ATYPICAL DIRICHLET FORMS ON P.C.F. FRACTALS AND THEIR SPECTRAL ASYMPTOTICS

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Abstract. We formulate a class of "homogeneous" Dirichlet forms (DF) that aims to those forms that do not satisfy the conventional energy self-similar identity. We give a thorough study of such DF and their spectral property. We also examine some atypical examples in this class in detail, emphasizing on the explicit constructions; a basic open question in [13] is also resolved.

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1. Introduction

Dirichlet forms play a central role in the analysis on fractals. There is a large literature on the topic based on Kigami's analytic approach on the *post-critically finite (p.c.f.) self-similar sets*, and the probabilistic approach of Lindstrøm on the nested fractals as well as Barlow and Bass on the Sierpinski carpet (see [1, 2, 9, 14, 15, 17, 19, 20, 22, 23, 24] and the references therein).

On a symmetric p.c.f. self-similar set K (nested fractal), the most celebrated and well-studied structure is the *energy self-similar identity*, which is a consequence of a regular harmonic structure that can be obtained by a (non-constructive) fixed point of certain nonlinear trace map of the resistances from V_1 to V_0 (V_0 is the boundary of the p.c.f. set, V_1 is its first iteration by the IFS) [20]. By equipping with a self-similar measure μ, the energy form defines a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, which yields a Laplacian. One of the important questions on $(\mathcal{E}, \mathcal{F})$ concerns the spectral distribution of the eigenvalues. In [21], Kigami and Lapidus proved that on the p.c.f. self-similar sets

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that admit an energy self-similar identity, the eigenvalue counting function $\rho(x)$ of the associated Laplacian satisfies the estimate

$$\rho(x) \asymp x^{d_S/2}, \quad x \geq 0$$

where d_S is the spectral dimension as the unique solution of $\sum_{i=1}^{N}(r_i\mu_i)^{d_S/2} = 1$, with r_i is the resistance factor in the energy self-similar identity, and μ_i is the weight of the self-similar measure. (We use $f \asymp g$ to mean $C^{-1}f \leq g \leq Cf$ for some $C > 0$, on the variables of f and g.)

On the other hand, much less is known for Dirichlet forms that do not satisfy the energy self-similar identity. To the authors' knowledge, the earliest work on such forms can be traced back to Hattori, Hattori and Watanabe [16]. They constructed a sequence of non-self-similar compatible networks of "one-parameter family" on the approximating graphs of the Sierpinski gasket (SG), which converges asymptotically to some one-dimensional diffusion; the heat kernel and the spectral distribution of the Laplacian were further investigated by Hambly and Jones in [12]. Furthermore Hambly and Kumagai [13] extended this consideration to the class of generalized SGs and Vicsek crosses, using some probabilistic estimation.

More recently, the authors gave a characterization of all the compatible networks on the SG (including all the atypical ones)[11], strengthened the construction in [16], and improved the spectral estimate in [12]. Also in [10], some new *asymmetric* p.c.f. fractals were introduced, and the connection of compatible sequence of resistances and the existence of local regular Dirichlet forms were investigated. In particular there are examples using a constructive way to obtain energy self-similar identity, and also DFs that arise from compatible sequences that do not satisfy such identity.

The goal of this paper is two-folds. First, by summing up the studies mentioned above, we formulate two conditions on the compatible sequence of conductances on the $V_n, n \geq 0$ (see Section 3): Condition (A1) is to assume that all the cells of the same level have the same conductance; it gives a *homogeneity* (or translation invariant) of the conductance in each level. Condition (A2) is such that for $p, q \in V_0, p \neq q$, the conductance $c_n(p, q)$ on the n-level has an exponential growth rate, i.e. $c_n(p, q) \asymp \kappa(p, q)^n$ for some constant $\kappa(p, q) > 1$, it allows anisotropic growth rate of the conductances.

In Section 3, we will give a detail study of the compatible sequence of conductances with (A1) and (A2), and to obtain a resistance metric, which define a resistance form and a Dirichlet form (Corollary 3.3). We call this a *homogenous resistance (Dirichlet) form*. For the estimation of the eigenvalue counting function, we need an additional condition (A3) (Section 4), and use a similar technique as in [21] to establish an estimation of the eigenvalue counting function (Theorem 4.6).

Our second goal is to investigate the construction aspect of the *atypical* homogeneous resistance forms. We assume (A1) and look for a compatible condition depend on the geometry of the underlying p.c.f. set. The basic approach is to determine the trace map Φ first. Then start on some initial conductance (or resistance) $\mathbf{x}_0$ on V_0, and use the inverse Φ^{-1} inductively to obtain a compatible sequence $\{\mathbf{x}_n\}_n$, and to verify it satisfies (A2). However there are technical difficulties arise in each step; a general solution is not available yet, and is still a challenging question. We will illustrate this thought through the following two examples.

The first one is on an asymmetric eyebolted Vicsek cross (adding two eyebolts to destroy the symmetric) [10]. For this, we need to introduce a " ⊠-X transform" for network of four vertices to evaluate the resistances (analogous to the well-known Δ-Y transform for triangles) (Theorem 5.1). This example shows that homogeneous DFs need not satisfy the energy self-similar identity. The second example is on the 3-dim SG. By considering certain initial conductances $\mathbf{x}_0$ on the six edges that give a "two-parameter family" of conductances, we can identify the region of $\mathbf{x}_0$ that (A2) is satisfied and give the explicit growth rates $\kappa(p, q)$, and identify those that give Dirichlet forms (Theorem 5.4).

In [13], Hambly and Kumagai investigated two examples on extending the asymptotic one-dimensional diffusion consideration of [16] to d-dimensional ℓ-level SG and Vicsek checkerboards. They made an assumption on the growth rate of the conductances to guarantee their models will yield Dirichlet forms, and showed that in some cases the assumption holds. The assumption remains unsolved for all cases. In here, we use an analytic approach in our setup and settle the problem; moreover, we find the exact growth rate of the conductances.

We organize the paper as follows. In Section 2, we give some preliminaries on the resistance forms and the p.c.f. sets. In Section 3, we study the compatible sequence of resistance forms with conditions (A1), (A2), and prove the estimation of the resistance metric. We then use this and an additional condition (A3) in Section 4 to study the spectral asymptotics of the homogeneous Dirichlet forms. In Section 5, we examine in detail the construction of homogenous resistance forms mentioned in the above. The final section is a brief remark on the examples in [13].

2. Preliminaries

Following from Kigami [20], we let X be a set, and $\ell(X)$ be the space of functions on X. A pair $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on X if it satisfies the following conditions.

(RF1). $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constant functions, and $\mathcal{E}$ is a nonnegative symmetric bilinear form on $\mathcal{F}$. In addition, $\mathcal{E}(u) := \mathcal{E}(u, u) = 0$ if and only if u is constant.

(RF2). Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3). If $x \neq y$, then there is $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

(RF4). For any $p, q \in X$,

$$R(p, q) := \sup\left\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u)} : u \in \mathcal{F}, \mathcal{E}(u) > 0\right\} \tag{2.1}$$

is finite.

(RF5). For $u \in \mathcal{F}$, let $\bar{u} = (u \vee 0) \wedge 1$, then $\bar{u} \in \mathcal{F}$, and $\mathcal{E}(\bar{u}) \leq \mathcal{E}(u)$.

The $R(p, q)$, $p, q \in X$ in (2.1) actually defines a metric on X and is called an *effective resistance metric* on X. There is another useful equivalent expression of this metric:

$$R(p, q)^{-1} = \inf\{\mathcal{E}(u) : u \in \mathcal{F}, u(p) = 1, u(q) = 0\}. \tag{2.2}$$

Let $\{(X_n, E_n)\}_{n=0}^{\infty}$ be a sequence of resistance forms, where the X_n's are finite sets, $X_n \subset X_{n+1}$ for all $n \geq 0$ and the energy is given by

$$E_n(u) = \sum\nolimits_{p,q\in X_n} c_{pq}^{(n)}(u(p) - u(q))^2, \quad u \in \ell(X_n). \tag{2.3}$$

where $c_{pq}^{(n)} \geq 0$. We call these $c_{pq}^{(n)} > 0$ the *conductance* of p and q of on (X_n, E_n). The sequence is called *compatible* if the restriction (or trace) of E_{n+1} on $\ell(X_n)$ equals to (X_n, E_n), $n \geq 0$. The pair (X_n, E_n) in (2.3) corresponds to an electrical network G_n with *resistance* $r_{pq}^{(n)} = (c_{pq}^{(n)})^{-1}$. An equivalent definition of compatibility on the sequence of networks $\{G_n\}_{n=0}^{\infty}$ is that the trace $P(G_n)$ on X_{n-1} equals G_{n-1}, $n \geq 1$ [20].

Let $\{(X_n, E_n)\}_{n=0}^{\infty}$ be a compatible sequence, we have for any $n \geq 1$, $u \in \ell(X_{n-1})$,

$$\min\{E_n(v) : v \in \ell(X_n), v|_{X_{n-1}} = u\} = E_{n-1}(u),$$

we call the v that attains the minimum the harmonic extension of u. For $u_n \in \ell(X_n)$, by applying the harmonic extension inductively, we obtain a sequence $\{u_m\}_{m\geq n}$ of harmonic extensions on X_m with $E_m(u_m) = E_n(u_n)$.

For any function u on $X_* = \cup_n X_n$, we note that $\{E_n(u|_{X_n})\}_n$ is an increasing sequence, so that

$$\mathcal{E}(u) := \lim_{n\to\infty} E_n(u|_{X_n}) \tag{2.4}$$

exists (may equals $+\infty$). Define

$$\mathcal{F} = \{u \in \ell(X_*) : \ \mathcal{E}(u) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{F})$ is a resistance form, which induces an effective resistance metric R on X_*. Denote by X the completion of X_* with respect to R, it follows that $(\mathcal{E}, \mathcal{F})$ extends to a resistance form on X [20, Theorem 2.3.10]. We call this a *compatible resistance form*. Further if μ is a σ-finite Borel measure on X, then $\mathcal{F} \cap L^2(X, \mu)$ is complete under the $\mathcal{E}_1^{1/2}$-norm, so that $(\mathcal{E}, \mathcal{F} \cap L^2(X, \mu))$ is a Dirichlet form on $L^2(X, \mu)$, where

$$\mathcal{E}_1(u) = \mathcal{E}(u) + \int_X u^2 d\mu, \qquad u \in \mathcal{F} \cap L^2(X, \mu).$$

Next, let $\{F_i\}_{i=1}^N$ be an iterated function system (IFS) on $\mathbb{R}^d$ such that

$$F_i(x) = \varrho(x - b_i) + b_i, \quad 1 \leq i \leq N, \tag{2.5}$$

where $0 < \varrho < 1$ and $b_i \in \mathbb{R}^d$. Let $K = \bigcup_{i=1}^N F_i(K)$ be the corresponding self-similar set, and let μ be the self-similar measure on K defined by $\mu = \frac{1}{N}\sum_{i=1}^N \mu \circ F_i^{-1}$. If the IFS satisfies the open set condition (OSC), i.e. there is a nonempty bounded open set O such that $F_i(O) \subset O$ and $F_i(O) \cap F_j(O) = \emptyset$ for $i \neq j$, then the Hausdorff dimension of K is $\dim_H(K) = \alpha = \frac{\log N}{|\log \varrho|}$, and μ is the α-dimensional Hausdorff measure normalized on K, which is α-regular in the sense that

$$\mu(B(x, r)) \asymp r^{\alpha},$$

for $0 < r < \operatorname{diam}(K)$, and $x \in K$ with $B(x, r) := \{y \in K : \ |x - y| < r\}$.

We always assume that K is connected. We define the symbolic space of K as usual. Let $\Sigma = \{1, \cdots, N\}$ be the alphabet, Σ_n be the set of all the words with length n, and Σ_∞ be the set of infinite words $\omega = \omega_1\omega_2\cdots$; let $\pi : \Sigma_\infty \to K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n\geq 1} K_{\omega_1\cdots\omega_n}$, a symbolic representation of $x \in K$ by ω, where $K_{\omega_1\cdots\omega_n} = F_{\omega_1} \circ \cdots \circ F_{\omega_n}(K)$.

Following Kigami [20], we define the *critical set* C and the *post-critical set* $\mathcal{P}$ for K by

$$C = \pi^{-1}\Big(\bigcup\nolimits_{1\le i<j\le N}(K_i \cap K_j)\Big), \quad \mathcal{P} = \bigcup\nolimits_{m\ge 1}\tau^m(C),$$

where $K_i = F_i(K)$, $\tau : \Sigma_\infty \to \Sigma_\infty$ is the left shift by one index. If $\mathcal{P}$ is a finite set, we call $\{F_i\}_{i=1}^N$ a *post-critically finite* (p.c.f.) IFS, and K a p.c.f. (self-similar) set. The *boundary* of K is defined to be $V_0 = \pi(\mathcal{P})$. (We always assume $\#(V_0) \ge 2$ to avoid triviality.) We use $(K, \{F_i\}_{i=1}^N, V_0)$ to denote a p.c.f. triple.

We also define

$$V_n = \bigcup_{i\in\{1,\cdots,N\}} F_i(V_{n-1}), \quad V_* = \bigcup_{n\ge 1} V_n.$$

It is clear that $\{V_n\}_{n=0}^\infty$ is an increasing sequence of sets, and K is the closure of V_*. For any $\omega \in \Sigma_n$, we call $K_\omega := F_\omega(K)$ a *cell* of K, where $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$.

It is known that a p.c.f. IFS in (2.5) satisfies the OSC [6]. (More generally, this is true if the associate similar matrices A_i of F_i (instead of the ϱ in (2.5)) are commensurable, i.e. there exists a matrix A and integers n_i such that $A_i = A^{n_i}$; but it is claimed in [25] that it is not true without this assumption. Hence the p.c.f. self-similar set K in (2.5) has dimension α, and is associated with a self-similar measure μ that is α-regular.

The following was proved in [9]:

Proposition 2.1. *A p.c.f. IFS in* (2.5) *satisfies the following separation property:*
***(H)**: there exists $\delta > 0$ such that for any integer $n \ge 1$ and any two words ω and τ with length n and $K_\omega \cap K_\tau = \emptyset$,*

$$\operatorname{dist}(K_\omega, K_\tau) \ge \delta\varrho^n.$$

We recall that on p.c.f. sets, a regular harmonic structure induces a compatible network sequence which converges to a resistance form satisfying the energy self-similar identity

$$\mathcal{E}(u) = \sum_{i=1}^N \frac{1}{r_i}\mathcal{E}(u \circ F_i), \tag{2.6}$$

with $0 < r_i < 1$ for $1 \le i \le N$. In [21], Kigami and Lapidus gave a detail study of the spectral asymptotic estimate of Laplacians of Dirichlet forms satisfying the energy self-similar identity, and with self-similar measures. Let $\rho(t)$ be the eigenvalue counting function of the Laplacian on K, with Dirichlet or Neumann boundary conditions. They proved [21, Theorem 2.4]

$$0 < \liminf_{x\to\infty} \rho(x)x^{-\frac{d_S}{2}} \le \limsup_{x\to\infty} \rho(x)x^{-\frac{d_S}{2}} < \infty,$$

where d_S is the unique positive number satisfying $\sum_{i=1}^N (r_i\mu_i)^{\frac{d_S}{2}} = 1$, and μ_i is the weight of the self-similar measure. Also they gave a refinement of this limit expression using Feller's renewal formula.

