DIRICHLET FORMS ON UNCONSTRAINED SIERPINSKI CARPETS

SHIPING CAO AND HUA QIU

Abstract. We construct symmetric self-similar Dirichlet forms on unconstrained Sierpinski carpets, which are natural extension of planner Sierpinski carpets by allowing the small cells to live off the $1/k$ grids. We also drop the non-diagonal assumption in this recurrent setting. The proof of the existence is purely analytic. A uniqueness theorem is also provided. Moreover, the additional freedom of unconstrained Sierpinski carpets allows us to slide the cells around. In this situation, we view unconstrained Sierpinski carpets as moving fractals, and we prove that the self-similar Dirichlet forms will vary continuously in a Γ-convergence sense.

1. Introduction

In this paper, we study the self-similar Dirichlet forms on the unconstrained Sierpinski carpets ($\mathcal{USC}$ for short). See Figure 1 for some examples.

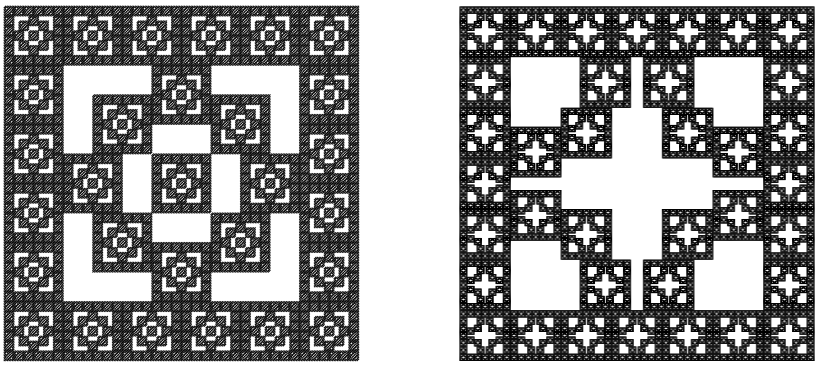


Figure 1. Unconstrained Sierpinski carpets ($\mathcal{USC}$).

To see the meaning of the adjective "*unconstrained*", let's first recall the definition of the classical Sierpinski carpets. Dividing the unit square into $k^2, k \geq 3$ identical small squares and deleting a few symmetrically (keeping all that boarding the boundary), we get an iterated function system (i.f.s. for short) consisting of similarities that map the unit square to a remaining small square. The attraction of the i.f.s is a Sierpinski carpet, as long as the fractal is connected. See Figure 2 for the standard Sierpinski carpet, where $k = 3$.

2010 *Mathematics Subject Classification.* Primary 28A80.

Key words and phrases. unconstrained Sierpinski carpets, Dirichlet forms, diffusions, self-similar sets.

The research of Qiu was supported by the National Natural Science Foundation of China, Grant 12071213.

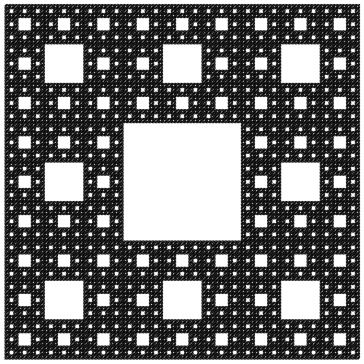


FIGURE 2. The standard Sierpinski carpet.

Clearly, all the squares are constrained on the $1/k$ grids in a classical Sierpinski carpet. The larger family of fractals, the unconstrained Sierpinski carpets, are exactly those self-similar sets generated in a similar manner, but the constraint removed by allowing the $1/k$ sized squares to move around. See Definition 2.1 for precise details. We will construct and study the diffusions on unconstrained Sierpinski carpets.

The study of diffusion processes on fractals emerged as an independent research field in the late 80's. Initial interest in such processes came from mathematical physicists working in the theory of disordered media [1, 24, 42]. On self-similar sets, the pioneering works are the constructions of Brownian motions on the Sierpinski gasket [19, 33] originated by Goldstein and Kusuoka independently and later [9] by Barlow and Perkins. The method features the study of a compatible sequence of graphs [30], and is extend to various fractals with finitely ramified cell structures [45]. See [10, 11, 13, 14, 17, 28, 29, 36, 37, 38, 40, 43, 45] and books [2, 30, 44] for the Dirichlet forms on such fractals.

The Sierpinski carpets are typical self-similar sets that are not finitely ramified. The construction of diffusions on Sierpinski carpets [3], initiated by Barlow and Bass, is a milestone in analysis on fractals.

In the pioneering work [3], the diffusion is defined to be a weak limit of renormalized Brownian motions on domains in $\mathbb{R}^2$, and the method later extends to generalized Sierpinski carpets (carpets in $\mathbb{R}^d, d \geq 3$) [7]. An alternative approach, with the language of Dirichlet forms, was given by Kusuoka and Zhou [34]. It remained a difficult question whether the diffusions resulted from the two methods are the same until 2010 [8]. See [4, 5, 6, 7] for more properties about the diffusions on Sierpinski carpets. Also see [20] for a recent work on the standard Sierpinski carpet based on the resistance estimate [5] and Γ-convergence [15, 39].

More or less, the constructions depend on delicate structure of the Sierpinski carpets. In [3], the non-diagonal condition is required since Brownian motion on $\mathbb{R}^2$ does not hit a single point. Also, the corner moves and slide moves [3, 7] depend on the exact way that cells intersect. In [20], the resistance estimate used transformations among different graphs, which essentially depends on the local structure of the fractal. A more general framework is provided in [34]. However, some extra conditions **(GB)**, **(LS)** and the Knight move method were used to fulfill the story.

Although it may not be possible to define a good diffusion on an arbitrary fractal, it is still one of the central question in this field to see how far we can loosen the geometric restrictions [2]. One of our main goal in this paper is to extend the constrcutions to the more general unconstrained Sierpinski carpets. In particular, we break the conditions **(GB)** and **(LS)** of [34]. Since we are dealing the recurrent case, the non-diagonal condition [3, 7, 8] is also dropped.

Theorem 1. *Let K be a $\mathcal{USC}$ and μ be the normalized Hausdorff measure on K. There is a unique (up to a constant multiplier) strongly local, regular, irreducible, symmetric, self-similar Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ such that $\mathcal{F}$ is compactly embedded in $L^2(K,\mu)$.*

Our approach to existence is purely analytic. We are inspired by [34], and adopt a few results there. To overcome the essential difficulty from the worse geometry, we develop a trace theorem. Let $\partial_n K$ be the collection of boundary of cells of side length k^{-n}, we use the trace theorem to estimate the energy of functions that are harmonic in $K\setminus\partial_n K$,

$$C_1\cdot[[h|_{\partial_n K}]]^2_{\Lambda^{\sigma(r)}_{2,2}(\partial_n K)}\le\mathcal{E}(h)\le C_2\cdot[[h|_{\partial_n K}]]^2_{\Lambda^{\sigma(r)}_{2,2}(\partial_n K)},\quad\forall h\in\mathcal{H}_n.$$

See Section 4 and Section 6 for details. Compared with [26], our story is on the opposite direction, we deduce important results including a uniform resistance estimate and an estimate of the energy measure on the boundary by the trace theorem. With all the preparation derived, the uniqueness is routine following the celebrated work [8].

The second goal, which also inspires the authors to break the constraint, is to study the family of $\mathcal{USC}$ as a whole. For example, one can perturbate the i.f.s. of the $\mathcal{USC}$ on the right hand side of Figure 1, and imagine this as squares sliding along edges. A natural question is whether or not the Dirichlet form varies continuously accordingly. We provide an answer to this in Section 9.

Theorem 2. *Let K, K_n, $n\ge1$ be $\mathcal{USC}$, and R, R_n, $n\ge1$ be the resistance metrics on them according to Theorem 1. Then $R_n\rightarrowtail R$ if and only if the geodesic metrics $d_{G,n}$ on K_n, $n\ge1$ are equicontinuous.*

The condition on the geodesic metric is essential since it prevents the creation of new shortest path for the diffusions. More precisely, we can not allow $x_n,y_n\in K_n$ that are bounded away in $d_{G,n}$, but get arbitrarily close in the Euclidean metric d. See Example 9.6 and Figure 7 for an illustration. The proof of Theorem 2 relies also on the trace theorem, as well as the uniqueness theorem.

At the end, we briefly introduce the structure of the paper. First, as the preliminary part, we introduce the precise definition and geometric properties of $\mathcal{USC}$ in Section 2. Later sections can be divided into two parts.

In Section 3-5 and also Appendix A,B, we focus on the existence of the self-similar Dirichlet forms. In Section 3, we introduce the Poincare constants from [34]. For convenience of readers, we slightly rewrite the proof of relationships between different Poincare constants in [34] in Appendix A; also, we provide a much shorter proof of the existence of a limit form of discrete forms, based on Γ-convergence in Appendix B. In Section 4, we use the idea of the trace theorem in the discrete situation to obtain an estimate of the discrete resistance metrics.

We use a quite different approach compared with [34], which is purely analytic. Finally, we construct the self-similar Dirichlet form in Section 5.

In Section 6-9, we prove the uniqueness and the continuity. The main tool to provide useful estimates is the trace theorem, developed in Section 6. In Section 7, we provide a uniform estimate of the resistance metrics with respect to the geodesic metrics. In Section 8, we prove the uniqueness of the self-similar Dirichlet forms following [8]. In Section 9, we prove Theorem 2. The key technique is to use the Γ-convergence developed by one of the author [10] (summarised in Appendix C), and to prove the energy measure on the boundary is 0, which is also an application of the trace theorem.

From time to time, we will write $a \asymp b$ for two functions (forms, and sequences) a, b, if there is a constant $C > 0$ so that $C^{-1} \cdot b \le a \le C \cdot b$.

2. The geometry of unconstrained Sierpinski carpets

In this section, we present a brief review of the geometric properties of $\mathcal{USC}$. Similar as the classical Sierpinski carpets, the $\mathcal{USC}$ are self-similar sets generated by removing small squares from an initial square and iterating.

Let $\square$ be a unit square in $\mathbb{R}^2$. We let

$$q_1 = (0,0), \quad q_1 = (1,0), \quad q_3 = (1,1), \quad q_4 = (0,1)$$

be the four vertices of $\square$, where we write $x = (x_1, x_2)$ for a point in $\mathbb{R}^2$. In addition, we will write

$$\overline{x, y} = \big\{(1-t)x + ty : 0 \le t \le 1\big\},$$

for the line segment connecting $x, y \in \mathbb{R}^2$.

For convenience, we denote the group of self-isometries on $\square$ by

$$\mathscr{G} = \big\{\Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}, id, \Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3}\big\},$$

where $\Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}$ are reflections (v for vertical, h for horizontal, d_1, d_2 for two diagonals),

$$\begin{aligned} &\Gamma_v(x_1, x_2) = (x_1, 1 - x_2), \qquad \Gamma_h(x_1, x_2) = (1 - x_1, x_2),\\ &\Gamma_{d_1}(x_1, x_2) = (x_2, x_1), \qquad \Gamma_{d_2}(x_1, x_2) = (1 - x_2, 1 - x_1), \end{aligned} \tag{2.1}$$

for $x = (x_1, x_2) \in \square$, id is the identity mapping, and $\Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3}$ are rotations

$$\Gamma_{r_1}(x_1, x_2) = (1 - x_2, x_1), \quad \Gamma_{r_2} = (\Gamma_{r_1})^2, \quad \Gamma_{r_3} = (\Gamma_{r_1})^3, \tag{2.2}$$

around the center of $\square$ counterclockwisely with angle $\frac{j\pi}{2}$, $j = 1, 2, 3$. In this paper, we will focus on the structures with $\mathscr{G}$-symmetry.

