and Apéry numbers, for $b, c \in \mathbb{Z}$ the author [S10d] introduced a new kind of numbers:

$$W_n(b, c) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n-k}{k}^2 b^{n-2k} c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{2k}{k}^2 b^{n-2k} c^k \quad (n \in \mathbb{N}).$$

Note that $W_n(-b, c) = (-1)^n W_n(b, c)$.

Conjecture A79 ([S10d]). *Let p be an odd prime.*

(i) *We have*

$$\begin{aligned} & \sum_{k=0}^{p-1} W_k(1, 1) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

If $p \equiv 1, 3 \pmod{8}$, then

$$\sum_{k=0}^{p-1} (16k + 3) W_k(1, 1) \equiv 8p \pmod{p^2}.$$

When $p \equiv 5, 7 \pmod{8}$ and $p \neq 7$, we have

$$\sum_{k=0}^{p-1} \frac{W_k(1, 1)}{(-7)^k} \equiv 0 \pmod{p^2}.$$

(ii) *We have*

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k W_k(1, -1) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} (6k+5)(-1)^k W_k(1, -1) \equiv p \left(2 + 3 \left(\frac{p}{3} \right) \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{n-1} (6k+5)(-1)^k W_k(1, -1) \equiv 0 \pmod{n} \quad \text{for all } n = 1, 2, 3, \dots$$

Remark. Let $p > 3$ be a prime. We also conjecture that

$$\sum_{k=0}^{p-1} \frac{W_k(1, -1)}{(-13)^k} \equiv 0 \pmod{p} \quad \text{if } p \equiv 2 \pmod{3},$$

and

$$\sum_{k=0}^{p-1} \frac{W_k(1, -1)}{(-3)^k} \equiv \sum_{k=0}^{p-1} \frac{W_k(1, -1)}{5^k} \equiv 0 \pmod{p} \quad \text{if } p \equiv 3 \pmod{4}.$$

Conjecture A80 ([S10d]). *Let p be an odd prime.*

(i) *We have*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{W_k(2, -1)}{(-2)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} (4k+3) \frac{W_k(2, -1)}{(-2)^k} \equiv 0 \pmod{p^2}.$$

(ii) *We have*

$$\begin{aligned} & \left(\frac{-1}{p} \right) \sum_{k=0}^{p-1} \frac{W_k(2, 1)}{(-2)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Also,

$$\sum_{k=0}^{p-1} (4k+3) \frac{W_k(2, -1)}{(-2)^k} \equiv p \left(2 \left(\frac{2}{p} \right) + \left(\frac{-1}{p} \right) \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{n-1} (4k+3) W_k(2, -1) (-2)^{n-1-k} \equiv 0 \pmod{n} \quad \text{for any } n = 1, 2, 3, \dots$$

Conjecture A81 ([S10d]). (i) *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{W_k(4, -1)}{(-4)^k} &\equiv \sum_{k=0}^{p-1} \frac{W_k(4, -9)}{4^k} \equiv \sum_{k=0}^{p-1} \frac{W_k(4, 9)}{16^k} \\ &\equiv \begin{cases} \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{8}. \end{cases} \end{aligned}$$

(ii) *For any $n \in \mathbb{Z}^+$ we have*

$$\begin{aligned} \sum_{k=0}^{n-1} (3k+2)W_k(4, -1)(-4)^{n-1-k} &\equiv 0 \pmod{2n}, \\ \sum_{k=0}^{n-1} (3k+2)W_k(4, 9)16^{n-1-k} &\equiv 0 \pmod{2n}, \end{aligned}$$

and

$$\sum_{k=0}^{n-1} (5k+4)W_k(4, -9)4^{n-1-k} \equiv 0 \pmod{2n}.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1} (3k+2) \frac{W_k(4, -1)}{(-4)^k} \equiv \frac{3\left(\frac{3}{p}\right) + \left(\frac{-1}{p}\right)}{2} p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} (3k+2) \frac{W_k(4, 9)}{16^k} \equiv 2p \pmod{p^2}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1} (5k+4) \frac{W_k(4, -9)}{4^k} \equiv \frac{3\left(\frac{3}{p}\right) + 5\left(\frac{-1}{p}\right)}{2} p \pmod{p^2}.$$