3. Homogeneous Dirichlet forms

Let $(K, \{F_i\}_{i=1}^N, V_0)$ be a p.c.f. triple with IFS defined by (2.5), and consider the compatible resistance forms $\{(V_n, E_n)\}_{n=0}^\infty$ with

$$E_n(u) = \sum_{\omega\in\Sigma_n} \sum_{p,q\in V_0} c(F_\omega(p), F_\omega(q))\Big(u(F_\omega(p)) - u(F_\omega(q))\Big)^2. \tag{3.1}$$

We assume that V_0 is *irreducible* in the sense that for any $p, q \in V_0$, there is a chain $p = p_1, \cdots, p_m = q$ in V_0 such that $c(p_i, p_{i+1}) > 0$ for $i = 1, \cdots, m-1$.

We impose the following two conditions on the compatible sequence $\{(V_n, E_n)\}_n$:

(A1): *For any $n \geq 0$, and any $p, q \in V_0$, $c(F_\omega(p), F_\omega(q))$ is independent of $\omega \in \Sigma_n$, that is*

$$c(F_\omega(p), F_\omega(q)) = c_n(p, q).$$

(A2): *For any distinct $p, q \in V_0$, either $c_n(p, q) = 0$ for all $n \geq 0$, or there exists a constant $\kappa(p, q) > 1$ such that $c_n(p, q) \asymp \kappa(p, q)^n$, i.e. there exists $C > 0$ such that $C^{-1}\kappa(p, q)^n \leq c_n(p, q) \leq C\kappa(p, q)^n$ for all $n \geq 0$.*

We remark that (A1) implies the conductances are *homogeneous* (or translation invariant) in the sense that all the cells on the same level have equal conductance; (A2) implies the $c_n(p, q)$ are asymptotically of the form $\kappa(p, q)^n$ depend on p, q, which allows anisotropic growth rate of the conductances. The condition that $\kappa(p, q) > 1$ ensures that the topology given by the effective resistance in the limit is equivalent to the topology by the Euclidean metric (see Theorem 3.3). Also the assumption that $\kappa(p, q) > 1$ can be defined slightly more general: for any $p, q \in V_0$, there exist a chain $p = p_1, p_2 \cdots, p_{k-1}, p_k = q$ with $\kappa(p_i, p_{i+1}) > 1$, and $c_n(p_i, p_{i+1}) \asymp \kappa(p_i, p_{i+1})^n$, $i = 1, \cdots, k-1$.

The most celebrated examples that satisfy (A1) and (A2) are the Sierpinski gasket and the Vicsek cross with regular harmonic structure, and $c_n(p, q) = \kappa^n$ for some $\kappa > 1$, and for all $p \neq q$ in V_0; in this case $r_1 = \cdots = r_N = \kappa^{-1}$ is the renormalization factor of the self-similar energy identity. Note that the regular harmonic structure in (2.6) with distinct r_i's will not satisfy (A1) as the cells in each level do not have equal conductances. Supplementing to the above examples, recently, two of the authors constructed [10, Proposition 7.1] on an "asymmetric" p.c.f. set (the eyebolted Vicsek cross) a homogeneous resistance form without self-similar energy identity. In this case, conditions (A1) and (A2) are satisfied, with $c_n(p, q) \asymp \kappa(p, q)^n$, but $\kappa(p, q)$ may be distinct among different choices of p, q. The condition that $\kappa(p, q) > 1$ ensures that the effective resistance metric in the limit is equivalent to the Euclidean metric (see Theorem 3.3).

We define $E_n^{(\ell)}(u)$ to be the form in (3.1) with conductances $c_{\ell+n}(p, q)$, a shift of ℓ-level of the conductances. Let $\mathcal{E}^{(0)}(u) = \mathcal{E}(u)$ as in (2.4), and for $\ell > 0$, define $\mathcal{E}^{(\ell)}(u)$ to be the limit of $E_n^{(\ell)}(u)$ as $n \to \infty$. The following is an analogue of the energy self-similar identity.

Proposition 3.1. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a p.c.f. set K defined by* (3.1) *and satisfies (A1). Then we have*

$$\mathcal{E}^{(0)}(u) = \sum\nolimits_{i=1}^N \mathcal{E}^{(1)}(u \circ F_i), \qquad u \in \mathcal{F} \tag{3.2}$$

and inductively $\mathcal{E}(u) = \sum_{\omega\in\Sigma_\ell} \mathcal{E}^{(\ell)}(u \circ F_\omega)$.

Proof. It follows directly from definition that

$$\sum_{i=1}^{N} E_n^{(1)}(u\circ F_i)=\sum_{i=1}^{N}\sum_{\omega\in\Sigma_n}\sum_{p,q\in V_0} c(F_{i\omega}(p),F_{i\omega}(q))\Big(u(F_{i\omega}(p))-u(F_{i\omega}(q))\Big)^2=E_{n+1}(u),$$

and (3.2) follows by taking limit on n. □

We define

$$\kappa_0=\max\left\{s:\ \begin{array}{c}\forall p\neq q \text{ in } V_0,\ \exists \text{ a chain } p=p_1,p_2,\cdots,p_m=q \text{ in } V_0\\ \ni \kappa(p_i,p_{i+1})\geq s,\ \forall\ 1\leq i\leq m-1\end{array}\right\}\ (>1)\ . \tag{3.3}$$

It is seen that $\kappa_0\geq\kappa_{\min}:=\min\{\kappa(p,q):p,q\in V_0,p\neq q\}$.

As in Section 2, we let $R(\cdot,\cdot)$ be the corresponding resistance metric on V_*, then R can be extended to the completion $\bar{V}_*$ with respect to R. In the following we see that $\bar{V}_*=K$. Let $\kappa_{\max}:=\max\{\kappa(p,q):p,q\in V_0\}$.

Theorem 3.2. *Suppose $(K,\{F_i\}_{i=1}^N,V_0)$ is a p.c.f. triple. Let $(\mathcal{E},\mathcal{F})$ be a compatible resistance form on V_* that satisfies (A1) and (A2), and let $R(\cdot,\cdot)$ be the resistance metric on V_*, then R satisfies the estimate*

$$C^{-1}|x-y|^{\frac{\log\kappa_{\max}}{|\log\varrho|}}\leq R(x,y)\leq C|x-y|^{\frac{\log\kappa_0}{|\log\varrho|}},\quad \forall x,y\in V_*. \tag{3.4}$$

Consequently, the R-closure of V_* is K, and $\mathcal{F}\subset C(K)$, the space of continuous functions on K with respect to R or the Euclidean distance.

Proof. For distinct points x,y in V_*, let ℓ be the integer such that $\delta\varrho^{\ell+1}\leq|x-y|<\delta\varrho^{\ell}$, where $\delta>0$ is a constant in property (H) in Proposition 2.1. We may assume that $\ell\geq 0$. Let ω and τ be two words with length ℓ such that $x\in K_\omega$ and $y\in K_\tau$, then by (H), we have

$$K_\omega\cap K_\tau\neq\emptyset.$$

To estimate the upper bound of (3.4), we pick $z\in F_\omega(V_0)\cap F_\tau(V_0)\subseteq K_\omega\cap K_\tau$ (it is possible that $\omega=\tau$), and estimate $R(x,z)$. We can find a decreasing sequence of cells $\{K_{\omega_k}\}_{k\geq 0}$ such that $\omega_0=\omega$, $\omega_k\in\Sigma_{\ell+k}$, $k\geq 0$, and $x\in\bigcap_{k\geq 0}K_{\omega_k}$. Choose an infinite chain $z=z_0,z_1,\cdots$ such that $\lim_{k\to\infty}z_k=x$ and z_k and z_{k+1} are contained in $F_{\omega_k}(V_0)$, $k\geq 0$.

Now for $u\in\mathcal{F}$, $\mathcal{E}(u)\neq 0$, consider $|u(z_k)-u(z_{k+1})|$. Note that $F_{\omega_k}^{-1}(z_k),F_{\omega_k}^{-1}(z_{k+1})$ are in V_0, then we can find a chain $F_{\omega_k}^{-1}(z_k)=x_1,\cdots,x_m=F_{\omega_k}^{-1}(z_{k+1})$ in V_0 such that $\kappa(x_j,x_{j+1})\geq\kappa_0$. Hence

$$\begin{aligned}|u(z_k)-u(z_{k+1})|&\leq\sum_{j=1}^{m-1}|u(F_{\omega_k}(x_j))-u(F_{\omega_k}(x_{j+1}))|\\ &\leq C_1\max_{\kappa(p,q)\geq\kappa_0}\left\{c_0(p,q)^{-1/2}\right\}\cdot\sum_{j=1}^{m-1}\left(\kappa_0^{-(\ell+k)}\mathcal{E}(u)\right)^{1/2}\\ &\leq mC_1\max_{\kappa(p,q)\geq\kappa_0}\left\{c_0(p,q)^{-1/2}\right\}\cdot\left(\kappa_0^{-(\ell+k)}\mathcal{E}(u)\right)^{1/2}\end{aligned}$$

(Note that m is uniformly bounded, and C_1 is a constant independent of z_k, z_{k+1} and the chain connecting them). Thus

$$|u(x)-u(z)| \le \sum_{k\ge 0}|u(z_k)-u(z_{k+1})| \le C_2 \max_{\kappa(p,q)\ge\kappa_0}\left\{c_0(p,q)^{-1/2}\right\}\cdot\sum_{k\ge 0}\left(\kappa_0^{-(\ell+k)}\mathcal{E}(u)\right)^{1/2}$$
$$\le C_3 \max_{\kappa(p,q)\ge\kappa_0}\left\{c_0(p,q)^{-1/2}\right\}\cdot\kappa_0^{-\ell/2}\mathcal{E}(u)^{1/2}. \tag{3.5}$$

It follows from (2.1) that

$$R(x,z) \le C_4 \max_{\kappa(p,q)\ge\kappa_0}\left\{c_0(p,q)^{-1}\right\}\cdot\kappa_0^{-\ell}. \tag{3.6}$$

Similarly, we have the same bound for $R(y,z)$. Hence

$$R(x,y) \le R(x,z)+R(y,z) \le C_5\kappa_0^{-\ell} \le C|x-y|^{-\frac{\log\kappa_0}{\log\varrho}}.$$

For the lower bound of (3.4), we can find a positive integer m_0, and an $(\ell+m_0)$-cell $K_{\widetilde{\omega}}$ containing x, such that $K_{\widetilde{\omega}}$ does not intersect any $(\ell+m_0)$-cell containing y. Then let h be the $(\ell+m_0)$-piecewise harmonic function with values 1 on $K_{\widetilde{\omega}}$, and 0 on any other points in $V_{\ell+m_0}$. Hence $h(y)=0$ and we have

$$\mathcal{E}(h) \le C_6 \max_{p,q\in V_0} c_{\ell+m_0}(p,q) \le C_7\kappa_{\max}^{\ell},$$

where C_6 and C_7 are constants independent of ℓ and x, y. Hence by (2.2), we have

$$R(x,y) \ge \mathcal{E}(h)^{-1} \ge C_7^{-1}\kappa_{\max}^{-\ell} \ge C^{-1}|x-y|^{-\frac{\log\kappa_{\max}}{\log\varrho}}.$$

That the closure of V_* equals K follows from the expression in (3.4). For $u\in\mathcal{F}$, $|u(x)-u(y)|^2 \le R(x,y)\mathcal{E}(u)$, $x,y\in K$. This implies u is continuous with respect to R, and also to the Euclidean distance by (3.4). □

Corollary 3.3. *With the assumptions in Theorem 3.2, let μ be a σ-finite Borel measure on (K,R) will full support, then $(\mathcal{E},\mathcal{F})$ is a local regular Dirichlet form on $L^2(K,\mu)$.*

Proof. The regularity of $(\mathcal{E},\mathcal{F})$ can be seen by using piecewise harmonic functions to approximate any $u\in C(K)$, hence $\mathcal{F}$ is dense in $C(K)(\subset L^2(K,\mu))$. For the locality, suppose $u,v\in\mathcal{F}$ have disjoint supports. Then $u|_{V_n}, v|_{V_n}$ have disjoint support for n large enough. An application of (3.1) shows that $E_m(u,v)=0$ for $m\ge n$. By taking limit on m, $\mathcal{E}(u,v)=0$. □

Definition 3.4. *On a p.c.f. self-similar set K, we call $(\mathcal{E},\mathcal{F})$ a homogenous resistance form if it is defined by a compatible resistance form satisfies (A1) and (A2) as in Theorem 3.2. Also we will call the local regular Dirichlet form associated with a measure μ a homogenous Dirichlet form, and use Δ_μ to denote the induced Laplacian.*

We need the following estimation in the next section.

Lemma 3.5. *Let $(\mathcal{E}, \mathcal{F})$ be a homogenous resistance form on a p.c.f. set K. and let $R^{(n)}$ be the resistance metric of $(\mathcal{E}^{(n)}, \mathcal{F})$, then $R^{(n)}$ is bounded on K and satisfies*

$$R^{(n)}(x, y) \le C\kappa_0^{-n}, \qquad x, y \in K \tag{3.7}$$

where C is independent of $n \ge 0$.

Proof. We first consider $n = 0$. For $x \in K$, let $B(x, \delta)$ be an open ball of radius δ (in the Euclidean distance), then referring to the proof Theorem 3.2 for $\ell = 0$, and by (3.6), we have

$$R(x, y) \le C_4 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p, q)^{-1}\right\}, \qquad x, y \in B(x, \delta) \cap K. \tag{3.8}$$

As K is compact, we can cover it by finitely many open balls $B(x, \delta)$'s. This implies

$$R(x, y) \le C, \qquad x, y \in K \tag{3.9}$$

for some $C > 0$.

For $\mathcal{E}^{(n)}, n > 0$, the conductances on V_0 are $c_n(p, q)$ and on V_k are $c_{n+k}(p, q)$. Similar to (3.8), we obtain

$$R^{(n)}(x, y) \le C_4 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_n(p, q)^{-1}\right\} \le C\kappa_0^{-n}, \qquad x, y \in K$$

for the same $C > 0$. □

4. Spectral asymptotics

In this section we will consider the homogeneous Dirichlet form associated with a measure μ and a Laplacian $\Delta = \Delta_\mu$. We first introduce one more assumption.