Definition 2.1 (Unconstrained Sierpinski carpets)**.**
Let $k \ge 3$ and $4(k-1) \le N \le k^2 - 1$. Let $\{F_i\}_{1 \le i \le N}$ be a finite set of similarities with contraction ratio k^{-1}. Assume the following holds:
(Nonoverlapping). *$F_i(\square) \cap F_j(\square)$ is either a line segment, or a point, or empty, $i \ne j$.*
(Connectivity). *$\bigcup_{i=1}^N F_i(\square)$ is connected.*
(Symmetry). *$\Gamma\big(\bigcup_{i=1}^N F_i(\square)\big) = \bigcup_{i=1}^N F_i(\square)$ for any $\Gamma \in \mathscr{G}$.*
(Boundary included). *$\overline{q_1, q_2} \subset \bigcup_{i=1}^N F_i(\square) \subset \square$.*

Then, call the unique compact subset $K \subset \square$ such that

$$K = \bigcup_{i=1}^{N} F_i K$$

an unconstrained Sierpinski carpet ($\mathcal{USC}$).

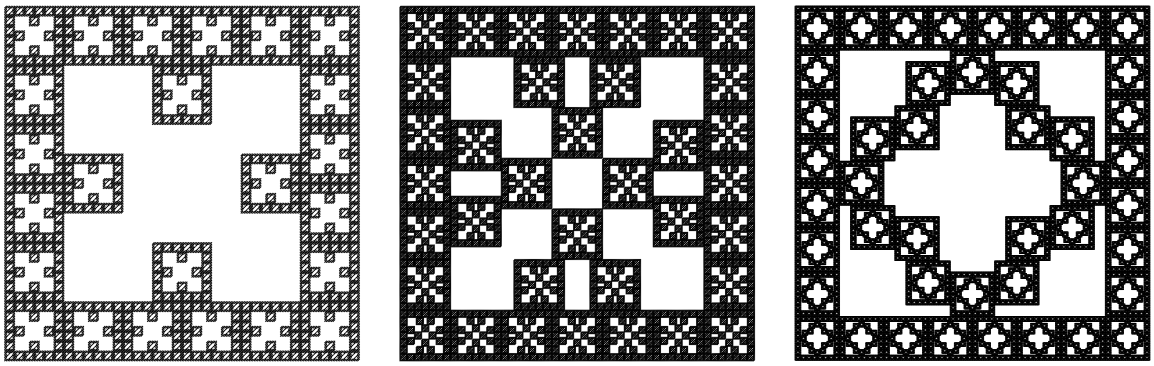


FIGURE 3. More unconstrained Sierpinski carpets ($\mathcal{USC}$).

See Figure 3 for more examples of $\mathcal{USC}$. Below we highlight some basic settings.

Basic settings and notations.

(1). Throughout the paper, k, N are fixed numbers. Most constants appear in Section 6 and Section 8 only depend on k.

(2). The squares $F_i(\square)$ along the boundary of $\square$ is useful, and we order them in the following way. For $0 \le j \le 3$ and $1 \le i \le k-1$, we stipulate that

$$F_{(k-1)j+i}(x) = \frac{1}{k}(x - q_{j+1}) + q_{j+1} + \frac{i-1}{k}(q_{j+2} - q_{j+1}) \tag{2.3}$$

with cyclic notation $q_5 = q_1$.

(3). For convenience, we denote the four sides of $\square$ by

$$L_1 = \overline{q_1, q_2}, \quad L_2 = \overline{q_2, q_3}, \quad L_3 = \overline{q_3, q_4}, \quad L_4 = \overline{q_4, q_1}. \tag{2.4}$$

(4). We write $\partial_0 K = \bigcup_{i=1}^{4} L_i$ for the square boundary of K.

In the rest of this section, we prove some geometric properties of $\mathcal{USC}$. Let's first introduce some more notations.

Definition 2.2. *Let $W_0 = \{\emptyset\}, W_1 = \{1, 2, \cdots, N\}$ and $W_n = W_1^n$ for $n \ge 1$. Also, write $W_* = \bigcup_{n=0}^{\infty} W_n$ for the space of finite words.*

(a). For each $w = w_1 \cdots w_n \in W_n$, we denote $|w| = n$ the length of w, and write

$$F_w = F_{w_1} \circ \cdots \circ F_{w_n}.$$

So each $w \in W_n$ represents a n-cell $F_w K$ in K.

(b). For $w, w' \in W_n$ with $n \geq 1$, we say $w \overset{n}{\sim} w'$ if $F_wK \cap F_{w'}K \neq \emptyset$. For $A, B \subset W_n$, we write $A \overset{n}{\sim} B$ if $w \overset{n}{\sim} w'$ for some $w \in A$ and $w' \in B$. In particular, we write $w \overset{n}{\sim} A$ if $\{w\} \overset{n}{\sim} A$.

(c). For $w = w_1 \cdots w_n \in W_n, w' = w'_1 \cdots w'_m \in W_m$, we write

$$w \cdot w' = w_1 \cdots w_n w'_1 \cdots w'_m \in W_{n+m}.$$

For $A \subset W_n, B \subset W_m$, we denote $A \cdot B = \{w \cdot w' : w \in A, w' \in B\}$.

(d). For $A \subset W_n$, we define a symmetric bilinear form $\mathcal{D}_{n,A}$ on $l(A)$ as,

$$\mathcal{D}_{n,A}(f,g) = \sum_{\substack{w,w' \in A \\ w \overset{n}{\sim} w'}} \big(f(w) - f(w')\big)\big(g(w) - g(w')\big), \quad \forall f, g \in l(A).$$

In particular, we write $\mathcal{D}_{n,A}(f) := \mathcal{D}_{n,A}(f,f)$, and $\mathcal{D}_n := \mathcal{D}_{n,W_n}$ for short.

(e). For $n \geq 0$, we define

$$\partial W_n = \big\{w \in W_n : F_wK \cap \partial_0 K \neq \emptyset\big\}.$$

Also, we write

$$W_{n,i} = \big\{w \in W_n : F_wK \cap L_i \neq \emptyset\big\}, \text{ for } i = 1, 2, 3, 4,$$

and $W_{*,i} = \bigcup_{n=0}^{\infty} W_{n,i}$.

In the following, we show that a $\mathcal{USC}$ K has the following geometric properties **(A1)**-**(A4)**. In particular, **(A1)**, **(A2)**, **(A4)** are the same as in [34], Section 2, while **(A3)** is relaxed for $\mathcal{USC}$.

(A1). *The open set condition: there exists a non-empty open set $U \subset \mathbb{R}^2$ such that $\bigcup_{i=1}^N F_iU \subset U$ and $F_iU \cap F_jU = \emptyset$, $\forall i \neq j \in \{1, \cdots, N\}$.*

(A2). *For $f \in l(W_n)$, $\mathcal{D}_n(f) = 0$ if and only if f is a constant function.*

(A3). *There is a constant $0 < c_0 < 1$ satisfying the following.*

(1). If $x, y \in K$ and $d(x,y) < c_0 k^{-n}$, then there exist $w, w', w'' \in W_n$ such that $x \in F_wK$, $y \in F_{w'}K$ and $F_wK \cap F_{w''}K \neq \emptyset$, $F_{w'}K \cap F_{w''}K \neq \emptyset$.

(2). If $w, w' \in W_n$ and there is no $w'' \in W_n$ so that

$$F_wK \cap F_{w''}K \neq \emptyset, \quad F_{w'}K \cap F_{w''}K \neq \emptyset.$$

Then $d(x,y) \geq c_0 k^{-n}$ for any $x \in F_wK$ and $y \in F_{w'}K$.

(A4). *$\partial W_n \neq W_n$ for $n \geq 2$.*

Remark 1. By **(A1)**, the Hausdorff dimension of K is $d_H = \frac{\log N}{\log k}$. Let μ be the normalized d_H-Hausdorff measure on K, i.e. μ is the unique self-similar measure on K such that $\mu = \frac{1}{N}\sum_{i=1}^N \mu \circ F_i^{-1}$.

Remark 2. In [34], **(A3)**-(2) was stated as:
If $w, w' \in W_n$ and $F_wK \cap F_{w'}K = \emptyset$, then $d(x,y) \geq c_0 k^{-n}$ for any $x \in F_wK$ and $y \in F_{w'}K$.

One can check **(A3)**-(2) does not hold if we allow two cells F_iK, F_jK, $i \neq j \in \{1, \cdots, N\}$ to have intersection length

$$|F_iK \cap F_jK| = \sum_{l=1}^{\infty} k^{-\frac{l(l+1)}{2}},$$

where $|F_iK \cap F_jK|$ denotes the length of $F_iK \cap F_jK$.

*Proof of **(A1)-(A4)**.*

(A1), **(A2)**, **(A4)** are obvious. It remains to verify **(A3)**.

(A3)-(1). Let

$$c_0' = \min\Big\{\frac{1}{2}, k\cdot\min\big\{d(F_iK, F_jK) : F_iK\cap F_jK = \emptyset, 1\le i,j\le N\big\}\Big\}.$$

We will show c_0' works for **(A3)**-(1). Let $x, y\in K$ with $d(x,y) < c_0'k^{-n}$ for some $n\ge 1$. Let

$$m_0 = 1+\min\big\{m\ge 0:\ \text{there is } \tau\in W_m \text{ such that } x,y\in F_\tau K\big\}.$$

We will see that either of the following two cases happens.

Case 1. There is $w\in W_n$ such that $x,y\in F_wK$.

Case 2. There are $w\neq w'\in W_{m_0}$ and $i,i'\in\{1,2,3,4\}$ such that

$$F_wK\cap F_{w'}K\neq\emptyset,$$

and

$$x\in\bigcup_{\tilde w\in L_{i,n-m_0}} F_{w\cdot\tilde w}K,\quad y\in\bigcup_{\tilde w\in L_{i',n-m_0}} F_{w'\cdot\tilde w}K.$$

In fact, if $n<m_0$, *Case 1* happens clearly. If $n\ge m_0$, there are $\tau\in W_{m_0-1}$ and $i\neq j$ so that $x\in F_{\tau\cdot i}K$, $y\in F_{\tau\cdot j}K$. Since $d(F_\tau^{-1}x, F_\tau^{-1}y) < c_0'k^{-n+(m_0-1)}\le c_0'k^{-1}$, by the definition of c_0', we have $F_iK\cap F_jK\neq\emptyset$, and so $F_{\tau\cdot i}K\cap F_{\tau\cdot j}K\neq\emptyset$. So *Case 2* happens by choosing $w=\tau\cdot i$ and $w'=\tau\cdot j$, noticing that $c_0'<1$.

Now if $n<m_0$, *Case 1* happens and **(A3)**-(1) holds immediately. Otherwise if $n\ge m_0$, *Case 2* happens, then we let v be the word in $w\cdot L_{i,n-m_0}$ such that $x\in F_vK$, and let v' be the word in $w'\cdot L_{i',n-m_0}$ such that $y\in F_{v'}K$. It is clear that if **(A3)**-(1) does not hold, then $d(x,y)\ge k^{-n} > c_0'k^{-n}$, a contradiction.

(A3)-(2). It is a consequence of **(A3)**-(1). In fact, let

$$c_0 = c_0'/2.$$

We can prove **(A3)**-(2) by contradiction. Let $w,w'\in W_n$, and assume there is no $w''\in W_n$ so that $F_wK\cap F_{w''}K\neq\emptyset, F_{w'}K\cap F_{w''}K\neq\emptyset$. If there are $x\in F_wK, y\in F_{w'}K$ so that $d(x,y) < c_0k^{-n}$, then we can find $x'\in F_w(K\setminus\partial_0K)$ and $y'\in F_{w'}(K\setminus\partial_0K)$ so that $d(x',y') < c_0'k^{-n}$. By **(A3)**-(1), we know there is $w''\in W_n$ satisfying $F_wK\cap F_{w''}K\neq\emptyset, F_{w'}K\cap F_{w''}K\neq\emptyset$. A contradiction. □

3. A limit form of Kusuoka and Zhou

In this section, we will introduce the celebrated results of Kusuoka and Zhou [34]. First, we define the *Poincare constants*, which was introduced in [34]. We modify the definition of R_n for the compatibility with the weaker version of **(A3)**-(2).