Conjecture A82 ([S10d]). (i) *For any prime $p \neq 2, 5$, we have*

$$\begin{aligned} &\sum_{k=0}^{p-1} W_k(1, -4) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \& \ p = x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \& \ 2p = x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases} \end{aligned}$$

For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (20k + 17)W_k(1, -4) \equiv 0 \pmod{n}.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1} (20k + 17)W_k(1, -4) \equiv p \left(10 \left(\frac{-1}{p} \right) + 7 \right) \pmod{p^2}.$$

(ii) For any prime $p > 5$, we have

$$\begin{aligned} & \sum_{k=0}^{p-1} W_k(1, 81) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ \& } p = x^2 + 10y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } p \equiv 7, 13, 23, 37 \pmod{40} \text{ \& } 2p = x^2 + 10y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-10}{p} \right) = -1. \end{cases} \end{aligned}$$

For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (10k + 9)W_k(1, 81) \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1} (10k + 9)W_k(1, 81) \equiv p \left(4 \left(\frac{-2}{p} \right) + 5 \right) \pmod{p^2}.$$

Conjecture A83 ([S10d]). (i) For any prime $p > 3$, we have

$$\begin{aligned} & \sum_{k=0}^{p-1} W_k(1, -324) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{13}{p} \right) = \left(\frac{-1}{p} \right) = 1 \text{ \& } p = x^2 + 13y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{13}{p} \right) = \left(\frac{-1}{p} \right) = -1 \text{ \& } 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-13}{p} \right) = -1. \end{cases} \end{aligned}$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (260k + 237)W_k(1, -324) \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1} (260k + 237)W_k(1, -324) \equiv p \left(130 \left(\frac{-1}{p} \right) + 107 \right) \pmod{p^2}.$$

Conjecture A84 ([S10d]). (i) *For any prime $p \neq 7$, we have*

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k W_k(1, -2^4) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ \& } p = x^2 + 7y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

For all $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (42k + 37)(-1)^k W_k(1, -2^4) \equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1} (42k + 37)(-1)^k W_k(1, -2^4) \equiv p \left(21 \binom{p}{7} + 16 \right) \pmod{p^2}.$$

(ii) *For any prime $p \neq 3, 7$, we have*

$$\begin{aligned} & \sum_{k=0}^{p-1} W_k(1, 7^4) \\ \equiv & \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

For all $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (40k + 37)W_k(1, 7^4) \equiv 0 \pmod{n}.$$

If $p \neq 7$ is a prime, then

$$\sum_{k=0}^{p-1} (40k + 37)W_k(1, 7^4) \equiv p \left(17 \binom{p}{3} + 20 \right) \pmod{p^2}.$$

Conjecture A85 ([S10c]). *Let k and l be positive integers. If $(ln + 1) \mid \binom{kn+ln}{kn}$ for all sufficiently large positive integers n , then each prime factor of k divides l . In other words, if k has a prime factor not dividing l then there are infinitely many positive integers n such that $(ln + 1) \nmid \binom{kn+ln}{kn}$.*

Remark. The author [S10c] noted that if k and l are positive integers then $\binom{kn+ln}{kn} \equiv 0 \pmod{(ln + 1)/(k, ln + 1)}$ for all $n \in \mathbb{Z}^+$.