Let κ_0 be as in (3.3). For $p, q \in V_0$, we define $p \sim_0 q$ if

$$\text{either } \ p = q \ \text{ or } \ \exists \text{ a chain } p = p_1, \cdots, p_m = q \ \ni \ \kappa(p_i, p_{i+1}) > \kappa_0.$$

This is an equivalence relation on V_0. We also consider the induced equivalence relation $\sim_1$ on V_1: the minimal equivalence relation on V_1 such that for any $k \in \Sigma$ and any $p \sim_0 q$, $F_k(p) \sim_1 F_k(q)$ (i.e. for $p, q \in V_1$, either $p = q$; or there exists a chain $p = p_1, \cdots, p_m = q$ in V_1 joining p, q, and for each i, there exists $k \in \Sigma$ such that $F_k^{-1}(p_i) \sim_0 F_k^{-1}(p_{i+1})$). The same way, we can define $\sim_n$ on V_n.

Let $C_i^{(0)}$ denote a nontrivial (non-singleton) connected component induced by $\sim_0$ (i.e. a non-trivial equivalent class in V_0), and let $\mathfrak{C}^{(0)}$ be the family of $C_i^{(0)}$'s. Similarly we let $C_j^{(n)}$, $\mathfrak{C}^{(n)}$ be the connect components and associated family of connected components of $\sim_n$ in V_n. The following are some simple observations.

(i) If $\mathfrak{C}^{(0)} = \emptyset$, then all the connected components are singletons and $\kappa_0 = \kappa_{\max}$; if $\mathfrak{C}^{(0)} \ne \emptyset$, then $C_i^{(0)} \subset V_0 \setminus \{p\}$ for some $p \in V_0$, following from the definition of κ_0;

(ii) For each map F_k in the IFS, $F_k(C_i^{(0)}) \subset C_j^{(1)}$ for some j. In general, equality does not hold, as $C_j^{(1)}$ can be a union of several $F_{k'}(C_{i'}^{(0)})$'s through the junction vertices of these cells.

We define an assumption on the homogeneous resistance form:

(A3) *there exists n_0 and an n_0-cell K_ω such that for any $p \in F_\omega(V_0)$, $q \in V_0$, $p \nsim_{n_0} q$.*

It is clear that if $\mathfrak{C}^{(0)} = \emptyset$, then (A3) is trivially satisfied for n_0 that has an n_0-cell not intersecting V_0 (by (i)). Also

Lemma 4.1. *If (A3) is satisfied at n_0, then (A3) is also satisfied for $n > n_0$.*

Proof. Suppose (A3) holds at n_0, let K_ω be the n_0-cell satisfying the condition in (A3), then pick arbitrarily an $i \in \Sigma$, the $(n_0 + 1)$-cell $K_{i\omega}$ will satisfy (A3) as well.

By repeating this argument to $n_0 + 1$, we have (A3) holds for $n_0 + 2$. Inductively, we have (A3) holds for any $n > n_0$. □

The following are some simple examples.

Example 1. (Sierpinski triangle). Let $\{p_1, p_2, p_3\}$ be the three vertices of a Sierpinski triangle, and let the nontrivial component be $\{p_1, p_2\}$ in V_0. Then the components in V_n are line segments in V_n parallel to $\overline{p_1 p_2}$. Clearly $(A3)$ is satisfied with $n_0 = 2$.

Example 2. (Pentagasket) Let $\{p_i\}_{i=1}^5$ be the vertices of the pentagasket. If we let $\{p_1, p_2\}$, $\{p_3, p_4, p_5\}$ be two components in V_0. Then the connected components in V_1, V_2 are as in the left two pictures Figure 1, and $n_0 = 2$.

If we take $\{p_1, p_2, p_3, p_4\}$ as the component in V_0, then the connected components in V_1, V_2 are as in the right two pictures in Figure 1, and $n_0 = 2$.

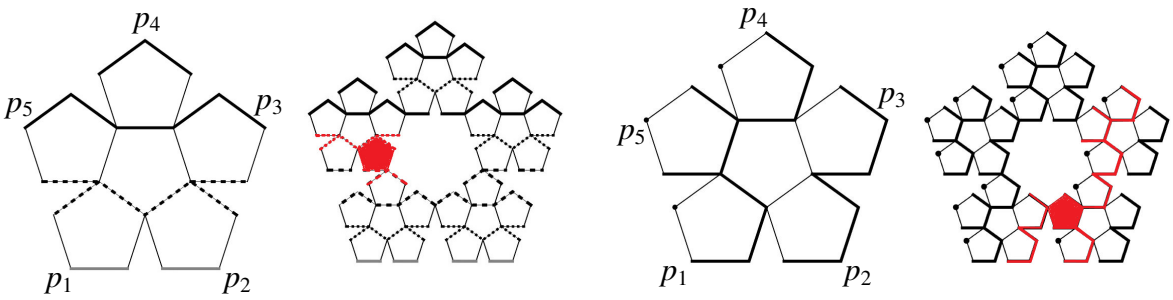


FIGURE 1. The connected components of V_1 and V_2 in boldface; the cell in red satisfies (A3).

Example 3. (Snowflake) Let $V_0 = \{p_i\}_{i=1}^6$ be the vertices of a hexagon. Let $\{F_i\}_{i=1}^7$ be as in Figure 2. Consider the component $\{p_i\}_{i=1}^5$ in V_0. Then there is only one non-trivial component in V_1, which meets all vertices in V_0 except one. By similarity, it is seen that it is the same for V_n. Hence (A3) is not satisfied.

In view of Theorem 3.2, it is known that under $\mathcal{E}_1^{1/2}$ norm, $\mathcal{F}$ can be compactly embedded into $L^2(K, \mu)$, and $-\Delta$ has a discrete spectrum which only has a limit point ∞ [20,

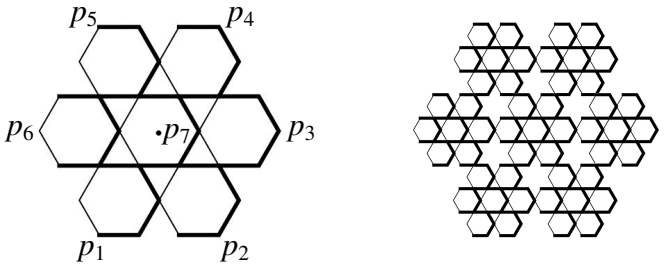


FIGURE 2. The connected components of V_1 and V_2 in boldface; the assumption (A3) fails.

Theorem 2.4.1]. We will consider the first eigenvalue λ_1 of $-\Delta$ with the Dirichlet boundary condition (i.e. on $\mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$). Note that λ_1 can be expressed by the cerebrated Rayleigh quotient:

$$\lambda_1 = \inf_{u\in\mathcal{F}_0, u\neq 0} \frac{\mathcal{E}(u)}{\|u\|_2^2}. \tag{4.1}$$

Let $\mathcal{E}^{(0)} = \mathcal{E}$, and let $\mathcal{E}^{(n)}$ be defined as before Proposition 3.1. Let $\lambda_1^{(n)}, n \geq 0$ be the first eigenvalue of $(\mathcal{E}^{(n)}, \mathcal{F})$.

Lemma 4.2. *Let K be a p.c.f. set satisfying (2.5) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}, \mathcal{F})$ be a homogeneous Dirichlet form in $L^2(K, \mu)$ satisfying (A3). Then there is $C > 0$ such that for $(\mathcal{E}^{(n)}, \mathcal{F})$, $n \geq 0$, we have*

$$C^{-1}\kappa_0^n \leq \lambda_1^{(n)} \leq C\kappa_0^n. \tag{4.2}$$

Proof. Let $n \geq 0$. By (A3), we can find an integer $n_0 \geq 1$ and an n_0-cell K_ω, such that each component containing a point in $F_\omega(V_0)$ is disjoint from V_0. Denote by Q the union of all such components, then $F_\omega(V_0) \subseteq Q$ and $Q \cap V_0 = \emptyset$. Let $u_0 \in \ell(V_{n_0})$ by taking 1 on Q, 0 on $V_{n_0} \setminus Q$, and extent harmonically to K. Clearly,

$$\|u_0\|_2^2 \geq \int_{F_\omega(K)} u_0^2 \, d\mu \;=\; \mu(K_\omega) = N^{-n_0}.$$

Define the boundary ∂Q of Q in V_{n_0} to be those $x \in Q$ that has a neighbor $y \in V_{n_0} \setminus Q$ such that $c_{n+n_0}(x, y) > 0$ (recall that the energy $\mathcal{E}^{(n)}$ is defined using the conductance $c_{n+m}(p, q)$ and letting $m \to \infty$), let Q_x denote those y. Note that x and y are in different components, hence $c_{n+n_0}(x, y) \leq C\kappa_0^{n+n_0}$, where C is independent of n; also note that $\#Q_x$ is uniformly bounded by some $C_0 > 0$, and $\#\partial Q$ is bounded by $(\#(V_0)N)^{n_0}$. Hence by (3.1),

$$\mathcal{E}^{(n)}(u_0) = \sum_{x\in\partial Q, y\in Q_x} c_{n+n_0}(x, y)(u_0(x) - u_0(y))^2 \leq C_0(\#(V_0)N)^{n_0} \cdot C\kappa_0^{n+n_0} =: C_1\kappa_0^n.$$

Therefore

$$\lambda_1^{(n)} \leq \frac{\mathcal{E}^{(n)}(u_0)}{\|u_0\|_2^2} \leq C_1\kappa_0^n \cdot N^{n_0} \leq C_2\kappa_0^n$$

for some $C_2 > 0$.

To estimate the lower bound, we let $u \in \mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$, $u \neq 0$, then

$$|u(x) - u(y)|^2 \leq R^{(n)}(x, y)\mathcal{E}^{(n)}(u), \quad x, y \in K.$$

By choosing y to be some $p \in V_0$, we have

$$|u(x)|^2 \le R^{(n)}(x,p)\mathcal{E}^{(n)}(u), \quad \forall x \in K.$$

Integrating both sides with respect to μ, we obtain

$$\|u\|_2^2 \le \int_K R^{(n)}(x,p)d\mu(x) \cdot \mathcal{E}^{(n)}(u).$$

Using Lemma 3.5, we have $C_3 > 0$ such that

$$C_3\kappa_0^n \le \frac{\mathcal{E}^{(n)}(u)}{\|u\|_2^2}.$$

Since u is arbitrary, this implies that $C_3\kappa_0^n \le \lambda_1^{(n)}$. Hence $C^{-1}\kappa_0^n \le \lambda_1^{(n)} \le C\kappa_0^n$ for some $C > 0$. □

Denote by $\Lambda_{\mathcal{D}}(\lambda_1)$ the Dirichlet eigensubspace of λ_1. In the following we show that $\dim \Lambda_{\mathcal{D}}(\lambda_1) = 1$ in a general setting. This result is implicitly implied in the monographs of Davies [4, Theorems 7.2, 7.3] and [5, Propostion 1.4.3] in the study of heat kernels and heat semigroups, $\{P_t\}_{t\ge 0}$, i.e. $\{e^{-t\Delta}\}_{t\ge 0}$. For completeness, we will give a modified proof here following his main idea.

We say that a linear subspace $\mathcal{L}$ of L^2 is a *sublattice* if $u \in \mathcal{L}$ implies $|u| \in \mathcal{L}$.

Proposition 4.3. *Let K be a compact connected set and ν a σ-finite Borel measure on K with full support. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(K,\nu)$ with $\mathcal{F} \subset C(K)$. Suppose $\mathcal{L} \subset \mathcal{F}$ is a closed linear sublattice of $L^2(K,\nu)$, and there exists $C > 0$ such that*

$$P_t u \le Cu, \qquad \forall\, t > 0,\ \ u \ge 0,\ \ u \in \mathcal{L}. \tag{4.3}$$

Then $\mathcal{L}$ has dimension at most one.

Proof. Suppose $\mathcal{L}$ is nontrivial, let $u \ge 0$ be any non-zero element in $\mathcal{L}$, then $u \in C(K)$. Let $U = \{x \in K :\ u(x) \ne 0\}$, U is open. We claim that $U = K$. Indeed, if $v \in C(K)$ and $|v| \le \alpha u$ for some $\alpha \ge 0$, then by the Markovian property of $\{P_t\}_{t>0}$ and (4.3), we have

$$|P_t v| \le P_t|v| \le \alpha P_t u \le \alpha C u.$$

Hence for

$$\mathcal{G} = \{v \in C(K) : |v| \le \alpha u \ \text{ for some } \alpha \ge 0\},$$

$P_t(\mathcal{G}) \subseteq \mathcal{G}$ for all $t \ge 0$. As U is an open set, $\mathcal{G}$ contains all the continuous functions that are compactly supported in U. Then the L^2-closure of $\mathcal{G}$ is the set of all $v \in L^2(K,\nu)$ with $v = 0$ on $K \setminus U$. So U is an *invariant* set of the semigroup $\{P_t\}_{t>0}$. (A ν-measurable set $B \subset K$ is said to be P_t-invariant if $P_t(1_B f) = 1_B P_t f$ ν-a.e. for any $f \in L^2$ and $t > 0$.) This implies $1_U = K$ ν-a.e. [4, Theorems 7.2]. On the other hand, by [8, Theorem 1.6.1], $1_U \in \mathcal{F}$, which is contained in $C(K)$ by assumption. Since K is connected, we have $U = K$ or $U = \emptyset$. Since u is nonzero, we conclude that $U = K$, and the claim follows.

Now, if $u \in \mathcal{L}$, then u^+ and u^- are in $\mathcal{L}$ and have disjoint supports. It follows from the claim that one of them must vanish. Hence $u \in \mathcal{L}$ implies $u \ge 0$ or $-u \ge 0$. If u, u' are two distinct positive elements of $\mathcal{L}$, then $u + \eta u'$ is either positive or negative for all $\eta \in \mathbb{R}$. But the sum must change sign as η increases through $\mathbb{R}$. Hence there is η such that $u + \eta u' = 0$. Hence $\mathcal{L}$ is one dimensional. □

Proposition 4.4. *With the same assumption as Proposition 4.3,* $\dim \Lambda_{\mathcal{D}}(\lambda_1) = 1$.

Proof. We make use of the Rayleigh quotient again. Let $u \in \mathcal{F}$ that attains the infimum in (4.1), and all such functions must be eigenfunctions of $\mathcal{D}$-eigenvalue λ_1.

We show that $\Lambda_1 := \Lambda_{\mathcal{D}}(\lambda_1)$ is a closed sublattice. Indeed for any nonzero $u \in \Lambda_1 \subseteq \mathcal{F}_0$, we see by the Markovian property of the Dirichlet form that $|u| \in \mathcal{F}_0$, and $\mathcal{E}(|u|) = \mathcal{E}(u)$ (use the equivalence of Markovian property and the normal contraction condition in [8, Theorem 1.4.1]). Hence $|u|$ attains the infimum in (4.1), and is also an eigenfunction of λ_1. This shows that Λ_1 is a linear sublattice.