Definition 3.1 (Poincare constants [34])**.** *For $n\ge 0$, $A\subset W_n$ and $f\in l(A)$, we write*

$$[f]_A = (\#A)^{-1}\sum_{w\in A} f(w).$$

(a). For $n \geq 1$, define

$$\lambda_n = \sup\Big\{ \sum_{w\in W_n} \big(f(w) - [f]_{W_n}\big)^2 : f \in l(W_n), \mathcal{D}_n(f) = 1\Big\}.$$

(b). For $n \geq 1$, define

$$\lambda_n^{(D)} = \sup\big\{N^n [f]^2_{W_n} : f \in l(W_n), f|_{\partial W_n} = 0, \mathcal{D}_n(f) = 1\big\}.$$

(c). For $A, B \subset W_n$, define

$$R_n(A, B) = \max\big\{\big(\mathcal{D}_n(f)\big)^{-1} : f \in l(W_n), f|_A = 0, f|_B = 1\big\}.$$

For $w \in W_*$, we write

$$\mathcal{N}_w = \bigcup\big\{w' \in W_{|w|} : \text{there is } w'' \in W_{|w|} \text{ so that } F_wK \cap F_{w''}K \neq \emptyset, F_{w'}K \cap F_{w''}K \neq \emptyset\big\}.$$

Then, for $m \geq 1$, define

$$R_m = \inf\big\{R_{|w|+m}(w \cdot W_m, (W_{|w|} \setminus \mathcal{N}_w) \cdot W_m) : w \in W_* \setminus \{\emptyset\}\big\}.$$

(d). For $m \geq 1$ and $w \overset{n}{\sim} w'$, define

$$\sigma_m(w, w') = \sup\big\{N^m\big([f]_{w\cdot W_m} - [f]_{w'\cdot W_m}\big)^2 : f \in l(W_{|w|+m}), \mathcal{D}_{|w|+m,\{w,w'\}\cdot W_m}(f) = 1\big\}.$$

For $m \geq 1$, define

$$\sigma_m = \sup\big\{\sigma_m(w, w') : n \geq 1, w, w' \in W_n, w \overset{n}{\sim} w'\big\}.$$

One of the important results in [34], based on the conditions **(A1)**-**(A4)**, is the comparison of the Poincare constants. Below, we copy the theorem ([34], Theorem 2.1) for convenience of readers.

Proposition 3.2 ([34], Theorem 2.1)**.** *Assume **(A1)**-**(A4)**. There is a constant $C > 0$ such that*

$$\begin{cases} \lambda_n^{(D)} \leq C \cdot \lambda_n, & n \geq 1,\\ N^m R_m \lambda_n \leq C \cdot \lambda_{n+m}, & n, m \geq 1,\\ \lambda_{n+m} \leq C \cdot \lambda_n \sigma_m, & n, m \geq 1,\\ \sigma_{n+m} \leq C \cdot \sigma_n \sigma_m, & n, m \geq 1,\\ (k^2 N^{-1})^n \leq C \cdot R_n, & n \geq 1. \end{cases}$$

In particular, in our setting, **(A3)**-(2) is a little weaker than the original one in [34]. However, the proof in [34] still works perfectly. For convenience of readers, in Appendix A, we rewrite the proof of the inequality $N^m R_m \lambda_n \leq C \cdot \lambda_{n+m}$. The other inequalities are less or not affected by the change of **(A3)**.

On the other hand, to get a self-similar form on K, we would expect $N^{-n}\lambda_n \asymp r^{-n}$ for some $r > 0$. Proposition 3.2 is still not enough for this purpose. We need to investigate the geometry of the fractals more carefully for the following two inequalities.

(B1). *There is a constant $C > 0$ such that $\sigma_n \leq C \cdot \lambda_n^{(D)}$, for $n \geq 1$.*
(B2). *There is a constant $C > 0$ such that $\lambda_n \leq C \cdot N^n R_n$, for $n \geq 1$.*

In the following Lemma 3.3, 3.4, we list some important consequences of **(B1)** and **(B2)**.

Lemma 3.3. *Assume **(A1)-(A4)** and **(B1)**, then there are $C_1, C_2 > 0$ such that*

$$C_1 \cdot \lambda_n^{(D)} \le \lambda_n \le C_2 \cdot \lambda_n^{(D)}, \quad \forall n \ge 1.$$

Lemma 3.4. *Assume **(A1)-(A4)** and **(B1)-(B2)**, then there are $r > 0$ and $C_1, C_2 > 0$ such that*

$$C_1 \cdot r^{-n} \le N^{-n}\lambda_n \le C_2 \cdot r^{-n}, \quad \forall n \ge 1.$$

In addition, $r \le \frac{N}{k^2}$.

Proof. It suffices to show that there are constants $C_3, C_4 > 0$ so that

$$C_3 \cdot \lambda_n \lambda_m \le \lambda_{n+m} \le C_4 \cdot \lambda_n \lambda_m$$

for any $n, m \ge 1$.

First, by Proposition 3.2 and **(B1)**, we have, for some constant $C > 0$, $\sigma_m \le C \cdot \lambda_m^{(D)} \le C^2 \cdot \lambda_m$, and thus $\lambda_{n+m} \le C \cdot \lambda_n \sigma_m \le C^3 \cdot \lambda_n \lambda_m$.

Second, by the second inequality in Proposition 3.2 and **(B2)**, we know, for some constant $C > 0$, $\lambda_{n+m} \ge C \cdot N^m R_m \lambda_n \ge C^2 \cdot \lambda_n \lambda_m$.

Finally, by the second and last inequalities of Proposition 3.2, we know, for some constant $C > 0$, $N^{-m}\lambda_m \ge C \cdot R_m \ge C^2 \cdot (k^2 N^{-1})^m$. So $r \le \frac{N}{k^2}$. □

We call r the *renormalization factor* in this paper. For $\mathcal{USC}$, we always have

$$r \le \frac{N}{k^2} \le 1 - \frac{1}{k^2}.$$

In [34], it is shown that a limit form of the discrete energies $\mathcal{D}_n$ can be well defined under the above assumptions. We combine Theorem 5.4 and Theorem 7.2 in [34] together into the following Theorem 3.5.

In the following, with some abuse of notations, for $f \in L^1(K, \mu)$, we write $\mathcal{D}_n(f) = \mathcal{D}_n(P_n f)$, where $P_n : L^1(K, \mu) \to l(W_n)$ is defined as $(P_n f)(w) = N^n \int_{F_w K} f(x)\mu(dx), \forall w \in W_n$.

Theorem 3.5. *Assume **(A1)-(A4),(B1)-(B2)** and $0 < r < 1$. Let*

$$\mathcal{F} = \{f \in L^2(K, \mu) : \sup_{n \ge 1} r^{-n}\mathcal{D}_n(f) < \infty\}.$$

(a). For any $\beta \in (0, -\frac{\log r}{\log k})$, there is a constant $C > 0$ such that

$$|f(x) - f(y)|^2 \le C \cdot d(x, y)^\beta \big(\sup_{n \ge 1} r^{-n}\mathcal{D}_n(f)\big), \quad \forall x, y \in K, \forall f \in \mathcal{F}.$$

In particular, $\mathcal{F} \subset C(K)$.

(b). There are a regular symmetric Dirichlet form $(\bar{\mathcal{E}}, \mathcal{F})$, $C_1, C_2 > 0$ such that

$$C_1 \cdot \sup_{n \ge 1} r^{-n}\mathcal{D}_n(f) \le \bar{\mathcal{E}}(f) \le C_2 \cdot \liminf_{n \to \infty} r^{-n}\mathcal{D}_n(f), \quad \forall f \in \mathcal{F}.$$

Theorem 3.5 (b) was proved with a probabilistic approach in [34]. It is also possible to prove it directly by the Γ-convergence method, when we have the condition $r < 1$. See Appendix B for a short proof of Theorem 3.5.

It remains to verify **(B1)**,**(B2)** in the $\mathcal{USC}$ setting. **(B1)** is easy, while **(B2)**, which is essentially linked with the regularity of the form is hard. In [34], the method of "Knight

moves" of Barlow and Bass [] is applied to show **(B2)**. However, the same idea fails essentially on $\mathcal{USC}$. We will develop a method that does not depend on "Knight moves", or on the probabilistic consideration, to prove **(B2)**. The method will involve a rough idea of the trace theorem, which will be stated in detail in Section 6.

In the rest of this section, we will verify **(B1)**. The new method to prove **(B2)** will be postponed to the next section.

3.1. **Verification of (B1).** Since we allow two cells to intersect at a single point, we will take the advantage that we are just focusing on 2-dimensional carpets (which is essential since we need the process hinting points). In the following lemma, we will use the facts that $\liminf_{n\to\infty} \frac{1}{n}\log R_n \geq 2\log k - \log N > 0$ and $N^{-n}\lambda_n \geq C\cdot N^{-l}\lambda_l R_{n-l}$ for some $C>0$, which are provided by Proposition 3.2.

Lemma 3.6. *There is $C>0$ such that $\big|f(w)-[f]_{W_n}\big|^2 \leq C\cdot N^{-n}\lambda_n \mathcal{D}_n(f)$ for any $n\geq 1, w\in W_n$ and $f\in l(W_n)$.*

Proof. Let $w=w_1\cdots w_n$ be the word appears in the lemma. Let $B_n=W_n$ and $B_l = w_1\cdots w_{n-l}\cdot W_l$ for $0\leq l\leq n-1$. By applying the definition of λ_l, we have

$$\begin{aligned}\big|[f]_{B_{l-1}}-[f]_{B_l}\big|^2 &\leq N^{-l+1}\sum_{w'\in B_{l-1}}\big(f(w')-[f]_{B_l}\big)^2\\ &\leq N^{-l+1}\lambda_l\mathcal{D}_n(f).\end{aligned}$$

Thus, $\big|f(w)-[f]_{W_n}\big| \leq \sum_{l=1}^n \sqrt{N^{-l+1}\lambda_l}\cdot\sqrt{\mathcal{D}_n(f)} \leq C\cdot\sqrt{N^{-n}\lambda_n\mathcal{D}_n(f)}$. The lemma then follows from Proposition 3.2. □

Now we proceed to prove **(B1)**. For convenience, for a function $f\in l(W_{n+m})$ and $w\in W_n$, we use the notation $f\circ F_w$ to denote the function in $l(W_m)$ with $f\circ F_w(\tau)=f(w\cdot\tau)$.

*Proof of **(B1)**.* Let $m\geq 1$ and choose $n\geq 1$, $w,w'\in W_n$ such that $w\overset{n}{\sim}w'$ and $\sigma_m(w,w')>\sigma_m/2$. Let f be the function on $A=\{w,w'\}\cdot W_m$ such that $\mathcal{D}_{n+m,A}(f)=1$, and

$$\big([f]_{w\cdot W_m}-[f]_{w'\cdot W_m}\big)^2 = N^{-m}\sigma_m(w,w').$$

Let $R:A\to A$ be the symmetry with respect to the middle point of $F_wK\cap F_{w'}K$. By symmetry, we can assume that $f=-f\circ R$ and $f|_{w\cdot W_m}\geq 0$, so that

$$[f]_{w\cdot W_m}=\frac{1}{2}\sqrt{N^{-m}\sigma_m(w,w')}.$$

Pick $\eta\in w\cdot W_m$ and $\eta'\in w'\cdot W_m$ so that $\eta\overset{n+m}{\sim}\eta'$ and $\eta=R(\eta')$. Clearly, we have $|f(\eta)-f(\eta')|\leq 1$ and so $f(\eta)\leq\frac{1}{2}$. Next, by Proposition 3.2 and Lemma 3.6, for m large enough, we can find l independent of m such that

$$\big|[f]_{\iota\cdot W_{m-l}}-f(\eta)\big| < \frac{1}{4}[f]_{w\cdot W_m},$$

where ι is the unique word in W_{n+l} such that $\eta\in\iota\cdot W_{m-l}$. Thus by choosing m large,

$$[f]_{\iota\cdot W_{m-l}} < \frac{1}{2}[f]_{w\cdot W_m},$$

noting that by Proposition 3.2, $N^{-m}\sigma_m\geq C\cdot R_m\to\infty$ for some constant $C>0$.