Conjecture A86 ([S10b]). For $n = 0, 1, 2, \dots$ set

$$s_n = \frac{\binom{6n}{3n} \binom{3n}{n}}{2(2n+1) \binom{2n}{n}}.$$

Then, for any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{s_k}{108^k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -1 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

Also, there are positive integers t_1, t_2, t_3, \dots such that

$$\sum_{k=0}^{\infty} s_k x^{2k+1} + \frac{1}{24} - \sum_{k=1}^{\infty} t_k x^{2k} = \frac{\cos(\frac{2}{3} \arccos(6\sqrt{3}x))}{12}$$

for all real x with $|x| \leq 1/(6\sqrt{3})$. Moreover, $t_p \equiv -2 \pmod{p}$ for any prime p .

Remark. The author [S10b] showed that $s_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \dots$. Using **Mathematica** the author found that

$$\sum_{k=0}^{\infty} s_k x^k = \frac{\sin(\frac{2}{3} \arcsin(6\sqrt{3}x))}{8\sqrt{3}x} \quad \left(0 < x < \frac{1}{108}\right)$$

and in particular

$$\sum_{k=0}^{\infty} \frac{s_k}{108^k} = \frac{3\sqrt{3}}{8}.$$

Recall that the Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined as follows:

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+1} = F_n + F_{n-1} \quad (n = 1, 2, 3, \dots).$$

Conjecture A87 (Sun and Tauraso [ST]). Let $p \neq 2, 5$ be a prime and let $a \in \mathbb{Z}^+$. Then

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \left(1 - 2F_{p^a - (\frac{p^a}{5})}\right) \pmod{p^3}.$$

Remark. The congruence mod p^2 was proved by the author in [S09b].

Conjecture A88 ([S09i]). *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{(-1)^{\lfloor n/5 \rfloor - 1}}{(2n+1)n^2 \binom{2n}{n}} \sum_{k=0}^{n-1} F_{2k+1} \binom{2k}{k} \equiv \begin{cases} 6 \pmod{25} & \text{if } n \equiv 0 \pmod{5}, \\ 4 \pmod{25} & \text{if } n \equiv 1 \pmod{5}, \\ 1 \pmod{25} & \text{if } n \equiv 2, 4 \pmod{5}, \\ 9 \pmod{25} & \text{if } n \equiv 3 \pmod{5}. \end{cases}$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geq b$ then the sum

$$\frac{1}{5^{2a}} \sum_{k=0}^{5^a - 1} F_{2k+1} \binom{2k}{k}$$

modulo 5^b only depends on b .

Remark. In [S09i] the author proved that if $p \neq 2, 5$ is a prime then

$$\sum_{k=0}^{p-1} F_{2k} \binom{2k}{k} \equiv (-1)^{\lfloor p/5 \rfloor} \left(1 - \left(\frac{p}{5}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} F_{2k+1} \binom{2k}{k} \equiv (-1)^{\lfloor p/5 \rfloor} \left(\frac{p}{5}\right) \pmod{p^2}.$$

Recall that the usual q -analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{0 \leq k < n} q^k$$

which tends to n as $q \rightarrow 1$. For any $n, k \in \mathbb{N}$ with $n \geq k$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{0 < r \leq n} [r]_q}{(\prod_{0 < s \leq k} [s]_q)(\prod_{0 < t \leq n-k} [t]_q)}$$

is a natural extension of the usual binomial coefficient $\binom{n}{k}$. A q -analogue of Fibonacci numbers introduced by I. Schur [Sc] is defined as follows:

$$F_0(q) = 0, F_1(q) = 1, \text{ and } F_{n+1}(q) = F_n(q) + q^n F_{n-1}(q) \quad (n = 1, 2, 3, \dots).$$

Conjecture A89 ([S09i]). *Let a and m be positive integers. Then, in the ring $\mathbb{Z}[q]$, we have the following congruence*

$$\sum_{k=0}^{5^a m - 1} q^{-2k(k+1)} \begin{bmatrix} 2k \\ k \end{bmatrix}_q F_{2k+1}(q) \equiv 0 \pmod{[5^a]_q^2}.$$

Conjecture A90 ([S09i]). *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{(-1)^{n-1}}{n^2(n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} u_{k+1}(4, 1) \binom{2k}{k} \equiv \begin{cases} 1 \pmod{9} & \text{if } n \equiv 0, 2 \pmod{9}, \\ 4 \pmod{9} & \text{if } n \equiv 5, 6 \pmod{9}, \\ -2 \pmod{9} & \text{otherwise.} \end{cases}$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geq b - 1$ then the sum

$$\frac{1}{3^{2a}} \sum_{k=0}^{3^a-1} u_{k+1}(4, 1) \binom{2k}{k}$$

modulo 3^b only depends on b .