For the closedness of Λ_1, we claim that

$$\Lambda_1 = \{u \in L^2 : \ P_t u = e^{-t\lambda_1} u \text{ for all } t \geq 0\}\,, \tag{4.4}$$

then Λ_1 will be closed. To see the claim, on the one hand, for any $u \in \Lambda_1$, we have $u \in \operatorname{dom} \Delta$ and $\Delta u = -\lambda_1 u$. Hence

$$P_t u = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta^n u = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-\lambda_1)^n u = e^{-t\lambda_1} u\,. \tag{4.5}$$

On the other hand, for $u \in L^2$ satisfying the right hand side of (4.4), then the following limit exists

$$\lim_{t \to 0+} \frac{P_t u - u}{t} = -\lambda_1 u\,,$$

which implies that $u \in \operatorname{dom} \Delta$ and $\Delta u = -\lambda_1 u$ and thus $u \in \Lambda_1$. The claim is proved.

By (4.5), we see that $P_t u \leq Cu$. Then Proposition 4.3 implies Λ_1 is of dimension one. □

For the discrete eigenvalues of $-\Delta$, we use $\rho_b(t)$ to denote the b-eigenvalue counting function of $-\Delta$, where b stands for $\mathcal{D}$ or $\mathcal{N}$, the Dirichlet or Neumann boundary condition:

$$\rho_b(t) = \sum_{\lambda \leq t} \dim \Lambda_b(\lambda),$$

where $\dim \Lambda_b(\lambda)$ is the dimension of the λ-eigensubspace. We use $\rho_b^{(n)}(t)$ to denote the b-eigenvalue counting function corresponding to $(\mathcal{E}^n, \mathcal{F})$.

Lemma 4.5. *Let K be a p.c.f. self-similar set satisfying (2.5), and let μ be the normalized Hausdorff measure on K. Suppose $(\mathcal{E}, \mathcal{F})$ is a homogeneous Dirichlet form in $L^2(K, \mu)$. Then for each $(\mathcal{E}^{(n)}, \mathcal{F}), n \geq 0$, we have*

$$N^k \rho_{\mathcal{D}}^{(n)}\left(\frac{t}{N^k}\right) \leq \rho_{\mathcal{D}}^{(0)}(t), \quad \text{and} \quad \rho_{\mathcal{N}}^{(0)}(t) \leq N^k \rho_{\mathcal{N}}^{(n)}\left(\frac{t}{N^k}\right) \tag{4.6}$$

for all $t \geq 0$ and $k \geq 0$.

Proof. The main idea of proof is based on [21, Propsitions 6.2, 6.3]. Let $\mathcal{F}_1 := \{u \in \mathcal{F} : u|_{V_1} = 0\}$. Denote by $\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right)$ the corresponding eigenvalue counting function. We first show that

$$\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right) = N \rho_{\mathcal{D}}^{(1)}\left(\frac{t}{N}\right). \tag{4.7}$$

Let λ be a $\mathcal{D}$-eigenvalue of $\mathcal{E}^{(0)}$ with eigenvector $\varphi \in \mathcal{F}_1$. Let $\varphi_i = \varphi \circ F_i$ on K, then for $g \in \mathcal{F}_1$, write $g_i = g \circ F_i \in \mathcal{F}_0 (= \{u \in \mathcal{F} : u|_{V_0} = 0\})$. By Proposition 3.1,

$$\sum_{i=1}^{N} \mathcal{E}^{(1)}(\varphi_i, g_i) = \mathcal{E}^{(0)}(\varphi, g) = \lambda(\varphi, g)_{L^2} = \lambda \sum_{i=1}^{N} \int_{K_i} \varphi g \, d\mu$$

$$= \lambda \sum_{i=1}^{N} \int_{K_i} (\varphi_i \circ F_i^{-1})(g_i \circ F_i^{-1}) \, d\mu = \lambda \sum_{i=1}^{N} \int_{K} \varphi_i g_i \, d\mu \circ F_i = \frac{\lambda}{N} \sum_{i=1}^{N} \int_{K} \varphi_i g_i \, d\mu.$$

This implies $\mathcal{E}^{(1)}(\varphi_i, g) = \frac{\lambda}{N}(\varphi_i, g)$ for all $i \in \Sigma$ and $g \in \mathcal{F}_0$. It follows that φ_i is a $\frac{\lambda}{N}$-eigenvalue of $\mathcal{E}^{(1)}$. Also $\frac{\lambda}{N}$ has multiplicity N with respect to $(\mathcal{E}^{(1)}, \mathcal{F}_0)$.

Similarly, let ψ be a λ-eigenvector of $(\mathcal{E}^{(1)}, \mathcal{F}_0)$, we can show that $\psi_i := (\psi \circ F_i^{-1}) \cdot \chi_{K_i}$ is an $N\lambda$-eigenvector of $(\mathcal{E}^{(0)}, \mathcal{F}_1)$, and $N\lambda$ has multiplicity N in $(\mathcal{E}^{(0)}, \mathcal{F}_1)$.

Combining the above proofs, we obtain (4.7). Using Proposition 3.1 on $\mathcal{E}^{(n)}$ and the same proof, we can show that

$$\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_1\right) = N\rho_{\mathcal{D}}^{(n+1)}\left(\frac{t}{N}\right). \tag{4.8}$$

Note that for all $n \geq 0$, we have $\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_1\right) \leq \rho_{\mathcal{D}}^{(n)}(t)$, together with (4.7), the first inequality in (4.6) follows.

For the Neumann eigenvalues, we consider the domain

$$\mathcal{F}_2 = \{u \in \ell(K \setminus V_1) : u \circ F_i := u_i \text{ on } K \setminus V_0 \text{ for some } \ u_i \in \mathcal{F}\} \ (\supset \mathcal{F}),$$

(see [21, Propositions 5.1, 6.2]). By applying a similar proof as the above, we can show that

$$\rho_{\mathcal{N}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_2\right) = N\rho_{\mathcal{N}}^{(n+1)}\left(\frac{t}{N}\right),$$

and observe that $\rho_{\mathcal{N}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_2\right) \geq \rho_{\mathcal{N}}^{(n)}(t)$, the second inequality in (4.6) follows. □

Theorem 4.6. *Let K be a p.c.f. self-similar set satisfying (2.5) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}, \mathcal{F})$ be a homogeneous Dirichlet form on $L^2(K, \mu)$ satisfying (A3). Then for $t_0 = \inf\{t : \rho(t) > 0\}$,*

$$\rho_{\mathcal{D}}(t) \asymp t^{\frac{\log N}{\log(N\kappa_0)}}, \qquad t > t_0.$$

Similarly, the same inequality holds when $\rho_{\mathcal{D}}(t)$ is replaced by $\rho_{\mathcal{N}}(t)$ and for any $t_0 > 0$.

Proof. We denote by $\lambda_1^{(n)}$ the first $\mathcal{D}$-eigenvalue of the Laplacian associated with $\mathcal{E}^{(n)}$. Then by Proposition 4.4, we have $\rho_{\mathcal{D}}^{(n)}\left(\lambda_1^{(n)}\right) = 1$, and $\rho_{\mathcal{N}}^{(n)}\left(\lambda_1^{(n)}\right) \leq \rho_{\mathcal{D}}^{(n)}\left(\lambda_1^{(n)}\right) + N = N + 1$ [21, Lemma 2.3]. Then by Lemma 4.5, we have

$$\rho_{\mathcal{N}}^{(0)}\left(N^n\lambda_1^{(n)}\right) \leq C \cdot N^n.$$

Letting $t = N^n\lambda_1^{(n)}$, by Lemma 4.2, we have $t \asymp N^n\kappa_0^n$, hence $N^n \asymp t^{\log N/\log(N\kappa_0)}$. It follows that

$$\rho_{\mathcal{D}}(t) = \rho_{\mathcal{D}}^{(0)}(t) \leq \rho_{\mathcal{N}}^{(0)}(t) \leq Ct^{\frac{\log N}{\log(N\kappa_0)}}$$

for some $C > 0$. The same argument yields the other inequality. □

5. Construction of homogeneous forms

In this section, we consider the construction of the homogeneous resistance forms. Let G_n denote the resistance network on V_n. The basic idea is the following: we use $\mathbf{x}_n$ to denote the resistance (or conductance) on the edges of G_n under the homogeneous assumption (A1). Then the compatible condition implies the existence of a map Φ (the trace map) such that $\Phi(\mathbf{x}_{n+1}) = \mathbf{x}_n, n \geq 0$. Inductively we have $\Phi^n(\mathbf{x}_n) = \mathbf{x}_0$. If Φ is invertible, then we have

$$\Phi^{-n}(\mathbf{x}_0) = \mathbf{x}_n, \quad n \geq 1.$$

In other word, for a given $\mathbf{x}_0$, we can get a compatible sequence of resistance (or conductance) ([16], [11]). We call this a *reverse recursive construction*.

This simple idea is by no mean easy to implement. First to establish the map Φ, it may need to use some network reduction techniques such as resistances in series and in parallel, and the Δ-Y transform as well as its analogs. Secondly, even the inverse Φ^{-1} exists, it may be impossible to find or too complicate to make it useful. Thirdly, it is not easy to determine the domain of $\mathbf{x}_0$ so that the resulting sequences $\mathbf{x}_n$ exists and satisfies (A2).

In the following, we present two examples, which give the atypical resistance forms, to illustrate the above points. The first one is the construction of such a form on the asymmetric eyebolted Vicsek cross [10]; the other one is on the 3-dimensional SG, which provides some new classes of atypical homogeneous resistance forms.

Example 1. Eyebolted Vicsek cross(EVC)

In $\mathbb{R}^2$, let $\{p_1, p_2, p_3, p_4\}$ be the four vertices of the unit square S as in Figure 3. Divide S into a mesh of sub-squares of size $1/9$, and pick 21 sub-squares as shown in Figure 3.

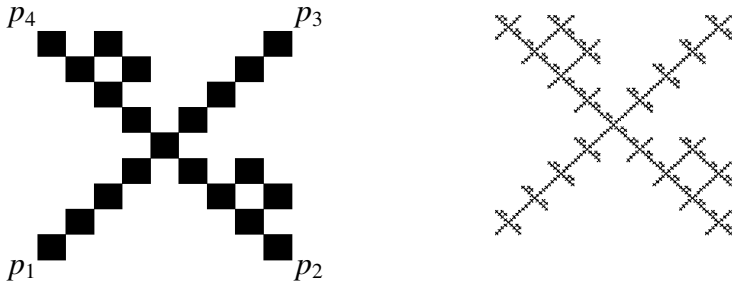


Figure 3. The eyebolted Vicsek cross.

For each sub-square Q, let $F_Q : S \to S$ be given by

$$F_Q(x) = x/9 + p_Q$$

where p_Q is chosen so that $F_Q(S) = Q$. Renumber the maps F_Q by $\{F_i\}_{i=1}^{21}$. Let K be the unique nonempty compact set such that $K = \bigcup_{i=1}^{21} F_i(K)$. Then $(K, \{F_i\}_{i=1}^{21})$ is a p.c.f. self-similar set with boundary $V_0 = \{p_1, p_2, p_3, p_4\}$. We call this modified Vicsek cross an *eyebolted Vicsek cross* (EVC). The Hausdorff dimension of K is $\alpha = \log 21/\log 9$, and the self-similar measure with the natural weight is the normalized α-dimensional Hausdorff measure μ on K.

It was shown in [10, Theorem 5.1] that on the eyebolted Vicsek cross, there are two local regular Dirichlet forms that can be constructed. One satisfies the energy self-similar identity (2.6), the other one is from a reverse recursive construction and does not satisfy (2.6) but satisfies (3.2). Now we will continue the study of the second construction [10, Section 6].

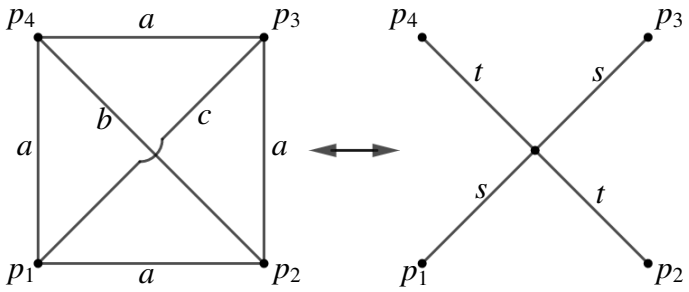


FIGURE 4. the ⊠-X transform of resistance networks.

Analogous to the well-known Δ-Y transform for resistance network with three vertices, in [10, Section 2], we introduced two equivalent networks with four vertices, call it ⊠-X transform, to handle resistance networks composed of aggregate of squares in the self-similar p.c.f sets. The transform is ([10, Lemma 2.7]): assume the resistance on G_0 is $bc = a^2$, then

$$a = 2(s+t), \quad b = \frac{2s}{t}(s+t), \quad \text{and} \quad c = \frac{2t}{s}(s+t) \tag{5.1}$$

(See Figure 5). In the following, for a given initial resistance on G_0 of the EVC (i.e. on the ⊠-side), we transform it to G_0' on the X-side. Then evaluate the compatible resistances on G_n', follow by a transform back to the G_n of EVC.

Theorem 5.1. *On the EVC, suppose we are given equal initial conductance $c_0(p_i, p_j), 1 \le i, j \le 4, i \ne j$, on each edge. Then there is a unique homogeneous resistance form with conductances*

$$c_n(p_1, p_3) \asymp 9^n, \quad c_n(p_2, p_4) \asymp \Big(\frac{64}{9}\Big)^n, \ \text{and} \ c_n(p_i, p_{i+1}) \asymp 8^n, \ 1 \le i \le 4 \ (p_5 = p_1).$$

Proof. We will consider the corresponding resistance through the ⊠-X transform. Without loss of generality, we can assume the initial resistance $\mathbf{y}_0 = (1, 1, 1, 1)$ on G_0'. Let $\mathbf{y}_n = (s_n, t_n, s_n, t_n)$ be the resistance on the n-level. As in [10, Lamma 5.1 and p.25], the compatible condition on the n-th level is $\Phi(\mathbf{y}_n) = \mathbf{y}_{n-1}$, and Φ is calculated to satisfy $\Phi(s, t, s, t) = (9s,\ 9t - \frac{t^2}{s+t},\ 9s,\ 9t - \frac{t^2}{s+t})$. It follows that $\Phi^n(\mathbf{y}_n) = \mathbf{y}_0$, hence $\Phi^{-n}(\mathbf{y}_0) = \mathbf{y}_n$. Solving the above equation for Φ^{-1}, we have

$$\mathbf{y}_n = \Phi^{-1}(\mathbf{y}_{n-1}) = (s_n, t_n, s_n, t_n), \quad \text{with} \ \ s_n = 9^{-n}, \ t_{n-1} = 9t_n - \frac{t_n^2}{9^{-n} + t_n}.$$

We claim that $t_n \asymp 8^{-n}$. For this, we let $x_n = 8^n t_n$, and using the above relationship of t_{n-1} and t_n, we have

$$x_{n-1} = x_n + \frac{1}{8} \cdot \frac{x_n}{1 + (9/8)^n x_n}. \tag{5.2}$$

Observe that $\{x_n\}$ is a non-negative, decreasing sequence, hence $\lim_{n\to\infty} x_n$ exists. We show that the limit is strictly positive, then the claim follows.