Now we consider two cases based on $[f]_\tau$, $\tau \in w \cdot \partial W_l$.

Case 1. $\max_{\tau \in w\cdot\partial W_l}[f]_{\tau\cdot W_{m-l}} < \frac{1}{2}[f]_{w\cdot W_m}$.

In this case, by a same reason as above, we have $f(\kappa) < \frac{3}{4}[f]_{w\cdot W_m}$ for all $\kappa \in w\cdot\partial W_l\cdot W_{m-l}$. Let u be the function such that $u|_{w\cdot\partial W_l\cdot W_{m-l}} = f|_{w\cdot\partial W_l\cdot W_{m-l}}$ and

$$\mathcal{D}_{n+m,w\cdot W_m}(u) = \inf\{\mathcal{D}_{n+m,w\cdot W_m}(u') : u'|_{w\cdot\partial W_l\cdot W_{m-l}} = f|_{w\cdot\partial W_l\cdot W_{m-l}}\}.$$

In other words u is the harmonic extension of $f|_{w\cdot\partial W_l\cdot W_{m-l}}$ onto $w\cdot W_m$. Let $\tilde{f} = (f-u)\circ F_w$, then one can see that $\tilde{f}|_{\partial W_m} = 0$, $[\tilde{f}]_{W_m} \geq \frac{1}{4}[f]_{w\cdot W_m} = \frac{1}{8}\sqrt{N^{-m}\sigma_m(w,w')}$ and $\mathcal{D}_m(\tilde{f}) \leq \frac{1}{2}$. This implies that $\lambda_m^{(D)} \geq C\cdot\sigma_m$ for some constant $C > 0$.

Case 2. $\max_{\tau \in w\cdot\partial W_l}[f]_{\tau\cdot W_{m-l}} \geq \frac{1}{2}[f]_{w\cdot W_m}$.

In this case, one can find $C_1 > 0$ (depending only on k, l) and $\iota_1, \iota_2 \in w\cdot\partial W_l$ such that $\iota_1 \overset{n+l}{\sim} \iota_2$ and $\left|[f]_{\iota_1\cdot W_{m-l}} - [f]_{\iota_2\cdot W_{m-l}}\right| > C_1\cdot\sqrt{N^{-m}\sigma_m(w,w')}$. As a consequence, one can see that there are $C_2 > 0$ depending only on l, and a pair $\gamma, \gamma' \in \partial W_l$ with $\gamma \overset{l}{\sim} \gamma'$, such that

$$\sigma_{m-l}(\gamma,\gamma') \geq C_2\cdot\sigma_m.$$

So we reach the standard case that is dealt with in Section 8 of [34].

Without loss of generality, we let $\gamma = 1111\cdots1$ and $\gamma' = 1111\cdots1(4k-4)$. So $F_{\gamma'}K$ is located above $F_\gamma K$, and share a common side with $F_\gamma K$. Let g be the unique function on $\{\gamma,\gamma'\}\cdot W_{m-l}$ so that $\mathcal{D}_{\{\gamma,\gamma'\}\cdot W_{m-l}}(g) = 1$ and $\left([g]_{\gamma\cdot W_{m-l}} - [g]_{\gamma'\cdot W_{m-l}}\right)^2 = N^{-m+l}\sigma_{m-l}(\gamma,\gamma')$. Still, we assume g is antisymmetric with respect to $F_\gamma K \cap F_{\gamma'}K$, symmetric with respect to the bisection of $F_\gamma K \cap F_{\gamma'}K$, and $g|_{\gamma\cdot W_{m-l}} \geq 0$, so that $[g]_{\gamma\cdot W_{m-l}} = \frac{1}{2}\sqrt{N^{-m+l}\sigma_{m-l}(\gamma,\gamma')}$.

Let $g_+ = g\circ F_\gamma \in l(W_{m-l})$. We will construct a function $h \in l(W_{m-l+3})$, $h|_{\partial W_{m-l+3}} = 0$ by using the copies of g_+. Recall the symmetries $\Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}$ defined in (2.1), which naturally act on W_{m-l}. We define $g_{d_1}, g_{d_2} \in l(W_{m-l})$ as follows

$$g_{d_1}(w) = \begin{cases} g_+(w), & \text{if } F_wK \cap \{(x_1,x_2) : x_2 \geq x_1\} \neq \emptyset, \\ g_+\circ\Gamma_{d_1}(w), & \text{if } F_wK \cap \{(x_1,x_2) : x_2 \geq x_1\} = \emptyset. \end{cases}$$

$$g_{d_2}(w) = \begin{cases} g_+(w), & \text{if } F_wK \cap \{(x_1,x_2) : x_2 \geq 1-x_1\} \neq \emptyset, \\ g_+\circ\Gamma_{d_2}(w), & \text{if } F_wK \cap \{(x_1,x_2) : x_2 \geq 1-x_1\} = \emptyset. \end{cases}$$

Next, we define a function $h' \in W_{m-l+1}$ as follows,

$$h'(w) = \begin{cases} g_{d_2}(w_2\cdots w_{m-l+1}), & \text{if } w_1 = 1, \\ g_+(w_2\cdots w_{m-l+1}), & \text{if } 2 \leq w_1 \leq k-1, \\ g_{d_1}(w_2\cdots w_{m-l+1}), & \text{if } w_1 = k, \\ 0, & \text{otherwise,} \end{cases}$$

where $w = w_1w_2\cdots w_{m-l+1}$ in the above formula. Finally, we define $h \in l(W_{m-l+3})$ as follows:

$$h(w) = \begin{cases} h'\circ\Gamma_h(w_3\cdots w_{m-l+3}), & \text{if } w_1 = 1, w_2 = k, \\ h'(w_3\cdots w_{m-l+3}), & \text{if } w_1 = 1, w_2 = k+1. \\ 0, & \text{otherwise.} \end{cases}$$

One can easily see that $h|_{\partial W_{m-l+3}} = 0$, $[h]_{W_{m-l+3}} \geq C_3 \cdot \sqrt{N^{-m+l}\sigma_{m-l}(\gamma,\gamma')}$ and $\mathcal{D}_{m-l+3}(h) \leq C_4$ for some $C_3, C_4 > 0$ independent of m. This implies that $\lambda_m^{(D)} \geq C_5 \cdot \lambda_{m-l+3}^{(D)} \geq C_6 \cdot \sigma_m$ for some $C_5, C_6 > 0$. □

4. An extension algorithm and its application to **(B2)**

The essential difficulty of **(B2)** for $\mathcal{USC}$ is caused by the unconstrained style of neighbourhood $\mathcal{N}_w$, $w \in W_*$, which may be of infinitely many different types. Recall that in [], the condition **(B2)** is verified by the "Knight moves" method due to Barlow and Bass [] for a limited range of local symmetric fractals including the standard Sierpinski carpets. However, there are very limited results about slide moves and corner moves in the $\mathcal{USC}$ setting, when the process travels near the intersection of cells. Also, we can not construct functions of controllable energy simply by only using symmetry as before.

We will invent a new technique to construct functions with nice boundary condition, for both the discrete case and the continuous case. The argument in this section will be purely analytic, and will not involve the "Knight moves" method in [].

We start this section with an introduction to the technique, and leave some estimates to the last part of this section, from which the condition **(B2)** follows as an application.

4.1. **Building bricks.**

We will break the fractal K near boundary into infinitely many pieces (building bricks), and glue functions on building bricks together to arrive desired boundary values.

Definition 4.1. *Let K be a $\mathcal{USC}$.*
(a). For $m \geq 1$, we define $T_{m,K} = \bigcup_{w\in W_{m,1}} F_w K$.
(b). Let $B_K = cl\big(T_{1,K} \setminus T_{2,K}\big)$.

For each $w \in W_{*,1} = \bigcup_{n=0}^{\infty} W_{n,1}$, the set $F_w B_K$ is called a level-$|w|$ *building brick*. We can decompose $T_{1,K}$ as the union of infinitely many bricks, or of lower wall $T_{m,K}$ and finitely many bricks,

$$T_{1,K} = \Big(\bigcup_{w\in W_{*,1}} F_w B_K\Big)\bigcup L_1 = \Big(\bigcup_{n=0}^{m-2}\bigcup_{w\in W_{n,1}} F_w B_K\Big)\bigcup T_{m,K}, \quad \text{for } m \geq 2. \tag{4.1}$$

See Figure 4 for an illustration. In this section, since we are considering the discrete energies $\mathcal{D}_n$ on cell graphs, we focus on the finite decomposition of $T_{1,K}$. We will consider the infinite decomposition of $T_{1,k}$ in the continuous case in Section 6.

We introduce some new notations for cell graph case.

Definition 4.2. *Let K be a $\mathcal{USC}$.*
(a). For $m \geq 1$ and $n \geq m$, we define $\bar{T}_{m,n} = \{w \in W_n : F_w K \subset T_{m,K}\}$.
(b). For $n \geq 2$, we define $\bar{B}_n = \{w \in W_n : F_w K \subset B_K\}$.

We can rewrite the second decomposition in (4.1) as

$$\bar{T}_{1,n} = \Big(\bigcup_{l=0}^{n-2}\bigcup_{w\in W_{l,1}} w\cdot \bar{B}_{n-l}\Big)\bigcup \bar{T}_{n,n}, \qquad n \geq 2. \tag{4.2}$$

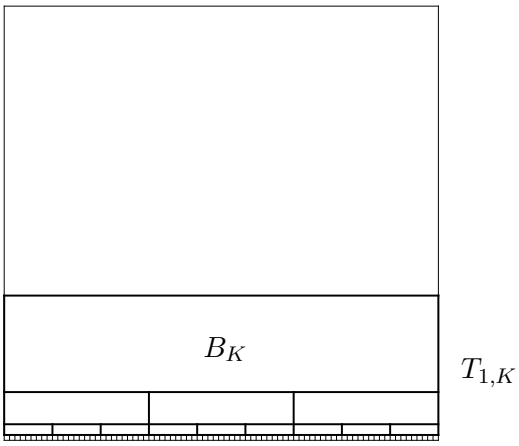


FIGURE 4. A finite decomposition of $T_{1,K}$ with $m = 4$.

4.2. **Functions with linear boundary values.** We will use the decomposition (4.2) to construct functions in $l(W_n)$, with good boundary condition on ∂W_n, $n \geq 2$.

To start with, let's define a *building brick function* b_n on $\bar{B}_n, n \geq 2$. This takes several steps.

Step b.1. Let h_n be the unique function on W_n such that

$$\begin{cases} h_n|_{W_{n,4}} = 0, \quad h_n|_{W_{n,2}} = 1, \\ \mathcal{D}_n(h_n) = \min\left\{\mathcal{D}_n(f) : f|_{W_{n,4}} = 0, f|_{W_{n,2}} = 1, f \in l(W_n)\right\}. \end{cases}$$

Let $A_n = \{w \in W_n : w \overset{n}{\sim} \bar{T}_{1,n}\}$, h'_n be the unique function on W_n such that

$$\begin{cases} h'_n|_{W_{n,3}} = 1, \quad h'_n|_{A_n} = 0, \\ \mathcal{D}_n(h'_n) = \min\left\{\mathcal{D}_n(f) : f|_{W_{n,3}} = 1, f|_{A_n} = 0, f \in l(W_n)\right\}. \end{cases}$$

Step b.2. Next, we define $b_n^{(1)} \in l(\bar{B}_n)$ and $b_n^{(2)} \in l(W_n \setminus \bar{T}_{1,n})$ as

$$b_n^{(1)}(i \cdot w) = h'_{n-1}(w) \cdot h_n(i \cdot w), \quad \forall 1 \leq i \leq k, \forall w \in W_{n-1} \setminus \bar{T}_{1,n-1},$$

and

$$b_n^{(2)}(w) = (1 - h'_n(w)) \cdot h_n(w), \quad \forall w \in W_n \setminus \bar{T}_{1,n}.$$

Step b.3. Lastly, we define $b_n \in l(\bar{B}_n)$ as

$$b_n(i \cdot w) = b_n^{(1)}(i \cdot w) + \frac{1}{k} b_{n-1}^{(2)}(w) + \frac{i-1}{k}, \quad \forall 1 \leq i \leq k, \forall w \in W_{n-1} \setminus \bar{T}_{1,n-1}.$$

We present some important observations of b_n in the following lemma.