Remark. In [S09i] the author proved that if $p > 3$ is a prime then

$$\sum_{k=0}^{p-1} u_k(4, 1) \binom{2k}{k} \equiv 2 \left(\left(\frac{p}{3} \right) - \left(\frac{-1}{p} \right) \right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} u_{k+1}(4, 1) \binom{2k}{k} \equiv \left(\frac{p}{3} \right) \pmod{p^2}.$$

Conjecture A91 ([S09a]). *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv -3p q_p^2(2) \pmod{p^2},$$

and

$$p \sum_{k=1}^{p-1} \frac{1}{k 2^k \binom{3k}{k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ -3/5 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

We also have

$$p \sum_{k=0}^{(p-1)/2} \frac{25k-3}{2^k \binom{3k}{k}} \equiv \left(\frac{-1}{p} \right) - \left(\frac{2}{p} \right) \frac{5p}{2} \pmod{p^2}$$

and

$$2p \sum_{k=0}^{p-1} \frac{25k-3}{2^k \binom{3k}{k}} \equiv 3 \left(\frac{-1}{p} \right) + (E_{p-3} - 9)p^2 \pmod{p^4}.$$

Remark. Gosper announced in 1974 that $\sum_{k=0}^{\infty} (25k-3)/(2^k \binom{3k}{k}) = \pi/2$. In [ZPS] Zhao, Pan and Sun proved that $\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv 0 \pmod{p}$ for any odd prime p .

Conjecture A92 ([S09f]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p^a \equiv 1, 2 \pmod{5}$, or $a > 1$ and $p \not\equiv 3 \pmod{5}$,*

$$\sum_{k=0}^{\lfloor \frac{4}{5}p^a \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a} \right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod{5}$, or $a > 1$ and $p \not\equiv 2 \pmod{5}$, then

$$\sum_{k=0}^{\lfloor \frac{3}{5}p^a \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a} \right) \pmod{p^2}.$$

Thus, if $p^a \equiv 1 \pmod{5}$ then

$$\sum_{\frac{3}{5}p^a < k < \frac{4}{5}p^a} (-1)^k \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

Conjecture A93 ([S09f]). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{3}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{5}{8}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a} \right) \pmod{p^2}.$$

Remark. The author [S09f] proved that $\sum_{k=0}^{\lfloor p^a/2 \rfloor} \binom{2k}{k}/16^k \equiv \left(\frac{3}{p^a} \right) \pmod{p^2}$ for any odd prime p and $a \in \mathbb{Z}^+$.

Conjecture A94 ([S09f]). *For any nonnegative integer n we have*

$$\frac{1}{(2n+1)^2 \binom{2n}{n}} \sum_{k=0}^n \frac{\binom{2k}{k}}{16^k} \equiv \begin{cases} 1 \pmod{9} & \text{if } 3 \mid n, \\ 4 \pmod{9} & \text{if } 3 \nmid n. \end{cases}$$

. Also,

$$\frac{1}{3^{2a}} \sum_{k=0}^{(3^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv (-1)^a 10 \pmod{27}$$

for every $a = 1, 2, 3, \dots$

Conjecture A95 ([S09f]). *Let p be an odd prime and let $m \equiv 4 \pmod{p}$.*

Then

$$\nu_p \left(\sum_{k=0}^n \frac{\binom{2k}{k}}{m^k} \right) \geq \nu_p \left((2n+1) \binom{2n}{n} \right) \quad \text{for any } n \in \mathbb{Z}^+.$$