Fix any $n_0 > 0$, and for $n \geq n_0$, we see by (5.2),

$$\begin{aligned} x_n &= x_{n_0} + \sum_{k=n_0}^{n-1} (x_{k+1} - x_k) \\ &= x_{n_0} - \frac{1}{8} \cdot \sum_{k=n_0}^{n-1} \frac{x_{k+1}}{1 + y_{k+1}} \qquad (\text{letting } \ y_k = 9^k t_k) \\ &\geq x_{n_0}\Big(1 - \frac{1}{8} \cdot \sum_{k=n_0}^{\infty} \frac{1}{y_{k+1}}\Big). \end{aligned} \tag{5.3}$$

Again, by checking the relation of t_{n-1} and t_n, we see that $\{y_n\}_n$ satisfies

$$y_{n-1} = y_n - \frac{y_n^2}{9(1 + y_n)}. \tag{5.4}$$

It is an increasing sequence, and $\lim_{n\to\infty} y_n = \infty$ (if otherwise, $\lim\limits_{n\to\infty} y_n = y$ is finite, then letting $n \to \infty$, we have $y = y - \frac{y^2}{9(1+y)}$, this implies $y = 0$, contradicting that $y_0 = 1$ and y_n is increasing). Hence we have

$$\frac{y_n}{y_{n-1}} = \left(1 - \frac{y_n}{9(1 + y_n)}\right)^{-1} > \tau$$

for some $\tau > 1$. This yields $y_n \geq \tau^n$. Substituting this into (5.3), we have

$$x_n \geq \frac{1}{2} x_{n_0},$$

proving that $t_n \asymp 8^{-n}$.

Finally, we transform the resistances s_n and t_n back to G_n; by (5.1), the resistances on G_n satisfies

$$\begin{aligned} &c_n(p_1, p_3)^{-1} = 2(s_n + \frac{s_n^2}{t_n}), \qquad c_n(p_2, p_4)^{-1} = 2(t_n + \frac{t_n^2}{s_n}), \\ &c_n(p_i, p_{i+1})^{-1} = 2(s_n + t_n), \ i = 1, 2, 3, 4, \ (p_5 = p_1). \end{aligned}$$

Substituting $s_n \asymp 9^{-n}$ and $t_n \asymp 8^{-n}$ in the above, we obtain the compatible conductance on G_n as stated. □

Corollary 5.2. *Let K be the EVC, μ the normalized Hausdorff measure on K, and let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form. Then the eigenvalue counting function $\rho(t)$ of the corresponding Laplacian has the estimate*

$$\rho(t) \asymp t^{\frac{\log 21}{\log 168}}.$$

Proof. We can check the conductances in Theorem 5.1 that (A1), (A2) and (A3) are satisfied, and that $N = 21$ and $\kappa_0 = 8$. Hence Theorem 4.6 yields the conclusion. □

Example 2. Homogenous forms on 3-dimensional SG

In [11], the authors studied the compatible resistance forms on the 2-dim SG. Let (a_0, b_0, c_0) be the conductance on V_0, and let (a_n, b_n, c_n) be the conductance on V_n. It is shown that in order to have a compatible resistance form, it is necessary and sufficient that the initial conductance satisfies $a_0 \geq b_0 = c_0$ (or their alternations). In the case that $a_0 = b_0 = c_0$, then it gives the canonical form that satisfies the energy self-similar identity; if $a_0 > b_0 = c_0$, then the compatibility of resistances yields

$$a_n \asymp 2^n, \qquad b_n = c_n \asymp \left(\frac{3}{2}\right)^n.$$

They define a homogeneous form satisfying (A1)-(A3), but does not satisfy the energy self-similar identity.

In the following, we consider the 3-dim SG. The situation is far more complicated. We can not characterize all the homogeneous forms; but we are able to identify a large class of them based on the initial conductances.

Let p_1, p_2, p_3, p_4 be the four vertices of a unit tetrahedron in $\mathbb{R}^3$. We define the 3-dim SG K to be the unique non-empty compact set in $\mathbb{R}^3$ with the contractions $\{F_i\}_{i=1}^4$ on $\mathbb{R}^3$ such that $F_i(x) = \frac{1}{2}(x-p_i)+p_i$, $1 \leq i \leq 4$. Then K has boundary points $V_0 = \{p_1, p_2, p_3, p_4\}$ and six edges.

We consider the initial conductance (a_0, b_0, c_0), with a_0, c_0 on two opposite edges, and b_0 on the other four edges (see Figure 5). Then let (a_n, b_n, c_n) be the conductance on the n-level cells an in (A1). We will estimate (a_n, b_n, c_n) by the compatible condition. Let G_n be the corresponding resistance network on V_n.

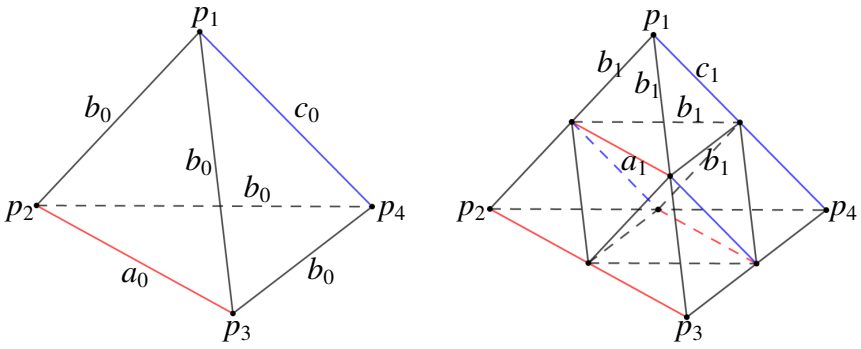


FIGURE 5. The conductances on G_0 and G_1.

To simplify notations, we write (a_0, b_0, c_0) as (A, B, C) and (a_1, b_1, c_1) as (a, b, c) on V_1. According to [20], the associated Laplacian matrix is

$$H_0 = \begin{pmatrix} -2B-C & B & B & C \\ B & -A-2B & A & B \\ B & A & -A-2B & B \\ C & B & B & -2B-C \end{pmatrix},$$

and the trace of G_1 on G_0 is given by $H_1|_{V_0} = T - J^t X^{-1} J$, where

$$T = \begin{pmatrix} -2b-c & & & \\ & -a-2b & & \\ & & -a-2b & \\ & & & -2b-c \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & a & a & 0 \\ b & 0 & b & 0 \\ b & b & 0 & 0 \\ c & 0 & 0 & c \\ 0 & b & 0 & b \\ 0 & 0 & b & b \end{pmatrix},$$

and

$$X = \begin{pmatrix} -2a-4b & b & b & 0 & b & b \\ b & -a-4b-c & a & b & 0 & c \\ b & a & -a-4b-c & b & c & 0 \\ 0 & b & b & -4b-2c & b & b \\ b & 0 & c & b & -a-4b-c & a \\ b & c & 0 & b & a & -a-4b-c \end{pmatrix}.$$

By solving the compatible condition $H_1|_{V_0} = H_0$ (use Mathematica), we have

$$A = \frac{(a+b)(b^2c + 5abc + 2a^2c + 4b^3 + 9ab^2 + 3a^2b)}{2(a+2b)(4b^2 + 2ac + 3ab + 3bc)}, \tag{5.5}$$

$$B = \frac{2b(a+b)(b+c)}{3ab + 4b^2 + 2ac + 3bc}, \tag{5.6}$$

$$C = \frac{(b+c)(ab^2 + 5abc + 2ac^2 + 4b^3 + 9b^2c + 3bc^2)}{2(2b+c)(4b^2 + 2ac + 3ab + 3bc)}. \tag{5.7}$$

By letting $v = a/b, w = c/b$ in the expression (5.1), (5.3), we define $\Phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ (here $\mathbb{R}_+ := (0, \infty)$), $\Phi(v, w) = \big(\Phi_1(v, w), \Phi_2(v, w)\big)$ such that

$$\Phi_1(v, w) := \frac{w(2v^2 + 5v + 1) + 3v^2 + 9v + 4}{4(1+w)(2+v)}, \tag{5.8}$$

$$\Phi_2(v, w) := \frac{v(2w^2 + 5w + 1) + 3w^2 + 9w + 4}{4(1+v)(2+w)}. \tag{5.9}$$

Note that $\Phi_1(v, w) = \Phi_2(w, v)$ for any $v, w \in \mathbb{R}^2_+$.

For $n \geq 1$, by viewing $(a_{n-1}, b_{n-1}, c_{n-1})$ as (A, B, C), (a_n, b_n, c_n) as (a, b, c), and setting $v_n = \frac{a_n}{b_n}$ and $w_n = \frac{c_n}{b_n}$. The compatibility of the sequence (a_n, b_n, c_n) (to be constructed) implies

$$(v_{n-1}, w_{n-1}) = \Phi(v_n, w_n). \tag{5.10}$$

Remark 1. We will prove that Φ is a one-to-one map on $\mathbb{R}^2_+$ (Lemma 5.6(iii)), and is a homeomorphism on certain sub-region S_∞ (Lemma 5.7(i)). Normally, one will write (5.10) as $\Phi^{-1}(v_{n-1}, w_{n-1}) = (v_n, w_n)$, $\Phi^{-n}(v_0, w_0) = (v_n, w_n)$. However Φ^{-1} has no explicit expression, and is too hard to handle, we will stick to consider $(v_0, w_0) = \Phi^n(v_n, w_n)$ for a fixed initial data.

In the following, we first state a lemma and the main theorem; they will be proved after we give a detail study of the map Φ in the sequel. Let $S_0 = \{(v, w) : 1 < w < v\}$ and $S_1 = \Phi(S_0)$, then $S_1 \supset S_0$ (see proof of Lemma 5.7(i)). Inductively, for $n \geq 1$, let $S_n = \Phi(S_{n-1})(\supset S_{n-1})$, and let $S_\infty = \bigcup_{n\geq 0} S_n$; it is invariant under Φ (see Figure 6 for the geometry of this as well as the following lemma).

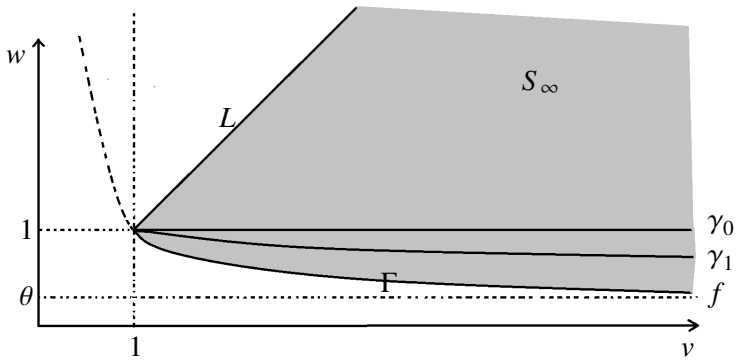


FIGURE 6. the map Φ on S_∞, the shaded region.

Lemma 5.3. *There exists a continuous non-increasing function $f : [1, \infty) \to (\theta, 1]$ with $\theta = \frac{\sqrt{17}-3}{4}$, and whose graph, denoted by Γ, is on the lower boundary of S_∞ and satisfies $\Phi(\Gamma) = \Gamma$.*

The following is the main theorem of this section. It classifies the initial conductances (a_0, b_0, c_0) that yield compatible resistance forms, and the growth rate of the corresponding conductances $\{(a_n, b_n, c_n)\}$ are also specified.

Theorem 5.4. *Suppose $a_0 \geq c_0 > 0$ and $b_0 > 0$. Then*

(i) if $\frac{c_0}{b_0} > f\left(\frac{a_0}{b_0}\right)$, then there exists a unique sequence $\{a_n, b_n, c_n\}_{n\geq 0}$ satisfying $a_n \geq c_n$, $\frac{c_n}{b_n} > f\left(\frac{a_n}{b_n}\right)$ and $a_n \asymp c_n \asymp 2^n$, $b_n \asymp 1$;

(ii) if $\frac{c_0}{b_0} = f\left(\frac{a_0}{b_0}\right)$, then there exists a unique positive solution $\{a_n, b_n, c_n\}_{n\geq 0}$ satisfying $a_n \geq c_n$, $\frac{c_n}{b_n} = f(\frac{a_n}{b_n})$; moreover, if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n a_0$; otherwise if $a_0 > c_0$, then $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$;

(iii) if $\frac{c_0}{b_0} < f\left(\frac{a_0}{b_0}\right)$, then there is no compatible sequence can be constructed.

It follows that for the cases in (ii), the $\{(a_n, b_n, c_n)\}$ defines resistance forms satisfying (A2).

Remark 2. As $v = \frac{a_0}{b_0}$, $w = \frac{c_0}{b_0}$, we see in Figure 6 that (i) is on the grey open region S_∞; (ii) is on the boundary curve Γ, with the first case at the point $(v, w) = (1, 1)$, and the second case are the rest of the points on Γ; (iii) is the region under $\overline{S}_\infty$.

Remark 3. For case (i), since $b_n \asymp 1$, it does not satisfy (A2), hence the induced compatible resistance form is not a homogeneous form. The case in (ii) with $a_0 = c_0$, the conductances $(3/2)^n$ corresponding to the renormalization factor of the standard energy form of 3-dim SG. (Recall that for the d-dim SG, the renormalization factor is $(d + 3)/(d + 1)$ [18].)

Remark 4. Note that a_0 and c_0 are conductance on symmetric positions of two edges on the SG, the above theorem holds the same for $c_0 \geq a_0 > 0$ by interchanging the role of a_0 and c_0.

Corollary 5.5. *Suppose $a_0 \geq c_0$ and $\frac{c_0}{b_0} = f\left(\frac{a_0}{b_0}\right)$ as in Theorem 5.4(ii). Then the compatible sequence $\{(a_n, b_n, c_n)\}_{n\geq 0}$ defines a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ where μ is the standard normalized Hausdorff measure on K. Let Δ be the Laplacian of the Dirichlet form and $\rho(t)$ be its eigenvalue counting function, then for t large enough, we have*

(i). if $a_0 = c_0$, then $\rho(t) \asymp t^{\frac{\log 4}{\log 6}}$;

(ii). if $a_0 > c_0$, then $\rho(t) \asymp t^{\log 4/\log\left(\frac{7+\sqrt{17}}{2}\right)}$.