Lemma 4.3. *Let $n \geq 2$ and $b_n \in l(\bar{B}_n)$ be the function defined in Step b.1-b.3.*
~~(a). $b_n|_{(i \cdot W_{n-1,4}) \cap \bar{B}_n} = \frac{i-1}{k}$ and $b_n|_{(i \cdot W_{n-1,2}) \cap \bar{B}_n} = \frac{i}{k}$, $\forall 1 \leq i \leq k$.~~
(a). $b_n|_{(1 \cdot W_{n-1,4}) \cap \bar{B}_n} = 0$, $b_n|_{(k \cdot W_{n-1,2}) \cap \bar{B}_n} = 1$.
(b). $b_n(w) = h_n(w)$, $\forall w \in W_{1,1} \cdot W_{n-1,3}$.

(c). $b_n(i\cdot w)=\frac{i-1}{k}+\frac{1}{k}h_{n-1}(w)$, $\forall 1\le i\le k, \forall w\in A_{n-1}\setminus \bar{T}_{1,n-1}$.
(d). $\mathcal{D}_{n,\bar{B}_n}(b_n)\le C\cdot\big(\mathcal{D}_n(h_n)+\mathcal{D}_{n-1}(h'_{n-1})\big)$, *where* $C>0$ *is independent of* n.

In particular, Lemma 4.3 (a) shows that we can glue small copies of b_n horizontally without introducing across energy. Lemma 4.3 (b),(c) imply that we can glue b_n with scaling copies of b_{n-1} vertically still without introducing across energy. ~~by adding a little energy (controlled by the energy of h_n, h'_{n-1}). ?~~

Next, we make a function that is linear on ∂W_n with controllable energy. For convenience, for each $f\in L^1(K,\mu)$ and $n\ge 1$, we define $P_{\partial_n}f\in l(\partial W_n)$ as the unique function such that

$$(P_{\partial_n}f)(w)=N^n\int_{F_wK}f(x)\mu(dx),\quad \forall w\in\partial W_n.$$

Lemma 4.4. *Let* $u\in l(K)$ *defined by* $u(x_1,x_2)=x_1$, $\forall x=(x_1,x_2)\in K$. *Then there is* $f_n\in l(W_n)$ *such that* $f_n|_{\partial W_n}=P_{\partial_n}u$ *and*

$$\mathcal{D}_n(f_n)\le C\cdot\big(k^{-n}+\sum_{l=0}^{n-2}k^{-l}\mathcal{D}_{n-l,\bar{B}_{n-l}}(b_{n-l})\big),\tag{4.3}$$

for some constant $C>0$ *independent of* $n\ge 2$.

Proof. First we consider $u'\in l(K)$ defined by $u'(x_1,x_2)=\frac{k^n}{k^n-1}(x_1-\frac{1}{2k^n})$ instead of u. Then we have $(P_{\partial_n}u')(w)=\frac{e(w)}{k^n-1}$, where $e(w)=\sum_{j=1}^n k^{n-j}(w_j-1)$ for $w=w_1\cdots w_n\in W_{n,1}$. In particular, $(P_{\partial_n}u')(1\cdots 1)=0$ and $(P_{\partial_n}u')(k\cdots k)=1$. Now we construct a function $g_n\in l(W_n)$ such that $g_n|_{\partial W_n}=P_{\partial_n}(u')$ by the following two steps.

Step 1. We first define g'_n *on* $\bar{T}_{1,n}$. Let

$$g'_n=P_{\partial_n}(u')\cdot 1|_{T_{n,n}}+\sum_{l=0}^{n-2}\sum_{w\in W_{l,1}}\frac{1}{k^l}\big(e(w)+b_{n-l}\circ F_w^{-1}\big)\cdot 1_{w\cdot\bar{B}_{n-l}},$$

where by a little abuse of the notation, $b_{n-l}\circ F_w^{-1}$ is the function supported on $w\cdot\bar{B}_{n-l}$ such that $b_{n-l}\circ F_w^{-1}(w\cdot\eta)=b_{n-l}(\eta)$ for any $\eta\in\bar{B}_{n-l}$.

Step 2. We use symmetry and h_n *in* ***Step b.1*** *to extend* g'_n *to* $g_n\in l(W_n)$. Recall the reflection $\Gamma_v(x_1,x_2)=(x_1,1-x_2)$ which naturally acts on W_n. We define

$$g_n(w)=\begin{cases}g'_n(w), & \text{if } w\in\bar{T}_{1,n},\\ g'_n\circ\Gamma_v(w), & \text{if } w\in\Gamma_v\bar{T}_{1,n},\\ h_n(w), & \text{otherwise.}\end{cases}$$

Clear, g_n satisfies the boundary condition $g_n|_{\partial W_n}=P_{\partial_n}(u')$.

It suffices to define $f_n=\frac{k^n-1}{k^n}g_n+\frac{1}{2k^n}$ such that $f_n|_{\partial W_n}=P_{\partial_n}u$. The energy estimate (4.3) of f_n follows from Lemma 4.3. □

In the next subsection, we will show the following two estimates.

$$\limsup_{n\to\infty}\tfrac{1}{n}\log(N^{-n}\lambda_n)\le\log k-\log 2,\tag{4.4}$$

$$\mathcal{D}_n(h_n)\le C\cdot N^n\lambda_n^{-1}\tag{4.5}$$

for some $C > 0$ independent of n. With these estimates, we have a more clear explanation of (4.3).

Corollary 4.5. *Assume (4.4) and (4.5) hold. Let f_n be the same as in Lemma 4.4. Then*

$$\mathcal{D}_n(f_n) \le C \cdot N^n \lambda_n^{-1} \tag{4.6}$$

for some constant $C > 0$ independent of n.

Lemma 4.4 and Corollary 4.5 are enough for us to verify **(B2)**.

Proposition 4.6. *Assume (4.4) and (4.5) hold. Then **(B2)** holds.*

Proof. Let $n, m \ge 1$ and $w \in W_m$. We need to estimate $R_{m+n}\big(w \cdot W_n, (W_m \setminus \mathcal{N}_w) \cdot W_n\big)$.

Let c_0 be the constant in **(A3)**, and let $p = (p_1, p_2)$ be the center of $F_w K$. We consider the following four linear functions on K,

$$u_1(x_1, x_2) = k^m c_1(x_1 - p_1) + c_2 \quad u_2(x_1, x_2) = k^m c_1(p_1 - x_1) + c_2,$$
$$u_3(x_1, x_2) = k^m c_1(x_2 - p_2) + c_2, \quad u_4(x_1, x_2) = k^m c_1(p_2 - x_2) + c_2,$$

for $x = (x_1, x_2) \in K$, where $c_1 = \sqrt{2}c_0^{-1}$ and $c_2 = 1 + \frac{1}{2}c_1$. In particular, let $u' = \min_{i=1}^4 P_{n+m}(u_i)$, one can check that $u'|_{w \cdot W_n} \ge 1$ and $u'|_{(W_m \setminus \mathcal{N}_w) \cdot W_n} \le 0$.

By Lemma 4.4 and Corollary 4.5, we can construct $g_i \in l(W_{m+n})$, $1 \le i \le 4$ so that $g_i(w' \cdot \tau) = P_n(u_i \circ F_{w'})(\tau), \forall w' \in W_m, \tau \in \partial W_n$ and $\mathcal{D}_{m+n, w' \cdot W_n}(g_i) \le C \cdot c_1^2 N^n \lambda_n^{-1}$, where C is the same constant in Corollary 4.5. In addition, since g_i is linear on $w' \cdot \partial W_n$ and well matched for any $w' \overset{m}{\sim} w''$, we have the estimate

$$\mathcal{D}_{m+n, A}(g_i) \le C' \cdot N^n \lambda_n^{-1} + C' \cdot k^{-n} \le C'' \cdot N^n \lambda_n^{-1},$$

where $A = \{v \in W_{m+n} : v \overset{m+n}{\sim} \mathcal{N}_w \cdot W_n\} \bigcup (\mathcal{N}_w \cdot W_n)$, and C', C'' are independent of n, m, w.

Finally, let $g = \big((\min_{i=1}^4 g_i) \vee 0\big) \wedge 1$. One can see that $g|_{w \cdot W_n} = 1$, $g|_{(W_m \setminus \mathcal{N}_w) \cdot W_n} = 0$ and $\mathcal{D}_{m+n}(g) \le 4C'' \cdot N^n \lambda_n^{-1}$. **(B2)** follows immediately. □

4.3. **Verification of the estimates.** We prove formulas (4.4) and (4.5) in this last subsection. The proof is essentially based on $\lambda_n^{(D)} \asymp \lambda_n$ (provided by Lemma 3.3 following **(B1)**) and Lemma 3.6.

First, we prove (4.5), which is equivalent to the effective resistance estimate

$$R_n(W_{n,2}, W_{n,4}) \ge C \cdot N^{-n} \lambda_n$$

for some $C > 0$ independent of n.

Let $n \ge 1, m \ge 1$. For a connected subset A of $W_{m,1}$ with $l = \#A \ge 2$, let $w_A^{(1)}$ be the most left one and $w_A^{(2)}$ be the most right one in A, viewing A as a chain of cells. Define

$$\tilde{R}_{n,m,A} = \max\big\{\mathcal{D}_{m+n,A}^{-1}(f) : f \in l(A \cdot W_n), f|_{w_A^{(1)} \cdot W_n} = 0, f|_{w_A^{(2)} \cdot W_n} = 1\big\}.$$

Clearly, $\tilde{R}_{n,m,A}$ depends only the length of A, i.e. $l = \#A$, and is independent of m. Denote $\tilde{R}_{n,m,A}$ as $\tilde{R}_{n,,l}$. The intuition of the following lemma comes from the series rule of resistances.

Lemma 4.7. *For $n \ge 1$, we have*

$$R_n(W_{n,1}, W_{n,3}) \ge \frac{1}{4l} \cdot \tilde{R}_{n,l}.$$

Proof. Clearly, we see that $\tilde{R}_{n,l}$ is increasing in l, i.e. $\tilde{R}_{n,l_1} \geq \tilde{R}_{n,l_2}$ if $l_1 \geq l_2$.

The lemma is obvious for $l = 2$, so we only consider the case $l \geq 3$.

First, we consider the case $l = 3$. In this case, $A = \{w_1, w', w_2\}$ such that $w_1 \overset{m}{\sim} w' \overset{m}{\sim} w_2$. Choose $f \in l(A \cdot W_n)$ so that $f|_{w_1 \cdot W_n} = 0, f|_{w_2 \cdot W_n} = 1$ and $\mathcal{D}_{m+n,A\cdot W_n}(f) = \tilde{R}^{-1}_{n,3}$. Define $g \in l(W_n)$ as

$$g(\tau) = 1_{W_{n,2}}(\tau) + f(w' \cdot \tau) \cdot 1_{W_n \setminus (W_{n,2} \cup W_{n,4})}, \quad \forall \tau \in W_n.$$

Then, we have $g|_{W_{n,4}} = 0$, $g|_{W_{n,2}} = 1$ and $\mathcal{D}_n(g) \leq 2 \cdot \mathcal{D}_{m+n,A\cdot W_n}(f)$. Thus,

$$R_n(W_{n,1}, W_{n,3}) \geq \frac{1}{2}\mathcal{D}^{-1}_{m+n,A\cdot W_n}(f) = \frac{1}{2}\tilde{R}_{n,3}.$$

Next, we consider the cases $l = 2^d + 2$, $d \geq 0$. By applying symmetry, one can easily see that

$$\tilde{R}_{n,2^{d+1}+2} \leq 2 \cdot \tilde{R}_{n,2^d+2}. \tag{4.7}$$

In fact, fix A with $\#A = 2^{d+1} + 2$, and choose $f \in l(A \cdot W_n)$ so that $f|_{w_1 \cdot W_n} = 0, f|_{w_2 \cdot W_n} = 1$ and $\mathcal{D}_{m+n,A\cdot W_n}(f) = \tilde{R}^{-1}_{n,2^{d+1}+2}$. We can see that

$$\tilde{R}^{-1}_{n,2^{d+1}+2} = \mathcal{D}_{m+n,A\cdot W_n}(f) \geq \mathcal{D}_{m+n,A\cdot W_n}(f \vee \frac{1}{2}) + \mathcal{D}_{m+n,A\cdot W_n}(f \wedge \frac{1}{2}) \geq 2 \cdot \frac{1}{4}\tilde{R}^{-1}_{n,2^d+2}.$$

Thus, $R_n(W_{n,1}, W_{n,3}) \geq \frac{1}{2^{d+1}}\tilde{R}_{n,2^d+2}$ by iterating (4.7).