Moreover, if $p > 3$ then

$$\frac{1}{p^a} \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv (-1)^{(p^a-1)/2} \pmod{p}.$$

Conjecture A96 ([S09d]). *Let p be an odd prime and let $h \in \mathbb{Z}$ with $2h - 1 \equiv 0 \pmod{p}$. If $a \in \mathbb{Z}^+$ and $p^a > 3$, then*

$$\sum_{k=0}^{p^a-1} \binom{hp^a-1}{k} \binom{2k}{k} \left(-\frac{h}{2}\right)^k \equiv 0 \pmod{p^{a+1}}.$$

Also, for any $n \in \mathbb{Z}^+$ we have

$$\nu_p \left(\sum_{k=0}^{n-1} \binom{hn-1}{k} \binom{2k}{k} \left(-\frac{h}{2}\right)^k \right) \geq \nu_p(n).$$

Conjecture A97 ([S09d]). *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod{3}$. Then*

$$\nu_3 \left(\frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k} \right) \geq \min\{\nu_3(n), \nu_3(m-1)\} - 1$$

for every $n \in \mathbb{Z}^+$. Furthermore,

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \binom{3^a-1}{k} (-1)^k \frac{\binom{2k}{k}}{m^k} \equiv -\frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for each integer $a > \nu_3(m-1)$. Also,

$$\sum_{k=0}^{3^a-1} \binom{3^a-1}{k} (-1)^k \binom{2k}{k} \equiv -3^{2a-1} \pmod{3^{2a}} \text{ for every } a = 2, 3, \dots$$

Conjecture A98 ([S09c]). *For any prime p and positive integer n we have*

$$\nu_p \left(\sum_{k=0}^{n-1} \binom{(p-1)k}{k, \dots, k} \right) \geq \nu_p(n)$$

and

$$\nu_p \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \binom{(p-1)k}{k, \dots, k} \right) \geq \nu_p(n).$$

Remark. The author [S09c] proved that an integer $p > 1$ is a prime if and only if

$$\sum_{k=0}^{p-1} \binom{(p-1)k}{k, \dots, k} \equiv 0 \pmod{p}.$$

He also showed that if $n \in \mathbb{Z}^+$ is a multiple of a prime p then

$$\sum_{k=0}^{n-1} \binom{(p-1)k}{k, \dots, k} \equiv 0 \pmod{p}.$$

Conjecture A99 (Discovered in 2007). *Let p be a prime and let $l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. If n or r is not divisible by p then we have*

$$\begin{aligned} & \nu_p \left(\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \binom{(k-r)/p}{l} \right) \\ & \geq \left\lfloor \frac{n-lp-1}{p-1} \right\rfloor + \nu_p \left(\binom{\lfloor (n-l-1)/(p-1) \rfloor}{l} \right). \end{aligned}$$

Remark. D. Wan [W] proved that the inequality holds if the last term on the right-hand side is omitted (see also Sun and Wan [SW]).

Conjecture A100. (i) (raised on Nov. 2, 2009 via a message to Number Theory List) *If $n > 1$ is an odd integer satisfying the Morley congruence*

$$\binom{n-1}{(n-1)/2} \equiv (-1)^{(n-1)/2} 4^{n-1} \pmod{n^3},$$

then n must be a prime.

(ii) ([S09b]) *If an odd integer $n > 1$ satisfies the congruence*

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(n-1)/2} \pmod{n^2},$$

then n must be a prime.

Remark. (a) In 1895 Morley [Mo] showed that $\binom{p-1}{(p-1)/2} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \pmod{p^3}$ for any prime $p > 3$. The author has verified part (i) of the conjecture for $n < 10^4$.

(b) The author [S09b] proved that if p is an odd prime then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

And he verified part (ii) of the conjecture for $n < 10^4$ via Mathematica. On the author's request, Qing-Hu Hou at Nankai Univ. finished the verification for $n < 10^5$.