Proof. (i) By Theorem 5.4(ii), if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n \cdot a_0$; a direct check shows that $\kappa_0 = \frac{3}{2}$, and the relation $\sim$ is the trivial relation (each point in V_0 is a single equivalent class). We see that condition (A3) holds. By Theorem 4.6, we obtain, with $N = 4$ and $\kappa_0 = \frac{3}{2}$, that

$$\rho(t) \asymp t^{\frac{\log 4}{\log 6}}.$$

(ii) If $a_0 > c_0$, then by the second conclusion of Theorem 5.4(ii), $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$, and thus $\kappa_0 = \frac{7+\sqrt{17}}{8}$, and the equivalent class of $\sim$ is given by $\{p_1\} \cup \{p_2, p_3\} \cup \{p_4\} (= V_0)$. We can check that condition (A3) holds. By Theorem 4.6, we obtain, with $N = 4$ and $\kappa_0 = \frac{7+\sqrt{17}}{8}$, that

$$\rho(t) \asymp t^{\log 4/\log \frac{7+\sqrt{17}}{2}}.$$

□

To prove Lemma 5.3 and Theorem 5.4, we start by studying some basic properties of the map Φ.

Lemma 5.6. *The map Φ satisfies:*

(i) for $(v, w) \in \mathbb{R}^2_+$, if $\frac{v}{w} = 1$, then $\frac{\Phi_1(v,v)}{\Phi_2(v,v)} = 1$; and if $\frac{v}{w} > 1$, then $1 < \frac{\Phi_1(v,w)}{\Phi_2(v,w)} < \frac{v}{w}$.

(ii) for $(v, w) \in \mathbb{R}^2_+$, the partial derivatives satisfy $\frac{\partial \Phi_1}{\partial v} > 0$, $\frac{\partial \Phi_1}{\partial w} < 0$, $\frac{\partial \Phi_2}{\partial v} < 0$, $\frac{\partial \Phi_2}{\partial w} > 0$, $\frac{\partial(\Phi_1 \cdot \Phi_2)}{\partial v} > 0$, $\frac{\partial(\Phi_1 \cdot \Phi_2)}{\partial w} > 0$.

(iii) Φ is one-to-one from $\mathbb{R}^2_+$ to its image.

Proof. The first statement of (i) follows trivially from (5.8), (5.9). The second statement of (i) follows from a direct calculation:

$$\frac{\Phi_1(v, w)}{\Phi_2(v, w)} = 1 + \frac{(v - w)(5 + 2v + 2w + vw)(4 + 3v + 3w + 2vw)}{(2 + v)(1 + w)(4 + v + 9w + 5vw + 3w^2 + 2vw^2)} > 1$$

and

$$\frac{v}{w} - \frac{\Phi_1(v, w)}{\Phi_2(v, w)} = \frac{(v - w)(2 + v + w)(4 + v + w)}{w(1 + w)(2 + v)(4 + v + 9w + 5vw + 3w^2 + 2vw^2)} > 0. \tag{5.11}$$

For (ii), let $(v, w) \in \mathbb{R}^2_+$, then a direct check yields

$$\begin{cases} \frac{\partial \Phi_1}{\partial v}(v, w) & = \frac{14+9w+v(4+v)(3+2w)}{4(2+v)^2(1+w)} > 0, \\ \frac{\partial \Phi_1}{\partial w}(v, w) & = -\frac{3+v}{4(1+v)(2+w)} < 0. \end{cases} \tag{5.12}$$

As $\Phi_2(v, w) = \Phi_1(w, v) = \Phi(\varphi(v, w))$ where $\varphi(v, w) = (w, v)$, then by the chain rule, $\frac{\partial \Phi_2}{\partial v}(v, w) = \frac{\partial \Phi_1}{\partial w}(w, v) < 0$; similarly, $\frac{\partial \Phi_2}{\partial w}(v, w) = \frac{\partial \Phi_1}{\partial v}(w, v) > 0$ also.

That $\frac{\partial(\Phi_1\Phi_2)}{\partial v}(v,w) > 0$ follows by a direct calculation (by Mathematica) that all the terms are positive (the lengthy expression is omitted). It follows that $\frac{\partial(\Phi_1\Phi_2)}{\partial w}(v,w) = \frac{\partial(\Phi_1\Phi_2)}{\partial v}(w,v) > 0$.

Now let us use (ii) to show (iii). Suppose there are two points (v,w) and $(\widetilde{v},\widetilde{w})$ in $\mathbb{R}^2_+$ such that $\Phi(v,w) = \Phi(\widetilde{v},\widetilde{w})$. Without loss of generality, assume $v \le \widetilde{v}$. Let $\eta : [0,1] \to R^2_+$ be the straight line connecting such two points, i.e. $\eta(s) = (s\widetilde{v} + (1-s)v, s\widetilde{w} + (1-s)w)$, $s \in [0,1]$. Then by applying Rolle's theorem to $\Phi_1(\eta(s))$ and $\Phi_1(\eta(s))\Phi_2(\eta(s))$ separately, there exist $s_1, s_2 \in (0,1)$ such that

$$\frac{d\Phi_1(\eta(s))}{ds}\Big|_{s=s_1} = 0, \qquad \frac{d(\Phi_1(\eta(s))\Phi_2(\eta(s)))}{ds}\Big|_{s=s_2} = 0.$$

To prove $(v,w) = (\widetilde{v},\widetilde{w})$, we have two possible cases: *Case (i)* $v = \widetilde{v}, w \ne \widetilde{w}$ or $v < \widetilde{v}, w \ge \widetilde{w}$: by applying the chain rule to $\frac{d\Phi_1(\eta(s))}{ds}\Big|_{s=s_1}$, and use (ii), we have

$$\frac{d\Phi_1(\eta(s))}{ds}\Big|_{s=s_1} = \frac{\partial\Phi_1}{\partial v}(\eta(s_1))\,(\widetilde{v}-v) + \frac{\partial\Phi_1}{\partial w}(\eta(s_1))\,(\widetilde{w}-w) \ne 0,$$

a contradiction. *Case (ii)* $v < \widetilde{v}$ and $w < \widetilde{w}$: by applying the chain rule, and use (ii), we have

$$\frac{d(\Phi_1(\eta(s))\Phi_2(\eta(s)))}{ds}\Big|_{s=s_2} = \frac{\partial(\Phi_1\Phi_2)}{\partial v}(\eta(s_2))\,(\widetilde{v}-v) + \frac{\partial(\Phi_1\Phi_2)}{\partial w}(\eta(s_2))\,(\widetilde{w}-w) > 0,$$

again a contradiction.

In view of the two cases, we conclude that $(v,w) = (\widetilde{v},\widetilde{w})$, and completes the proof of (iii). □

Next we use the above properties to prove another lemma which illustrates the dynamical behavior of Φ, and plays a key role in obtaining the existence of a compatible sequence $\{(a_n, b_n, c_n)\}_{n\ge 0}$.

Let $S_\infty = \cup_{n=0}^{\infty} S_n$ be the region defined preceding Lemma 5.3. Let $L = \{(v,v) : v \ge 1\}$ and $\gamma_0(t) = (t,1)$, $t \in (1,\infty)$ be the two half-lines, then the boundary of S_0 is $\partial S_0 = L \cup \gamma_0$.

Lemma 5.7. *The map Φ has the following properties:*

(i) Φ is a homeomorphism between S_∞ and S_∞;

(ii) for $(v,w) \in S_\infty$, we have $v > w$ and $w > \theta$, where $\theta = \frac{\sqrt{17}-3}{4}$ (< 1).

Proof. To prove (i), we note that Φ is a one-to-one continuous map (Lemma 5.6(iii)). By the well-known *Brouwer invariance of domain theorem* [3], we see that S_0 and $S_1 := \Phi(S_0)$ are homeomorphic. We claim that $S_0 \subset S_1$. Indeed, consider $\partial S_0 = L \cup \gamma_0$. First, Lemma 5.6(i) implies $\Phi(L) = L$; secondly, let $\gamma_1 = \Phi(\gamma_0)$, then by the expression of Φ, we have

$$\gamma_1(t) := (\alpha_1(t),\, \beta_1(t)) = \left(\frac{5+14t+5t^2}{8(2+t)},\ \frac{4+2t}{3+3t}\right), \quad t \in (1,\infty). \tag{5.13}$$

It is clear that the range of α_1 is $(1,\infty)$, and $\beta_1(t) < 1$ for all $t > 1$. We conclude that γ_1 is a curve located strictly lower than γ_0. This proves the claim $S_0 \subset S_1$. Inductively, we have $S_{n-1} \subset S_n$, so that $\Phi(S_\infty) = S_\infty$, and Φ is a homeomorphism between S_∞ and S_∞.

To prove (ii), we observe that the statement is trivial on S_0. Suppose the statement is true on S_{n-1}. For $(v,w) \in S_n$, we let $(v',w') \in S_{n-1}$ such that $(v,w) = \Phi(v',w')$. Induction hypothesis implies $v' > w'$, then the second part of Lemma 5.6(i) yields $v > w$.

To prove $w > \theta$, we note that

$$w = \Phi_2(v', w') = \frac{v'(2w'^2 + 5w' + 1) + 3w'^2 + 9w' + 4}{4(1 + v')(2 + w')} > \frac{(2w'^2 + 5w' + 1)}{4(2 + w')} =: h(w'). \tag{5.14}$$

Consider $h(x) = x$, by solving the equation, we obtain the fixed point $x = \theta = \frac{\sqrt{17}-3}{4}$, which is the positive root of $2x^2 + 3x - 1 = 0$. It is direct to check that $h'(x) > 0$, hence h is increasing. As $w' > \theta$ (induction hypothesis), we have by (5.10),

$$w > h(w') > h(\theta) = \theta.$$

Hence $w > \theta$ holds for all $(v, w) \in S_\infty = \cup_{n=0}^{\infty} S_n$. □

Proof of Lemma 5.3. We examine the boundary of S_∞ through the boundary of each S_n, and define a continuous function f on $(1, \infty)$ whose graph is the lower part of the boundary of S_∞ (see Figure 6).

For $n \geq 1$, let $\gamma_n(t) = \Phi(\gamma_{n-1}(t))$ and write $\gamma_n(t) = (\alpha_n(t), \beta_n(t))$, $t \in (1, \infty)$. Then use induction, and similar to (5.13), we see that $\alpha_n(t) \to \infty$ as $t \to \infty$. Hence $\alpha_n(t)$ is a homeomorphism from $(1, \infty)$ to itself (by Lemma 5.7(i)).

We claim that *for any $n \geq 1$ and for all $t \in (1, \infty)$, $\alpha_n'(t) > 0$, $\beta_n'(t) < 0$.* We prove this by induction. First, using (5.13) and a direct calculation, we have

$$\alpha_1'(t) = \frac{5}{8} + \frac{3}{8(2 + t)^2} > 0, \qquad \beta_1'(t) = -\frac{2}{3(1 + t)^2} < 0.$$

Inductively assume that for $k \geq 1$, $\alpha_k'(t) > 0$ and $\beta_k'(t) < 0$. Then by the chain rule and using Lemma 5.6(ii), we have

$$\alpha_{k+1}' = (\Phi_1(\alpha_k, \beta_k))' = \frac{\partial \Phi_1}{\partial v} \cdot \alpha_k' + \frac{\partial \Phi_1}{\partial w} \cdot \beta_k' > 0,$$

and

$$\beta_{k+1}' = (\Phi_2(\alpha_k, \beta_k))' = \frac{\partial \Phi_2}{\partial v} \cdot \alpha_k' + \frac{\partial \Phi_2}{\partial w} \cdot \beta_k' < 0.$$

This completes the proof of the claim.

For $n \geq 1$, since $\alpha_n(t)$ is a homeomorphism from $(1, \infty)$ to $(1, \infty)$, we can define $f_n(v) = \beta_n(\alpha_n^{-1}(v))$, $v \in (1, \infty)$, we claim that

$$|f_n'(v)| \leq C, \ v \in (1, \infty) \ \textit{for some } C > 0, \textit{ independent of } n \textit{ and } v.$$

Indeed, observe that

$$\left|f_{n+1}'\right| = \frac{-\beta_{n+1}'}{\alpha_{n+1}'} = \frac{\left|\frac{\partial \Phi_2}{\partial v}\right| \alpha_n' + \frac{\partial \Phi_2}{\partial w} \left|\beta_n'\right|}{\frac{\partial \Phi_1}{\partial v} \alpha_n' + \left|\frac{\partial \Phi_1}{\partial w}\right| \left|\beta_n'\right|} \leq \frac{\left|\frac{\partial \Phi_2}{\partial v}\right|}{\frac{\partial \Phi_1}{\partial v}} + \frac{\frac{\partial \Phi_2}{\partial w}}{\left|\frac{\partial \Phi_1}{\partial w}\right|}. \tag{5.15}$$

Since $(\alpha_n(t), \beta_n(t))$ is located in the area $(1, \infty) \times (\theta, 1]$, using (5.12), we see that there exists $C_1 > 0$ such that

$$C_1^{-1} \leq \frac{\partial \Phi_1}{\partial v}, \left|\frac{\partial \Phi_1}{\partial w}\right| \leq C_1, \quad \text{and} \quad \left|\frac{\partial \Phi_2}{\partial v}\right|, \frac{\partial \Phi_2}{\partial w} \leq C_1.$$

This proofs the claim.

For fixed $v > 1$, $f_n(v)$ is decreasing on n (as $S_n \subsetneq S_{n+1}$). Let

$$f(v) = \lim_{n \to \infty} f_n(v). \tag{5.16}$$

By Lemma 5.7(ii), $f_n(v) > \theta$ and clearly $f_n(v) \le 1$, we see that $\{f_n\}_{n\ge1}$ are uniformly bounded. By the last claim, $\{f_n\}_{n\ge1}$ are equi-continuous. Hence by the Arzelà-Ascoli theorem, $f(v)$ is continuous on $(1,\infty)$. Since each f_n is decreasing on v (as $f_n'(v) = \frac{\beta_n'(v)}{\alpha_n'(v)} < 0$), f is non-increasing on v. In addition, $\lim_{v\to1+} f(v) = 1$.

Finally, let $\Gamma := \{(v, f(v)),\, v \in (1,\infty)\}$ be the graph of f. Denote by $\overline{S}_\infty$ the closure of S_∞, then $\partial S_\infty = L \cup \Gamma$ where $L = \{(v,v) : v \ge 1\}$. By Lemma 5.7(i), Φ is a homeomorphism on S_∞, hence $\Phi(\partial\overline{S}_\infty) = \partial\overline{S}_\infty$. Observe that $\Phi(L) = L$ and $L \cap \Gamma = \emptyset$, we must have $\Phi(\Gamma) = \Gamma$. □

Proof of Theorem 5.4. We follow the preceding notations to write $v_n = \frac{a_n}{b_n},\ w_n = \frac{c_n}{b_n}$.