Finally, for general $l \geq 3$, the estimate follows from the monotonicity of $\tilde{R}_{n,l}$. □

Now, we prove (4.5). In the proof, we write $q_{i,n} = \{w \in W_n : q_i \in F_w K\}$ for $i = 1, 2, 3, 4$.

Lemma 4.8. *There exists $C > 0$ such that for any $n \geq 1$,*

$$R_n(W_{n,2}, W_{n,4}) = R_n(W_{n,1}, W_{n,3}) \geq C \cdot N^{-n}\lambda_n.$$

Proof. Fix a large n, and let $f \in l(W_n)$ such that $f|_{\partial W_n} = 0$, $[f]_{W_n} = \sqrt{N^{-n}\lambda_n^{(D)}}$ and $\mathcal{D}_n(f) = 1$. Let $\tau \in W_n$ such that

$$f(\tau) = \max_{w \in W_n} f(w) > [f]_{W_n}.$$

Choose $M_1 > 0$ so that $|f(\tau') - f(\tau)| < \frac{1}{3}[f]_{W_n}$ for any $\tau' \in W_n$ such that $\tau'_m = \tau_m$ for all $1 \leq m \leq M_1$. Noticing that by Lemma 3.6, M_1 can be chosen independently of n (for large n).

Next, we choose a sequence $\tau = \tau^{(0)}, \cdots, \tau^{(2L-1)} \in W_n$ such that $\tau^{(2L-1)} \in \partial W_n$, and

$$\begin{cases} \tau_1^{(2l)} \cdots \tau_{M_1}^{(2l)} = \tau_1^{(2l+1)} \cdots \tau_{M_1}^{(2l+1)}, & \text{for } 0 \leq l < L, \\ \tau^{(2l+1)} \overset{n}{\sim} \tau^{(2l+2)}, & \text{for } 0 \leq l < L-1. \end{cases}$$

See Figure 5 for an illustration for such sequence. Noticing that L (depending only on M_1) can be a fixed number for any large n, and $|f(\tau^{(2l+1)}) - f(\tau^{(2l+2)})| < 1$ which is ignorable compared with $[f]_{W_n}$. As a consequence of the choice of M_1, noticing that $f(\tau^{(2L-1)}) = 0$, we can find $1 \leq l < L$ such that $\left|f(\tau^{(2l)}) - f(\tau^{(2l+1)})\right| \geq \frac{1}{2L}[f]_{W_n}$. Now we fix this l and consider

$$g = f \circ F_{\tau_1^{(2l)} \cdots \tau_{M_1}^{(2l)}} \in l(W_{n-M_1}).$$

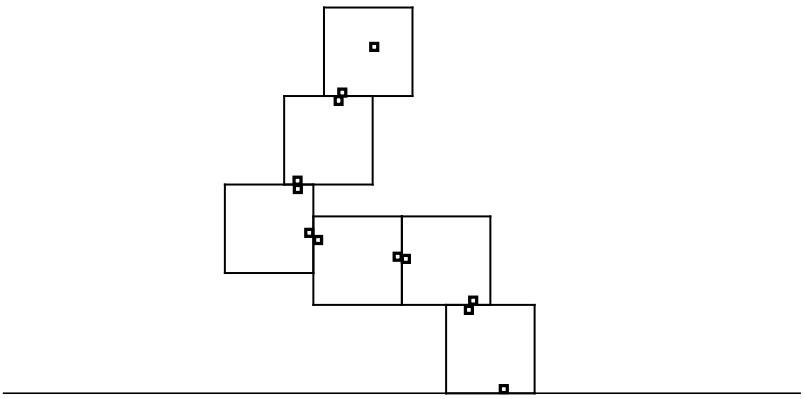


FIGURE 5. The sequence τ, where we use small squares to stand for n-cells in τ, and big squares to stand for the associated $(n-M_1)$-cells.

Clearly, $\mathcal{D}_{n-M_1}(g) \le 1$, and we can choose $1 \le j \le 4$, and η, η' from

$$W_{n-M_1,j} \bigcap \big\{\tilde{\tau}^{(2l)}, \tilde{\tau}^{(2l+1)}, q_{i,n-M_1}, i = 1,2,3,4\big\},$$

such that $g(\eta) - g(\eta') \ge \frac{1}{6L}[f]_{W_n}$, where $\tilde{\tau}^{(2l)} = \tau^{(2l)}_{M_1+1} \cdots \tau^{(2l)}_n$ and $\tilde{\tau}^{(2l+1)} = \tau^{(2l+1)}_{M_1+1} \cdots \tau^{(2l+1)}_n$.

Lastly, again by Lemma 3.6, we choose M_2 independently of large n and M_1 so that

$$\big|g(w) - g(\eta)\big| \le \frac{1}{24L}[f]_{W_n}, \quad \big|g(w') - g(\eta')\big| \le \frac{1}{24L}[f]_{W_n},$$

for any $w, w' \in W_{n-M_1}$ such that $w_1 \cdots w_{M_2} = \eta_1 \cdots \eta_{M_2}$ and $w'_1 \cdots w'_{M_2} = \eta'_1 \cdots \eta'_{M_2}$. Thus, by letting $\kappa = \eta_1 \cdots \eta_{M_2}$ and $\kappa' = \eta'_1 \cdots \eta'_{M_2}$, we have

$$\min_{w \in \kappa \cdot M_{n-M_1-M_2}} g(w) - \max_{w \in \kappa' \cdot M_{n-M_1-M_2}} g(w') \ge g(\eta) - g(\eta') - 2 \cdot \frac{1}{24L}[f]_{W_n} \ge \frac{1}{12L}\sqrt{N^{-n}\lambda_n^{(D)}}.$$

Observing that $\kappa, \kappa' \in W_{M_2,j}$, we conclude that $\tilde{R}_{n-M_1-M_2,l} \ge C' \cdot N^{-n}\lambda_n$ for some $C' > 0$, where l is the length of the finite chain in $W_{M_2,j}$ connecting κ, κ'. Since M_1, M_2 are independent of large n, and l is bounded above depending only on M_2, by using Lemma 4.7, the lemma follows. □

Finally, (4.4) is an easy consequence of Lemma 4.8.

Corollary 4.9. $\limsup_{n\to\infty} \frac{1}{n} \log(N^{-n}\lambda_n) \le \log k - \log 2$.

Proof. For $n \ge 1$, let $f \in l(W_n)$ such that $f|_{W_{n,1}} = 0$ and $f|_{W_{n,3}} = 1$ and $\mathcal{D}_n(f) = \big(R_n(W_{1,n}, W_{3,n})\big)^{-1}$. Let $A = \{W_{1,2}, W_{1,4}\} \cdots \{W_{1,2}, W_{1,4}\}$ where "$\cdot$" repeats n times. Geometrically, A stands for the collection of 2^n many straight chains of n-cells starting from $W_{n,1}$ and ending at $W_{n,3}$. By collecting the energy of f on A, it is directly to see that $\mathcal{D}_n(f) \ge C_1 \cdot 2^n k^{-n}$ for some $C_1 > 0$. Then by Lemma 4.8, we get $N^{-n}\lambda_n \le C_2 \cdot 2^{-n}k^n$ for some $C_2 > 0$, which immediately proves the corollary. □

*Proof of **(B2)**.* **(B2)** is a consequence of Proposition 4.6, Lemma 4.8 and Corollary 4.9. □

5. Self-similar forms

In this section, we prove the existence of a self-similar Dirchlet form on $\mathcal{USC}$. The proof is based on the existence of a limit form in Theorem 3.5.

First, we formally introduce the definition of self-similar Dirichlet forms on K. Recall that the *capacity* on K associated with $(\mathcal{E}, \mathcal{F})$ is defined as

$$Cap(U) = \inf\{\mathcal{E}(f) : f|_U \geq 1, \text{a.e.}, f \in \mathcal{F}\},$$

for open $U \subset K$. For general Borel sets B, $Cap(B) = \inf_{B\subset U} Cap(U)$, where the infimum is taken over open sets U containing B. A function $f \in l(K)$ is *quasi-continuous* if and only if for any $\varepsilon > 0$, there is open $U \subset K$ such that $Cap(U) < \varepsilon$ and $f|_{K\setminus U} \in C(K \setminus U)$.

Definition 5.1. *Let K be a $\mathcal{USC}$ and μ be the normalized Hausdorff measure on K. A regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K is called self-similar if the following holds:*

(1). If $f \in \mathcal{F}$, then $f \circ F_i \in F$ for $1 \leq i \leq N$. Conversely, if f is quasi-continuous on K, and $f \circ F_i \in \mathcal{F}$, then $f \in \mathcal{F}$.

(2). The self-similar identity of the energy holds,

$$\mathcal{E}(f) = \sum_{i=1}^{N} r^{-1}\mathcal{E}(f \circ F_i), \quad \forall f \in \mathcal{F}, \tag{5.1}$$

where $r > 0$ is called the renormalization factor.

In this paper, we have $\mathcal{F} \subset C(K)$ for most of times, so we do not need to worry about the quasi-continuity of the functions.

Theorem 5.2. *Let K be a $\mathcal{USC}$ and μ be the normalized Hausdorff measure on K. There is a local regular, recurrent, irreducible, symmetric, self-similar Dirichlet from $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$.*

In addition, there are constants $C, C' > 0$ such that

$$C \cdot \bar{\mathcal{E}}(f) \leq \mathcal{E}(f) \leq C' \cdot \bar{\mathcal{E}}(f), \quad \forall f \in \mathcal{F},$$

where $(\bar{\mathcal{E}}, \mathcal{F})$ is the same as that in Theorem 3.5.

Proof. Recall the form $(\bar{\mathcal{E}}, \mathcal{F})$ in Theorem 3.5. For $n \geq 1$, we define

$$\tilde{\mathcal{E}}_n(f) = \sum_{w\in W_n} r^{-n}\bar{\mathcal{E}}(f \circ F_w), \quad \forall f \in \mathcal{F}.$$

We claim that

$$C_1 \cdot r^{-n}\mathcal{D}_n(f) \leq \tilde{\mathcal{E}}_n(f) \leq C_2 \cdot \liminf_{m\to\infty} r^{-m}\mathcal{D}_m(f), \quad \forall f \in \mathcal{F} \tag{5.2}$$

for some $C_1, C_2 > 0$ independent of n.