PART B. CONJECTURES THAT HAVE BEEN CONFIRMED

Conjecture B1 (raised in an early version of [S10], and confirmed by Kasper Andersen). *For any positive integer n , the arithmetic mean*

$$s_n := \frac{1}{n} \sum_{k=0}^{n-1} (21k + 8) \binom{2k}{k}^3$$

is always an integer divisible by $4 \binom{2n}{n}$.

Remark. The author created the sequence $\{s_n\}_{n \geq 1}$ at OEIS as A173774 (cf. [S10a]). On Feb. 11, 2010, Andersen proved the conjecture by noting that $t_n := s_n / (4 \binom{2n}{n})$ coincides with

$$r_n := \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2.$$

Conjecture B2 (raised in [S09e], and confirmed by Zhi-Hong Sun [Su]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} ((-2)^{-k} - 4^{-k}) \binom{2k}{k}^2 \equiv 0 \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k}^2}{16^k} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 3 \pmod{4}$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 0 \pmod{p^2}.$$

Remark. In [S09e] the author proved those congruences modulo p except the first one.

Conjecture B3 (raised in [S09e], and confirmed by Roberto Tauraso). *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^2}{16^k} \equiv 12p^2 - 4 \pmod{p^3} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{kC_k^2}{16^k} \equiv 4 - 10p^2 \pmod{p^3}.$$

Also,

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} C_k}{64^k} \equiv p \pmod{p^2},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} C_k}{64^k} \equiv (-1)^{(p-1)/2} \frac{2}{3} p \pmod{p^2} \text{ provided } p > 3.$$

If $p > 3$, then

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} C_k}{27^k} \equiv p \pmod{p^2} \text{ and } \sum_{k=0}^{(p-1)/2} \frac{\binom{3k}{k} C_k}{27^k} \equiv \frac{p}{2} \left(\frac{p}{3}\right) \pmod{p^2}.$$

Remark. The author [S09e] showed that $\sum_{k=0}^{p-1} C_k^2/16^k \equiv -3 \pmod{p}$ for any odd prime p , and his PhD student Yong Zhang proved the first and the second congruences mod p^2 . *Mathematica* yields that

$$\sum_{k=2}^{\infty} \frac{27^k}{(k-1)k^2 \binom{3k}{k,k,k}} = \frac{81}{4} - 3\sqrt{3}\pi.$$

Conjecture B4 (raised in [S09e], and confirmed by the author's PhD student Yong Zhang). *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k+1}^2}{16^k} \equiv (-1)^{(p-1)/2} - 4 + p^2(8 + E_{p-3}) \pmod{p^3}.$$

If $p > 3$, then

$$\sum_{k=0}^{(p-1)/2} \frac{C_k C_{k+1}}{16^k} \equiv 8 \pmod{p^2}.$$

If $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{2k}{k+1}}{8^k} \equiv 0 \pmod{p}.$$

If $p \equiv 3 \pmod{4}$, then

$$\sum_{k=0}^{p-1} \frac{C_k C_{k+1}}{(-16)^k} \equiv -10 \pmod{p}.$$

Remark. As for the first congruence in Conjecture B5, the author [S09e] proved the congruence mod p and then his PhD student Yong Zhang showed the congruence mod p^2 . Following the author's recent method in [S10], Zhang confirmed the congruence with the help of the software Sigma.

Conjecture B5 (raised in [S09c], and confirmed by the author's PhD student Yong Zhang). *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod{3}$. Then*

$$\nu_3 \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{m^k} \right) \geq \min\{\nu_3(n), \nu_3(m-1) - 1\}$$

for every $n \in \mathbb{Z}^+$. Furthermore,

$$\frac{1}{3^a} \sum_{k=0}^{3^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for any integer $a \geq \nu_3(m-1)$.