Part (i). The assumption implies that $(v_0, w_0) \in S_\infty$, and the sequence $\{(v_n, w_n)\}_{n\ge0}$ are also in S_∞ (by Lemma 5.7(i)). To prove our result, we need

$$\lim_{n\to\infty} w_n = \infty. \tag{5.17}$$

Granting this, we see from the proof of Lemma 5.6(i) that $v_n > w_n$, hence $v_n \to \infty$ also; by (5.9) and (5.10), we have

$$w_{n-1} = \frac{v_n(2w_n^2 + 5w_n + 1) + 3w_n^2 + 9w_n + 4}{4(1+v_n)(2+w_n)}.$$

A direct estimation shows that for large n,

$$\frac{w_{n-1}}{w_n} \;=\; \frac{1}{2}\left(1 + \frac{\varphi(v_n, w_n)}{w_n}\right) \;<\; \frac{3}{4}$$

where $\varphi(v_n, w_n)$ includes all the terms of the above equation except the first term; it is a bounded function on $v_n, w_n, n \ge 0$, we use c to denote its bound. We can find $1 < \tau < \frac{4}{3}$ such that $w_n > \tau^n$ for all n. Hence for $n \ge 0$,

$$(1 - c\tau^{-n})\,\frac{w_n}{2} \;\le w_{n-1} \;\le (1 + c\tau^{-n})\,\frac{w_n}{2}. \tag{5.18}$$

This gives $w_n \asymp 2^n$. By using (5.8) and a similar argument as the above, we also have $v_n \asymp 2^n$.

Now we return to prove (5.17). To do so, we first show that there is $C > 0$ such that for all $n \ge 1$,

$$\frac{v_n}{w_n} \le C. \tag{5.19}$$

Suppose otherwise, by Lemma 5.6(i), $\frac{v_n}{w_n}$ is increasing in n, then we must have $\lim_{n\to\infty} \frac{v_n}{w_n} = \infty$, so that $\lim_{n\to\infty} v_n = \infty$. Then use (5.9) and by the same argument to obtain (5.18), we can conclude that $w_n \asymp 2^n$. The same way, we have $v_n \asymp 2^n$. This contradicts that $\lim_{n\to\infty} \frac{v_n}{w_n} = \infty$. Hence (5.19) holds.

That (5.17) follows by (5.19) can be shown by contradiction. If (5.17) does not hold, then $\{w_n\}_n$ is bounded, so is $\{v_n\}_n$ by (5.19). Let M be their bound. Consider (5.11) for the $\{v_n, w_n\}$. By factoring out the w_n from $(v_n - w_n)$, we can estimate the rest of the terms to be bounded below by a $C > 0$ dependent only on M. Hence

$$\frac{v_n}{w_n} - \frac{v_{n-1}}{w_{n-1}} \ge C\Big(\frac{v_n}{w_n} - 1\Big) \ge C\Big(\frac{v_0}{w_0} - 1\Big). \tag{5.20}$$

By summing up the telescoping sequence on the left, we see that $\frac{v_n}{w_n} \to \infty$ as $n \to \infty$, a contradiction to (5.19). Hence (5.17) holds.

To complete the proof of part (i), we convert the v_n, w_n back to a_n, b_n, c_n. By (5.6), we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n+1)(w_n+1)}{3v_n+4+2v_nw_n+3w_n} = 1 + o(\delta^n),$$

for some $\delta \in (0,1)$, which implies that $b_n \asymp 1$, and hence $c_n = b_n w_n \asymp 2^n$ and $a_n = b_n v_n \asymp 2^n$ as asserted.

Part (ii): As $\Phi(\Gamma) = \Gamma$ (Lemma 5.3), we see that $\{(v_n, w_n)\}_n$ are contained in Γ. For $v_0 > 1$, we first show that $\lim_{n\to\infty} w_n = \theta$. By Lemma 5.6(i), $\frac{v_n}{w_n}\left(= \frac{v_n}{f(v_n)}\right)$ is increasing on n; as f is non-increasing, we see that $(1 >)\{w_n\}_n$ is non-increasing on n. By Lemma 5.7(ii), $w_n \geq \theta$, thus $\lim_{n\to\infty} w_n = \theta' \geq \theta$ exists. Then by the argument in deriving (5.20), we see that $\frac{v_n}{w_n} \to \infty$ as $n \to \infty$. Hence $v_n \to \infty$. If $\theta' > \theta$, we make use of (5.9) that for large v_n,

$$\frac{w_{n-1}}{w_n} \to \frac{2w_n^2+5w_n+1}{4w_n(2+w_n)}.$$

Recall that θ is the fixed point of $h(x) = \frac{2x^2+5x+1}{4(2+x)}$ (Lemma 5.7(ii)), a direct calculation shows that $\frac{h(\theta')}{\theta'} < \frac{h(\theta)}{\theta} = 1$. Hence from the above, we have for large v_n, there is $0 < \delta < 1$ such that

$$\frac{w_{n-1}}{w_n} \to \frac{h(\theta')}{\theta'} < \delta < 1,$$

therefore $w_n \geq C\delta^{-n} \to \infty$, a contradiction, and we conclude that $\lim_{n\to\infty} w_n = \theta$.

By (5.8), for n large,

$$\frac{v_n}{v_{n-1}} \to \frac{4+4w_n}{3+2w_n} > 1,$$

so that there is some $\tau > 1$ and $C > 0$ such that for n large enough,

$$v_n \geq C^{-1}\tau^n. \tag{5.21}$$

Substituting this into

$$\frac{w_{n-1}}{w_n} = \frac{v_n(2w_n^2+5w_n+1)+3w_n^2+9w_n+4}{4w_n(1+v_n)(2+w_n)},$$

we have

$$\frac{2w_n^2+5w_n+1}{4w_n(2+w_n)} \geq 1 - C'\tau^{-n},$$

(It seems we need this to get that w_n goes to θ sufficiently fast)

which implies that

$$w_n - \theta \leq C''\tau^{-n}.$$

Therefore, from this and (5.21), we have for n large enough,

$$\frac{v_n}{v_{n-1}} = \frac{4+4w_n}{3+2w_n} + O(\tau^{-n}) \to \frac{4+4\theta}{3+2\theta},$$

which implies that $v_n \asymp \left(\frac{4+4\theta}{3+2\theta}\right)^n = \left(\frac{7-\sqrt{17}}{2}\right)^n$.

In the case $a_0 = c_0$, then we must have $b_0 = a_0 = c_0$ also (for if otherwise, $b_0 = \eta^{-1}a_0$, then $v_0 = w_0 = \eta$, but (η, η) is not on Γ); hence $v_n = w_n = 1, n \geq 0$. From (5.6), we have $\frac{b_{n-1}}{b_n} = \frac{2}{3}$. Hence $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n a_0$.

Assume that $a_0 > c_0$. By (5.6) and the above estimate, we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n+1)(w_n+1)}{3v_n+4+2v_nw_n+3w_n} = \frac{2+2w_n}{3+2w_n} + o(\delta^n) = \frac{2+2\theta}{3+2\theta} + o(\delta^n),$$

for some $\delta \in (0,1)$, we obtain $b_n \asymp \left(\frac{3+2\theta}{2+2\theta}\right)^n$ where $\theta = \frac{\sqrt{17}-3}{4}$. Hence $c_n = b_n w_n \asymp b_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$, and $a_n = b_n v_n \asymp \left(\frac{3+2\theta}{2+2\theta}\right)^n \left(\frac{4+4\theta}{3+2\theta}\right)^n \asymp 2^n$.

Proof of (iii): Let (v_0, w_0) be with $f(v_0) > w_0$, assume that there is a compatible sequence $\{(v_n, w_n)\}_{n\geq 0}$. As Φ is a homeomorphism on S_∞, we see that $1 > f(v_n) > w_n$. Consider $h(x) = \frac{2x^2+5+1}{4(2+x)}$ on $[0,1]$, then $0 < h'(x) < 1$, so that $h(x)$ is an increasing function. By (5.14), we see that $w_0 > h(w_1)$, and repeat applying h, we have

$$w_0 > h(w_1) > h^{(2)}(w_2) > \cdots > h^{(n)}(w_n).$$

Also note that

$$|h^{(n)}(w_n) - \theta| = |h^{(n)}(w_n) - h^{(n)}(\theta)| \leq c^n |w_n - \theta|$$

where $c = \max_{x\in[0,1]}\{h'(x)\} < 1$, which yields

$$w_0 > h^{(n)}(w_n) \to \theta.$$

Note that this is also true if we start from any $(v_m, w_m), m < n$ instead of (v_0, w_0). Hence we can conclude that

$$1 > f(v_n) > w_n \geq \theta. \tag{5.22}$$

We also note that by applying the same argument as in (5.20), then $v_n/w_n \to \infty$ as $n \to \infty$, hence $v_n \to \infty$ holds (as $w_n < 1$).

Next we define a sequence of positive numbers $d_n := |f(v_n) - w_n| = f(v_n) - w_n$ for $n \geq 0$. We want to show that there is $\delta \in (0,1)$ such that

$$d_n > \delta^{-1} d_{n-1}. \tag{5.23}$$

This will lead to a contradiction, as both $\{f(v_n)\}_n$ and $\{w_n\}_n$ are bounded. Denote by $p = (v_n, f(v_n))$, $q = (v_n, w_n)$. Then $f(v_n) > w_n \geq \theta$. Write

$$\Phi(p) = (\Phi_1(v_n, f(v_n)), \Phi_2(v_n, f(v_n))), \qquad \Phi(q) = (\Phi_1(v_n, w_n), \Phi_2(v_n, w_n)).$$

These are points satisfies $\Phi(p) \in \Gamma$, $\Phi(q) = (v_{n-1}, w_{n-1})$. (See Figure 7.)

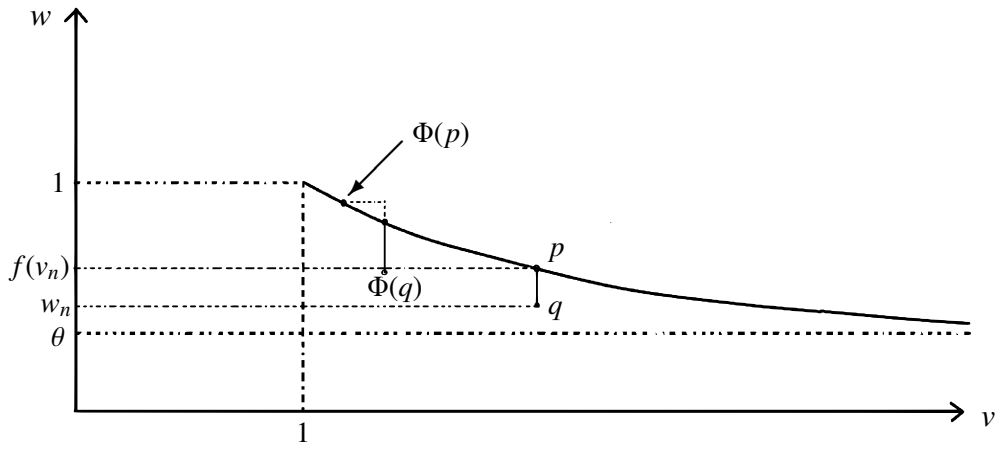


FIGURE 7. The position of p, q, $\Phi(p)$ and $\Phi(q)$.

Consider $\frac{\partial\Phi_2}{\partial w}$ (use (5.12) and $\Phi_2(v,w)=\Phi_1(w,v)$), it follows from $v\to\infty$ and $w\to\theta$ (from the last statement in the last paragraph) that

$$\frac{\partial\Phi_2}{\partial w}\to\frac{9+8\theta+2\theta^2}{4(2+\theta)^2}=\frac{5\theta+10}{10\theta+18}<1. \tag{5.24}$$

As $\frac{\partial\Phi_2}{\partial w}>0$, and by (5.24), for v large and w close to θ, there is $0<\delta<1$ such that

$$d_{n-1}=f(v_{n-1})-w_{n-1}=\int_{w_n}^{f(v_n)}\frac{\partial\Phi_2}{\partial w_n}dw<\delta(f(v_n)-w_n)=\delta d_n.$$

This proves (5.23), and completes the proof of part (iii). □

We remark that in the above proof of (iii), we consider $(v_0,w_0)\notin\overline{S}_\infty$ with $v_0>w_0$, and look at the reverse iteration $(v_n,w_n)=\Phi^{-n}(v_0,w_0)$, then from (5.23), we see after finitely many steps $\Phi^{-n}(v_0,w_0)$ will go outside the positive quadrant, hence for such initial condition, not even (A1) can be established.

In view of Remark 4 after Theorem 5.4, we have settled the existence of DF for all the cases $(v,w)\in\mathbb{R}^2_+$ except at $I=(0,1)^2$. However this part can also be resolved following our line of investigation. Indeed, if we consider Φ on $\mathbb{R}^2_+$, then $\Phi^n(\mathbb{R}^2_+)$ will shrink to $\mathcal{S}_\infty:=S_\infty\cup S^t_\infty\cup L$ where S^t_∞ is the reflection of S_∞ along the diagonal line $v=w$, and $\Phi^n(I)$ will converge to $\{(1,1)\}$. If we pick a $(v_0,w_0)\in I$, then $\Phi^{-n}(v_0,w_0)$ will eventually map outside $\mathbb{R}^2_+$ by a similar argument as in proof of (iii). (will the argument in (iii) work?. Please check (Yes, the proof of (iii) can deal the case in I)

6. An open question in Hambly and Kumagai [13]

In [13], the generalized d-dimensional ℓ-level SGs and the heat kernels of the "asymptotic lower-dimensional diffusions" were considered. The model was given conductances on G_0 that is 1 on the horizontal edges, and w on the slant edges, and by the homogeneity condition (A1), the conductances on the n-th iteration G_n is defined (see Figure 8), that is, the horizontal face of a subcell is a $(d-1)$-dim SG and has equal conductance on all its edges, and the conductance of the slant edges and the horizontal edges has ratio w and 1. This is the "one-parameter family" model (on w) in [13], extending the one in [16] on the 2-dim SG, which was proved in [11] that this is the only compatible resistance form on a SG (under the assumption of homogeneity in (A1)).