On one hand, by Theorem 3.5, there is $C_2 > 0$ such that $\bar{\mathcal{E}}(f) \leq C_2 \cdot \liminf_{m\to\infty} r^{-m}\mathcal{D}_m(f)$ for any $f \in \mathcal{F}$. It follows from the definition of $\mathcal{D}_m$, we have

$$\begin{aligned}\tilde{\mathcal{E}}_n(f) = \sum_{w\in W_n} r^{-n}\bar{\mathcal{E}}(f \circ F_w) &\leq C_2 \cdot \liminf_{m\to\infty} \sum_{w\in W_n} r^{-n-m}\mathcal{D}_m(f \circ F_w)\\ &\leq C_2 \cdot \liminf_{m\to\infty} r^{-m}\mathcal{D}_m(f).\end{aligned}$$

On the other hand, still by Theorem 3.5, $\mathcal{F} \subset C(K)$ and $\big|P_0 f - f(x)\big|^2 \le C_3 \cdot \bar{\mathcal{E}}(f), \forall x \in K, \forall f \in \mathcal{F}$, for some $C_3 > 0$, where $P_0 f = \int_K f(y)\mu(dy)$. Thus, for any $w \overset{n}{\sim} w'$ in W_n,

$$\big|(P_n f)(w) - (P_n f)(w')\big|^2 \le 2C_3 \cdot \big(\bar{\mathcal{E}}(f \circ F_w) + \bar{\mathcal{E}}(f \circ F_{w'})\big),$$

where $(P_n f)(w) = N^n \int_{F_w K} f(y)\mu(dy)$ as previous sections. By summing over all $w \overset{n}{\sim} w'$, we get $C_1 \cdot r^{-n}\mathcal{D}_n(f) \le \tilde{\mathcal{E}}_n(f)$ for some $C_1 > 0$.

Let $\hat{\mathcal{F}}$ be a $\mathbb{Q}$-vector subspace of $\mathcal{F}$, that is dense in $\mathcal{F}$ with respect to the norm $\|f\|_{\bar{\mathcal{E}}_1} := \sqrt{\bar{\mathcal{E}}(f) + \|f\|^2_{L^2(K,\mu)}}$. Then by a diagonal argument, there is a subsequence $\{n_l\}_{l\ge 1}$ such that the limit

$$\mathcal{E}(f) := \lim_{l\to\infty} \frac{1}{n_l} \sum_{m=1}^{n_l} \tilde{\mathcal{E}}_m(f)$$

exists for any $f \in \hat{\mathcal{F}}$. In addition, by (5.2) and Theorem 3.5, we know

$$C_4 \cdot \bar{\mathcal{E}}(f) \le \mathcal{E}(f) \le C_5 \cdot \bar{\mathcal{E}}(f), \quad \forall f \in \hat{\mathcal{F}},$$

for some constant $C_4, C_5 > 0$. Thus, we can extend $\mathcal{E}$ continuously to $\mathcal{F}$ with respect to $\bar{\mathcal{E}}_1$.

Immediately, $(\mathcal{E}, \mathcal{F})$ is a regular symmetric self-similar Dirichlet form on K. In particular, one can easily see that $f \in C(K)$ and $f \circ F_i \in \mathcal{F}, \forall 1 \le i \le N$ imply $f \in \mathcal{F}$. The local property follows from the self-similarity of the form. Since $1 \in \mathcal{F}$ and $\mathcal{E}(1) = 0$, by Theorem 1.6.3 of [], $(\mathcal{E}, \mathcal{F})$ is recurrent. Finally, since $\mathcal{F} \subset C(K)$, so $1_A \in \mathcal{F}$ implies $\mu(A) \in \{0, 1\}$. So $(\mathcal{E}, \mathcal{F})$ is irreducible. □

6. A uniform trace theorem

From now on, we turn to the second topic of this paper. We will consider whether the self-similar form $(\mathcal{E}, \mathcal{F})$ on a $\mathcal{USC}$ constructed in Theorem 5.2 is unique, and whether the form depends in a continuous way on the underlying $\mathcal{USC}$.

As a preparation, we will develop a uniform trace theorem for forms on $\mathcal{USC}$. Here uniform means the constants C appearing in this section do not depend on K (only depends on k). To achieve this, we need to normalized self-similar forms on $\mathcal{USC}$ in a natural way.

Basic settings. Throughout Section 6-9, we fix k and N. We introduce the following conditions on a form $(\mathcal{E}, \mathcal{F})$ on K for short.

(C1). *$(\mathcal{E}, \mathcal{F})$ is local regular, recurrent, irreducible and self-similar with a renormalization factor $0 < r \le \frac{N}{k^2}$. In addition,*

$$R(L_1, L_3) = 1, \tag{6.1}$$

where $R(A, B) = \big(\inf\{\mathcal{E}(f) : f|_A = 0, f|_B = 1\}\big)^{-1}$ for any $A, B \subset K$ with positive capacity.

(C2). *Every point in K has positive capacity. In particular, $R(x, y) := R\big(\{x\}, \{y\}\big)$ is well-defined for any $x \ne y$ in K.*

Clearly, the form constructed in Theorem 5.2 satisfies **(C1)** and **(C2)** if we normalize it with a constant so that $R(L_1, L_3) = 1$. In this section we will present an extension theorem in Subsection 6.3, where **(C1)** is quite enough, and a trace theorem in Subsection 6.5, where **(C2)** is needed.

6.1. **Besov spaces on** $[0,1]$**.** The Besov spaces on the line segment $[0,1]$ naturally appear in the trace theorem of Dirichlet spaces. See [] for related works. In this paper, we will admit the following discrete version of Besov spaces on $[0,1]$.

Definition 6.1. *Let* $\frac{1}{2} < \sigma < 1$. *For each* $u \in C[0,1]$, *define*

$$[[u]]_{\Lambda^{\sigma}_{2,2}} = \sqrt{\sum_{m=0}^{\infty} k^{(2\sigma-1)m} \sum_{l=0}^{k^m-1} \big(u(\frac{l}{k^m}) - u(\frac{l+1}{k^m})\big)^2},$$

and

$$\Lambda^{\sigma}_{2,2} = \big\{u \in C[0,1] : [[u]]_{\Lambda^{\sigma}_{2,2}} < \infty\big\}.$$

Throughout this section, we always choose $\sigma = \sigma(r) = -\frac{\log r}{2\log k} + \frac{1}{2}$, so that

$$[[u]]_{\Lambda^{\sigma(r)}_{2,2}} = \sqrt{\sum_{m=0}^{\infty} r^{-m} \sum_{l=0}^{k^m-1} \big(u(\frac{l}{k^m}) - u(\frac{l+1}{k^m})\big)^2}.$$

We remark that k is a fixed integer and $0 < r < 1$ is the factor in **(C1)**. We also write

$$\|u\|_{\Lambda^{\sigma(r)}_{2,2}} = \sqrt{[[u]]^2_{\Lambda^{\sigma(r)}_{2,2}} + \|u\|^2_{L^2([0,1])}}$$

for the norm on $\Lambda^{\sigma(r)}_{2,2}$, where $L^2[0,1]$ is with respect to the Lebesgue measure on $[0,1]$.

Finally, for a line segment $\overline{x,y}$, with $x \neq y$ in $\mathbb{R}^2$, we identify $\overline{x,y}$ with $[0,1]$ by the linear map $\gamma_{\overline{x,y}} : z \to \frac{z-x}{y-x}, z \in \overline{x,y}$. We simply write

$$[[u]]_{\Lambda^{\sigma(r)}_{2,2}(\overline{x,y})} = |\overline{x,y}|^{1/2-\sigma(r)} [[u \circ \gamma_{\overline{x,y}}]]_{\Lambda^{\sigma(r)}_{2,2}},$$

and $\Lambda^{\sigma(r)}_{2,2}(\overline{x,y}) = \big\{u \in C(\overline{x,y}) : [[u]]_{\Lambda^{\sigma(r)}_{2,2}(\overline{x,y})} < \infty\big\}$, where $|\overline{x,y}|$ is the length of $\overline{x,y}$. Noticing that $[[u]]_{\Lambda^{\sigma(r)}_{2,2}(\overline{x,y})} = [[u]]_{\Lambda^{\sigma(r)}_{2,2}(\overline{y,x})}$, the norm depends only on x, y, not on the order of them.

6.2. **Building bricks and boundary graph.** Let K be a $\mathcal{USC}$. For $m \geq 1$, recall that in Definition 4.1 we have defined $T_{m,K} = \bigcup_{w \in W_{m,1}} F_w K$, and $B_K = cl(T_{1,K} \setminus T_{2,K})$. In addition, $T_{1,K}$ can be decomposed into

$$T_{1,K} = \big(\bigcup_{w \in W_{*,1}} F_w B_K\big) \bigcup L_1,$$

where each $F_w B_K$ is called a level-$|w|$ *building brick*.

We introduce two graphs based on B_K and $T_{1,K}$.

Definition 6.2. *Let* K *be a* $\mathcal{USC}$.

(a). Define $V_{B_K} = \{F_1 q_4, F_k q_3\} \cup \{F_{i1} q_4, F_{ik} q_3\}_{i=1}^k$,

$$E_{B_K} = \big\{\{F_1 q_4, F_k q_3\}, \{F_1 q_4, F_{11} q_4\}, \{F_k q_3, F_{kk} q_3\}\big\} \bigcup \big\{\{F_{i1} q_4, F_{ik} q_3\}_{i=1}^k\big\},$$

and denote $G_{B_K} = (V_{B_K}, E_{B_K})$ *the associated finite graph.*

(b). Define $V_K = \bigcup_{w \in W_{*,1}} F_w V_{B_K}$, $E_K = \bigcup_{w \in W_{*,1}} F_w(E_{B_K})$, *and denote* $G_K = (V_K, E_K)$ *the associated infinite graph. See Figure 6 for an illustration.*

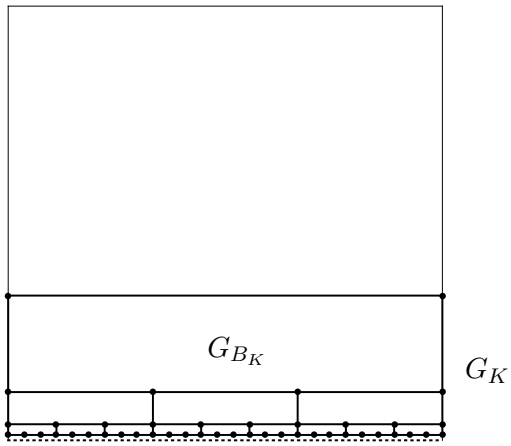


FIGURE 6. The infinite graph G_K.

The points in V_{B_K} are vertices on the boundary of B_K, while the set V_K is the collection of boundary points of all bricks, with the lower boundary L_1 as its limit. We call G_K the *boundary graph* of K.

In the following, we define an energy on G_K.

Definition 6.3. *(a). For $f \in l(V_{B_K})$, define*

$$\mathcal{D}_{G_{B_K}}(f) = \sum_{\{x,x'\}\in E_{B_K}} \big(f(x) - f(y)\big)^2.$$

(b). For $f \in l(V_K)$, define

$$\mathcal{D}_{G_K}(f) = \sum_{w\in W_{*,1}} r^{-|w|}\mathcal{D}_{G_{B_K}}(f \circ F_w),$$

*where r is the renormalization factor in **(C1)**.*

As before, we use the $\mathbb{R}^2$-coordinate to represent points in V_K. Readers can check that

$$V_{B_K} = \big\{(0, \frac{1}{k}), (1, \frac{1}{k})\big\} \bigcup \big\{(\frac{l}{k}, \frac{1}{k^2}) : 0 \le l \le k\big\},$$

and

$$V_K = \bigcup_{m=0}^{\infty} \big\{(\frac{l}{k^m}, \frac{1}{k^{m+1}}) : 0 \le l \le k^m\big\}.$$

6.3. **An extension theorem.** In this subsection, we always assume $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on K satisfying **(C1)**. We begin our story with an extension theorem. The method is similar to that in Section 4.2. First, we introduce a finite dimensional space of functions that support on B_K.

Definition 6.4. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on K satisfying **(C1)**.*

(a). Let h be the unique function such that $h|_{L_4} = 0$, $h|_{L_2} = 1$ and h is harmonic in $K \setminus (L_2 \cup L_4)$.

(b). Let h' be the unique function supported in $K\setminus T_{2,K}$ such that $h'|_{L_3}=0$, $h'_{\bigcup_{i=1}^k F_iL_3}=1$, and h' is harmonic in $K\setminus(T_{1,K}\cup L_3)$ and in each $B_K\cap F_iK\setminus F_i(L_3\cup T_{1,K}), i=1,\cdots,k$.