Conjecture B6 (raised in [S09f], and confirmed by Hao Pan and the author). *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{4}$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{3}{4} p^a \rfloor} \frac{\binom{2k}{k}}{(-4)^k} \equiv \left(\frac{2}{p^a} \right) \pmod{p^2}.$$

Conjecture B7 (raised in [S09h], and confirmed by the author's former student Hui-Qin Cao). *If p is a prime with $p \equiv 11 \pmod{12}$, then*

$$\sum_{k=0}^{p-1} \left(\frac{k}{3} \right) \frac{\binom{2k}{k}^2}{(-16)^k} \equiv 0 \pmod{p}.$$

Conjecture B8 (raised in [S09g], and confirmed by the author's former student Li-Lu Zhao (Hong Kong University)). *Let m be any positive even integer. If p is a prime with $p-1 \nmid 3m$, then*

$$\sum_{k=1}^{p-1} \frac{H_{k,m}^2}{k^m} \equiv 0 \pmod{p},$$

where $H_{k,m} := \sum_{j=1}^k 1/j^m$.

Remark. The author [S09g] proved the congruence in the case $2p/3 < m < p$. For a prime $p > 3$ the author [S09g] established the following basic congruences for harmonic numbers:

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}, \quad \sum_{k=1}^{p-1} H_k^2 \equiv 2p - 2 \pmod{p^2}, \quad \sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p} \quad \text{provided } p > 5,$$

where H_k denotes the harmonic number $\sum_{j=1}^k 1/j$.

Conjecture B9 (raised in [S09d], and confirmed by Hui-Qin Cao and the author [CS]). *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} ((-1)^k - (-3)^{-k}) \equiv \left(\frac{p}{3}\right) (3^{p-1} - 1) \pmod{p^3}.$$

If $p \equiv \pm 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} (-1)^k u_k(4, 1) \equiv (-1)^{(p-1)/2} u_{p-1}(4, 1) \pmod{p^3}.$$

If $p \equiv \pm 1 \pmod{8}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} \frac{u_k(4, 2)}{(-2)^k} \equiv (-1)^{(p-1)/2} u_{p-1}(4, 2) \pmod{p^3}.$$

Remark. Note that

$$u_k(4, 2) = \begin{cases} 2^{k/2} P_k & \text{if } k \text{ is even,} \\ 2^{(k-3)/2} Q_k & \text{if } k \text{ is odd.} \end{cases}$$

The author [S09h] showed that

$$\sum_{k=0}^{(p-1)/2} \frac{u_k(4, 2)}{16^k} \binom{2k}{k} \equiv \frac{(-1)^{\lfloor (p-4)/8 \rfloor}}{2} \left(1 - \left(\frac{2}{p}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{v_k(4, 2)}{16^k} \binom{2k}{k} \equiv 2(-1)^{\lfloor p/8 \rfloor} \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Recall that the n th Bell number b_n denote the number of partitions of a set of cardinality n . Bell numbers are also given by $b_0 = 1$ and $b_{n+1} = \sum_{k=0}^n \binom{n}{k} b_k$ ($n = 1, 2, 3, \dots$).

Conjecture B10 (Discovered on July 17, 2010, and confirmed by the author and D. Zagier [SZ]). *For any positive integer n there is a unique integer $s(n)$ such that for any prime p not dividing n we have*

$$\sum_{k=0}^{p-1} \frac{b_k}{(-n)^k} \equiv s(n) \pmod{p}.$$

In particular,

$$\begin{aligned} s(2) &= 1, & s(3) &= 2, & s(4) &= -1, & s(5) &= 10, & s(6) &= -43, \\ s(7) &= 266, & s(8) &= -1853, & s(9) &= 14834, & s(10) &= -133495. \end{aligned}$$

Remark. The author and D. Zagier [SZ] proved that $s(n) = (-1)^{n-1} D_{n-1} + 1$ for all $n = 1, 2, 3, \dots$, where D_m denotes the derangement number $m! \sum_{k=0}^m (-1)^k / k!$ (the number of fixed-point-free permutations of a set of cardinality m).

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