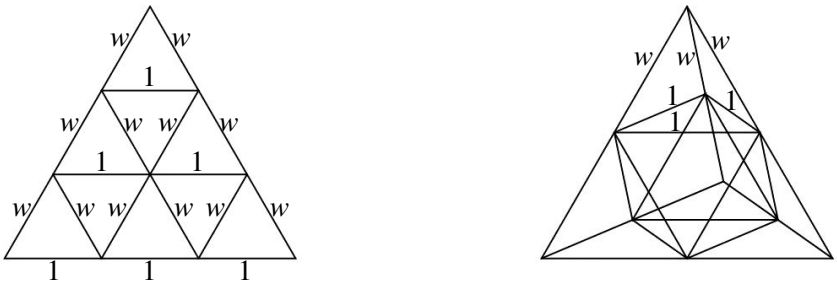


Figure 8. The d-dimensional SG's with ℓ-level, $(d,\ell)=(2,3)$ and $(3,2)$.

Denote this network by D_w, it was shown that one can write the trace map $T - J^t X^{-1} J = R(w)^{-1} D_{\alpha(w)}$, where $R(w)$ is the resistance scaling factor of the horizontal edges of the $(d-1)$-dim SG, and $\alpha(w)$ is the new conductance of edges connecting vertices of different levels. The map $\alpha : [0,1] \to [0,1]$ is a strictly increasing map with two fixed points 0 and 1, where the case $w = 1$ is corresponding to the standard resistance form on K, the existence of such form was proved in [22], [20].

In comparison with the construction in Section 5 with three types of conductances on the edges represented by (a_n, b_n, c_n). By factoring out b_n, we end up considering a "two-parameter family" $(v, w)(= (v, 1, w))$ associated with a map Φ. It is shown that the homogenous DFs are generated by the (v, w) on the curve Γ of the boundary of S_∞, which is represented as a function $f : [1, \infty] \to [\theta, 1]$ (Theorem 5.4(ii)). Note that this curve is remarkably resembled to the $\alpha(w)$, in that $f(1) = 1$ corresponds to $\alpha(1) = 1$, and $f(\infty) = \theta$ corresponds to $\alpha(0) = 0$.

To put the conductances of the model into the present notations, on each subcell in G_n, we let (a_n, b_n) denote the conductances of the horizontal edges of the $(d-1)$-dim SG, and the slant edges connecting each level respectively. It was estimated in [14, Lemma 3.11] that $a_n \asymp R_G^n$ where $R_G(> 1)$ is the reciprocal of the renormalization factor of the standard form on a $(d-1)$-dim SG. By factoring out a_n, the $\frac{b_n}{a_n}$ is related to the $\alpha(w)$ as follows: let $\alpha^{-1}(w)$ denote the inverse map of $\alpha(w)$. By using the reverse recursive construction, if we start at w, then $\frac{b_n}{a_n} = \alpha^{-n}(w)$. It was proved in [14, Lemma 3.9] that $\alpha^{-n}(w) \asymp \beta^n$ for some $\beta \in (0,1)$. Hence we have

$$a_n \asymp R_G^n, \qquad b_n \asymp (R_G\beta)^n. \tag{6.1}$$

In order that the compatible sequence of conductances defines a homogenous resistance form, it is necessary that $(R_G\beta) > 1$, which is condition (A2) in our setup. In [14, p.373], they make this as an assumption, and checked that this holds for cases of $d, \ell \le 3$; this assumption was used in their consequent developments. It remains to be answered:

Question: is $(R_G\beta) > 1$?

In the following, we give a positive answer to this question using some resistor network techniques. Moreover, we can identify the precise growth rate.

To this end, we fist give some notations. Let K be a d-dimensional SG with ℓ levels, $d \ge 2$ and $\ell \ge 2$. On V_1, for $1 \le i \le \ell$, let $\mathcal{N}(d, i)$ denote the number of horizontal $(d-1)$-dim SGs in the i-th level of V_1 (counting from the top downward), this equals the number of 1-cells in V_1 sitting on the i-th level. By convention, we take $\mathcal{N}(1, i) = 1$ for all $i \ge 1$. We see that for $d \ge 2$,

$$\mathcal{N}(d, 1) = 1, \quad \text{and} \quad \mathcal{N}(d, i) = \sum\nolimits_{j=1}^{i} \mathcal{N}(d-1, j), \qquad i \ge 1.$$

Let $H_i,\ 0 \le i \le \ell$ denote the vertices on the i-th level (H_0 is the top vertex of the d-dim SG), and denote by $x \sim y$ for $x \in H_{i-1}$, $y \in H_i$ if x and y are connected by a slant edge. Then it is seen that the number of such pairs of the two levels is $d\mathcal{N}(d, i)$.

Since we are fixing $d \ge 2$ in the following, we will write $\mathcal{N}_i := \mathcal{N}(d, i)$ for simplicity.

Proposition 6.1. *Let K be a d-dimensional ℓ-level SG, let $0 < w < 1$ and $(a_0, b_0) = (1, w)$ be the conductances on V_0. Then*

$$a_n \asymp R_G^n, \qquad b_n \asymp (R_G\beta)^n \asymp \left(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\right)^n.$$

It follows that $(R_G\beta) > 1$.

Proof. In view of (6.1), we only need to estimate b_n. By the homogeneity condition (A1), for simplicity, we consider network G_0 on V_0 with conductance (a_n, b_n), and G_1 on V_1 with conductance (a_{n+1}, b_{n+1}). By shorting the d points in V_0 contained in the bottom face, and by the compatibility, it is seen that the effective conductance between the top vertex P and the bottom in V_0 is $db_n =: C_n$ by the parallel law.

Recall the monotonicity law [7] says that if we decrease the resistance somewhere in the network, then the resistances of any two nodes will not increase. Hence the conductance C_{n+1} of G_1 becomes larger if we shorten the horizontal edges in each level of V_1 to a node, hence the trace on V_0 will have larger conductance than C_n, denote this effective conductance by C'_n. Then by using the parallel law, the resistance between any two levels H_{i-1}, H_i is $\frac{1}{b_{n+1}\cdot d\mathcal{N}_i}$, follow by the series law, the effective resistance on G_0 is $\sum_{i=1}^{\ell} \frac{1}{b_{n+1}\cdot d\mathcal{N}_i}$. Therefore we obtain

$$C'_n := db_{n+1}\left(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\right)^{-1}.$$

Hence by $C'_n \geq C_n$,

$$\frac{b_{n+1}}{b_n} \geq \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} \qquad n \geq 0. \tag{6.2}$$

Observe that $b_n = (b_n/b_{n-1})\cdots(b_2/b_1)\cdot b_1$, apply the above to $1 \leq k < n$, we have $b_n \geq b_0(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i})^n$. As $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} > \frac{1}{\mathcal{N}_1} = 1$, and $(R_G\beta)^n \asymp b_n$, we see that $R_G\beta \geq \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} > 1$. This confirms the Question in [14].

In the following we prove that $R_G\beta$ actually equals $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}$. For (a_{n+1}, b_{n+1}) on G_1, we consider the harmonic function u on V_1 with values 1 at the top vertex P and 0 at B_0, the d points in V_0 contained in the bottom face. Then the graph energy $E(u) = db_n$ by the compatibility. Consider a pair of horizontal edges x, y in a level H_i in G_1, we then have $a_{n+1}|u(x) - u(y)|^2 \leq E(u)$, and hence $|u(x) - u(y)|^2 \leq db_n/a_{n+1}$. As a consequence, $|u(x) - u(y)|^2$ is of order at most $O(\beta^n)$ for any $x, y \in H_i$.

For each $1 \leq i \leq \ell$, fix an $x_i \in H_i$. By the above estimate, it is direct to check that for $x \in H_{i-1}, y \in H_i$,

$$\begin{aligned} |u(x) - u(y)|^2 &= |u(x_{i-1}) - u(x_i) + (u(x) - u(x_{i-1}) + u(x_i) - u(y))|^2 \\ &\geq |u(x_{i-1}) - u(x_i)|^2 - |u(x_{i-1}) - u(x_i)| \cdot O(\beta^{n/2}) \end{aligned} \tag{6.3}$$

where $|O(\beta^{n/2})| \leq C\beta^{n/2}$ for some $C > 0$ (independent of n).

Write $E(u)$ into the summation of horizontal energies $E_i^{(h)}(u)$ on each layer H_i for $i = 1, \cdots, \ell$, and the slant energies $E_i^{(s)}(u)$ between two neighboring layers H_{i-1} and H_i for

$i = 1, \cdots, \ell$. Note that the numbers of the pairs $x \sim y$ for $x \in H_{i-1}$ and $y \in H_i$ is $d\mathcal{N}_i$. By using (6.3), we have,

$$\begin{aligned} db_n = E(u) &= \sum_{i=1}^{\ell} E_i^{(h)}(u) + \sum_{i=1}^{\ell} E_i^{(s)}(u) \\ &\geq \sum_{i=1}^{\ell} \sum_{\substack{x\in H_{i-1}, y\in H_i \\ x\sim y}} b_{n+1}(u(x) - u(y))^2 \\ &\geq \sum_{i=1}^{\ell} d\mathcal{N}_i \, b_{n+1}\Big(u(x_{i-1}) - u(x_i)\Big)^2 - b_{n+1} \cdot O'(\beta^{n/2}) \end{aligned} \tag{6.4}$$

where $|O'(\beta^{n/2})| \leq C'\beta^{n/2}$ for some $C' > 0$ independent of n.

Now we view $x_0(= P), x_1, \cdots, x_\ell$ as a one dimensional network with resistance $\frac{1}{\mathcal{N}_i}$ between two nodes, and the effective resistance on x_0 and x_ℓ is $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}$. By $u(x_0) = 1$, $u(x_\ell) = 1$, and the minimal energy property of harmonic extension, it follows that (treating $u(x_i)$'s as an extension of u),

$$\sum_{i=1}^{\ell} \mathcal{N}_i \, |u(x_{i-1}) - u(x_i)|^2 \geq \Big(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\Big)^{-1} =: Q^{-1}. \tag{6.5}$$

Also let $\varepsilon_n = C'd^{-1}Q\beta^{n/2}$, and substitute this into (6.4), we obtain

$$b_n \geq b_{n+1} \cdot Q^{-1}(1 - \varepsilon_n), \quad i.e. \quad \frac{b_{n+1}}{b_n} \leq Q(1 - \varepsilon_n)^{-1}.$$

Apply this to $1 \leq k < n$, and combine with (6.2), we have

$$Q^n b_0 \leq b_n \leq Q^n b_0 \prod_{k=0}^{n-1} (1 - \varepsilon_k)^{-1}.$$

It is standard to check that $\prod_{k=0}^{n-1}(1-\varepsilon_k)$ converge to a constant: $\prod_{k=0}^{n-1}(1-\varepsilon_k) = e^{\sum_{k=0}^{n-1} \log(1-\varepsilon_k)} \asymp e^{-\sum_{k=0}^{n-1} \varepsilon_k} \to e^{-c/(1-\beta^{1/2})}$ as $n \to \infty$ for some $c > 0$. Hence we have

$$b_n \asymp Q^n = \Big(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\Big)^n$$

as asserted. □

We then apply Theorem 4.6 to obtain the spectral asymptotics for this class of Dirichlet forms as following.

Corollary 6.2. *Let K be a d-dimensional level-ℓ SG with $d \geq 2$, $\ell \geq 2$. Let μ be the normalized Hausdorff measure on K and let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form of the homogeneous resistance form in Proposition 6.1. Let $\rho(t)$ be its eigenvalue counting function, then for t large enough,*

$$\rho(t) \asymp t^{\frac{\log \mathcal{N}(d+1,\ell)}{\log(\mathcal{N}(d+1,\ell)S(d,\ell))}},$$

where $S(d, \ell) = \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}(d,i)}$.

Proof. Given the conductances $\{(a_n, b_n)\}_{n\geq 0}$ for $(\mathcal{E}, \mathcal{F})$ on K, it is easy to check that conditions (A1), (A2) and (A3) are all satisfied. By Theorem 4.6 with $N = \mathcal{N}(d + 1, \ell)$ and $\kappa_0 = S(d, \ell)$, the estimate follows. □

In [13], they also consider another class of examples, the $(2k+1)$-Vicsek checkerboard (VC), where the conductance on the two diagonal is 1, and is w on the other edges containing the four vertices $V_0 = \{p_1, p_2, p_3, p_4\}$ of a unit square (see Figure 9).

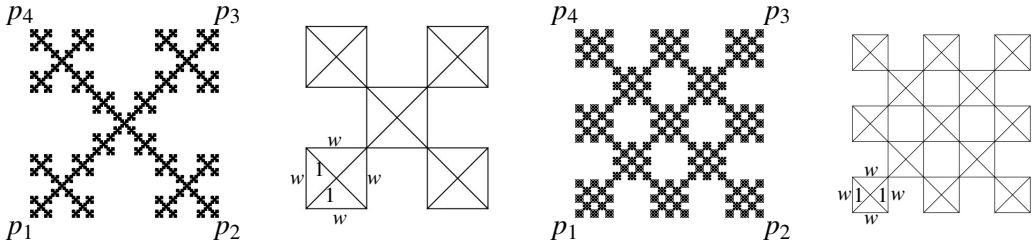


FIGURE 9. The $(2k+1)$-Vicsek checkerboard with $k = 1, 2$.

By using the same notations and strategy of proof as the above SG case, it was shown that [13, Lemma 3.9, Lemma 3.11]

$$a_n \asymp R_G^n = (2k+1)^n, \qquad b_n \asymp (R_G\beta)^n, \quad \beta \in (0,1),$$

where a_n is the conductance on the diagonal edges and b_n is the conductance on the rest edges. They also exhibited a few cases that $R_G\beta \le 1$. In the following, we prove

Proposition 6.3. *For the $(2k+1)$-VC, $R_G\beta \le 1$. Hence the compatible sequence of one-parameter family does not yield a homogeneous resistance form.*

Proof. On V_1 we put conductance (a_{n+1}, b_{n+1}), and by the compatibility, its trace on V_0 is (a_n, b_n). Let u be the harmonic function on V_1 such that $u(p_1) = u(p_3) = 0$ and $u(p_2) = u(p_4) = 1$, then it is clear that $E(u) = 4b_n$. By the monotonicity law, if we remove the cells that are not located on the two diagonals, then the new harmonic function, denoted by v, satisfies $E(v) \le E(u)$. By a direct computation, we have

$$E(v) = \frac{4(a_{n+1}+b_{n+1})b_{n+1}}{(a_{n+1}+b_{n+1}) + 4kb_{n+1}}. \tag{6.6}$$

By (6.6) and that $E(v) \le E(u) = 4b_n$, we obtain

$$\frac{b_{n+1}}{b_n} \le 1 + \frac{4kb_{n+1}}{a_{n+1}+b_{n+1}} = 1 + O(\beta^n).$$

This shows that $R_G\beta \le 1$, and hence the answer to the Question is negative for the Vicsek checkerboards. □

(Does it mean their consideration of one-dimension diffusion on VC do not work?)(Yes, you are right, their consideration on VCs does not work.)

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