(c). Let $\mathcal{B}^{(1)},\mathcal{B}^{(2)}\in C(K)$ be defined as

$$\mathcal{B}^{(1)}(x)=\big(1-h(x)\big)\cdot h'(x),\quad \mathcal{B}^{(2)}(x)=h(x)\cdot h'(x),\quad x\in K.$$

(d). For each $f\in l(V_{B_K})$, we define $\mathcal{B}_f\in C(B_K)$ as

$$\begin{aligned}\mathcal{B}_f(x)=&f(F_1q_4)\mathcal{B}^{(1)}(x)+f(F_kq_3)\mathcal{B}^{(2)}(x)\\&+\sum_{i=1}^k\big(f(F_{i1}q_4)\mathcal{B}^{(1)}\circ F_i^{-1}(x)+f(F_{ik}q_3)\mathcal{B}^{(2)}\circ F_i^{-1}(x)\big),\quad x\in B_K,\end{aligned}$$

where we admit the convention

$$g\circ F_w^{-1}(x)=\begin{cases}g(F_w^{-1}x), & \text{if } F_w^{-1}x \text{ is in the support of } g,\\ 0, & \text{otherwise.}\end{cases}$$

Next, for convenience, on a simple set $A=\bigcup_{i=1}^{M_A}F_{w^{(i)}}K$, where $M_A<\infty, w^{(i)}\in W_*$ and $\mu(F_{w^{(i)}}K\cap F_{w^{(i')}}K)=0$ for any $1\le i\ne i'\le M_A$, we define

$$\mathcal{F}_A=\big\{f\in C(A): f\circ F_{w^{(i)}}\in\mathcal{F}, 1\le i\le M_A\big\},$$

and

$$\mathcal{E}_A(f,g)=\sum_{i=1}^{M_A}r^{-|w^{(i)}|}\mathcal{E}(f\circ F_{w^{(i)}},g\circ F_{w^{(i)}}),\quad \forall f,g\in\mathcal{F}_A.$$

It is easy to check that $(\mathcal{E}_A,\mathcal{F}_A)$ is a Dirichlet form on $L^2(A,\mu)$. Write $\mathcal{E}_A(f):=\mathcal{E}_A(f,f)$ for short.

We immediately have the following lemma. In particular, Lemma 6.5 (b) is a consequence of the construction of $\mathcal{B}_f$ and Lemma 6.5 (a).

Lemma 6.5. *Let $f,f'\in l(V_{B_K})$, and $\mathcal{B}_f$, $\mathcal{B}_{f'}$ be the functions defined in Definition 6.4.*

(a). If $f\equiv c$ on V_{B_K}, then $\mathcal{B}_f\equiv c$ on B_K.

(b). $\mathcal{B}_f\in\mathcal{F}_{B_K}$ and there exists $C>0$ depending only on k, so that

$$\mathcal{E}_{B_K}(\mathcal{B}_f)\le C\cdot\mathcal{D}_{G_{B_K}}(f).$$

(c). If $f(F_kq_3)=f'(F_1q_4)$ and $f(F_{kk}q_3)=f'(F_{11}q_4)$, then

$$\mathcal{B}_f(x)=\mathcal{B}_{f'}\big(x-(1,0)\big),\qquad \forall x\in L_2\cap B_K.$$

(d). If $f(F_1q_4)=f'(F_{i1}q_4)$ and $f(F_kq_3)=f'(F_{ik}q_3)$ for some $1\le i\le k$, then

$$\mathcal{B}_{f'}\circ F_i(x)=\mathcal{B}_f(x),\qquad \forall x\in\overline{F_1q_4,F_kq_3}.$$

In addition,

$$\mathcal{B}_f(x)=f(F_1q_4)\cdot\big(1-h(x)\big)+f(F_kq_3)\cdot h(x),\qquad \forall x\in\overline{F_1q_4,F_kq_3}.$$

Remark. Lemma 6.5 (c),(d) imply that the functions $\mathcal{B}_f$ can be scaled and glued together easily by only looking at the values of f on V_{B_K}, and Lemma 6.5 (b) provides a good energy estimate of $\mathcal{B}_f$ on B_K in terms of that of f on V_{B_K}.

Theorem 6.6. *Let $f \in l(V_K)$ with $\mathcal{D}_{G_K}(f) < \infty$, and define $g \in C(T_{1,K} \setminus L_1)$ as*

$$g(x) = \mathcal{B}_{f\circ F_w} \circ F_w^{-1}(x), \text{ if } x \in F_w B_K, w \in W_{*,1}.$$

Then, g extends continuous to $g \in \mathcal{F}_{T_{1,K}} \subset C(T_{1,K})$. In addition, there is $C > 0$ depending only on k, such that

$$\mathcal{E}_{T_{1,K}}(g) \le C \cdot \mathcal{D}_{G_K}(f).$$

Proof. Let h be the unique function such that $h|_{L_4} = 0$, $h|_{L_2} = 1$ and h is harmonic in $K \setminus (L_2 \cup L_4)$. For $m \ge 2$, define g_m on $T_{1,K}$ as follows,

$$g_m(x) = \begin{cases} g(x), & \text{if } x \in T_{1,K} \setminus T_{m,K}, \\ \big(f(F_w F_1 q_4)(1-h) + f(F_w F_k q_3)h\big) \circ F_w^{-1}(x), & \text{if } x \in F_w T_{1,K}, \text{ for some } w \in W_{m-1,1}. \end{cases}$$

By Lemma 6.5, it is easy to see that $g_m \in \mathcal{F}_{T_{1,K}}$ and

$$\mathcal{E}_{T_{1,K}}(g_m) \le C \cdot \mathcal{D}_{V_K}(f),$$

for some $C > 0$ depending only on k. Clearly, $\{\|g_m\|_{L^2(T_{1,K},\mu)}\}$ is uniformly bounded, so we can find a subsequence of $\{g_m\}$ converging weakly to g in $\mathcal{F}_{T_{1,K}}$. Thus, $g \in \mathcal{F}_{T_{1,K}}$, and the desired estimate holds. □

Corollary 6.7. *Let $u \in \Lambda_{2,2}^{\sigma(r)}(L_1)$. There exists $g \in \mathcal{F}_{T_{1,K}}$ such that*

$$\begin{cases} g|_{L_1} = u, \\ \mathcal{E}_{T_{1,K}}(g) \le C \cdot [[u]]^2_{\Lambda_{2,2}^{\sigma(r)}(L_1)} \end{cases}$$

for some constant $C > 0$ depending only on k. In addition, if $u(q_1) = u(q_2) = c$, we can require $g|_{L_2 \cap T_{1,K}} = g|_{L_4 \cap T_{1,K}} = g|_{F_i L_3} = c, \forall 1 \le i \le k$.

Proof. Define $f \in l(V_K)$ as $f(x_1, x_2) = u(x_1)$ for any $x = (x_1, x_2) \in V_K$. It is easy to check that $\mathcal{D}_{G_K}(f) \le (1+r)[[u]]^2_{\Lambda_{2,2}^{\sigma(r)}(L_1)}$. The corollary then follows from Theorem 6.6 and Lemma 6.5. □

We will deal with $L_i, i = 1, 2, 3, 4$ together at the end of this section after we have established the associated trace theorem.

6.4. **Resistance between q_1, q_2, q_3, q_4.** Next, we will consider the converse direction to the last subsection, the trace of $\mathcal{F}$ to $L_i, i = 1, 2, 3, 4$. Before doing that, we need to consider the resistance between points.

In the remaining of this section, we will always assume that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on K satisfying both **(C1)** and **(C2)**. A uniform estimate will be provided, independent of K and the form $(\mathcal{E}, \mathcal{F})$. To achieve this, we will apply the celebrated method of *slide moves* and *corner moves* by Barlow and Bass, see [] for details. Readers can refer to Section 4.2 in [] for a more general version.

Following [], let's write $(\Omega, \mathcal{M}, X_t, P_x)$ for the Hunt process associated with $(\mathcal{E}, \mathcal{F})$ and $L^2(K, \mu)$. By **(C2)**, we have

$$P_x(\dot{\sigma}_A < \infty) = 1, \forall x \in K, A \subset K$$

where

$$\dot{\sigma}_A = \inf\{t \geq 0 : X_t \in A\}.$$

Since a $\mathcal{USC}$ is complicated in its inner cells, we only consider the slide and corner moves on the boundary cells, i.e. cells intersection $L_i, 1 \leq i \leq 4$. The following result is a simple case of Proposition **??**4.5? in [], noticing that $(\mathcal{E}, \mathcal{F})$ satisfies both **(C1)** and **(C2)**. Readers can also read [] for a proof which is essentially an argument of symmetry.

Proposition 6.8 ([]). *Let $n \geq 1$, $1 \leq l \leq k^n - 1$ and*

$$x \in \big([\frac{2l-1}{2k^n}, \frac{2l+1}{2k^n}] \times \{0\}\big) \bigcup \big(\{\frac{l}{k^n}\} \times [0, \frac{1}{2k^n}]\big).$$

Let

$$\begin{aligned} Z &= \big(\{\frac{l-1}{k^n}\} \times [0, \frac{1}{k^n}]\big) \bigcup \big(\{\frac{l+1}{k^n}\} \times [0, \frac{1}{k^n}]\big) \bigcup \big([\frac{l-1}{k^n}, \frac{l+1}{k^n}] \times \{\frac{1}{k^n}\}\big), \\ X_i &= [\frac{2l+i-3}{2k^n}, \frac{2l+i-2}{2k^n}] \times \{0\}, \qquad i = 1, 2, 3, 4, \\ X_5 &= \{\frac{l}{k^n}\} \times [0, \frac{1}{2k^n}]. \end{aligned}$$

Then, there is $p > 0$ which does not depend on anything such that

$$P_x(\dot{\sigma}_{X_i} < \dot{\sigma}_Z) \geq p, \qquad \forall 1 \leq i \leq 5.$$

Using Proposition 6.8, one can easily get the following estimate.

Lemma 6.9. *Let $n \geq 2$, $x \in \{\frac{1}{k^n}\} \times [0, \frac{1}{2k^n}]$ and $Y_1 = \{0\} \times [0, \frac{1}{2k^n}]$, $Y_2 = [0, \frac{1}{2k^n}] \times \{0\}$ and $Z = \{\frac{1}{2}\} \times [0, 1]$. Then,*

$$P_x(\dot{\sigma}_{Y_1} < \dot{\sigma}_Z, \dot{\sigma}_{Y_2} < \dot{\sigma}_Z) \geq p^3,$$

where p is the same as in Proposition 6.8.

Using Lemma 6.9, we can have an estimate of the effective resistance $R(q_1, q_2)$.

Proposition 6.10. *Let $(\mathcal{E}, \mathcal{F})$ be a form on K satisfying **(C1)** and **(C2)**. Then $R(q_1, q_2) \leq C$ for some $C > 0$ depending only on k.*

Proof. Let h be the unique function on K such that

$$h(q_1) = 1, h(q_2) = -1, \mathcal{E}(h) = \frac{4}{R(q_1, q_2)},$$

i.e. h is harmonic in $K \setminus \{q_1, q_2\}$. We need to show that $\mathcal{E}(h) \geq C \cdot \mathcal{E}(h')$, where $C > 0$ depends only on k and h' is the unique function such that

$$h'|_{L_4} = 0, h'|_{L_2} = 1, \mathcal{E}(h') = \frac{1}{R(L_2, L_4)} = 1.$$

First, noticing that $r \leq \frac{N}{k^2} \leq \frac{k^2-1}{k^2}$, we can find n depending only on k so that $r^n < \frac{1}{16}$. Then, one can see that $h(F_1^n q_2) > \frac{1}{2}$. By a routine stopping time argument as in the proof of Theorem 3.1 in [], one can find a path $\gamma : [0, 1] \to K$ such that

$$\gamma(0) = F_1^n q_2, \gamma(1) = q_1, \text{ and } h\big(\gamma(t)\big) > \frac{1}{2}, \forall 0 \leq t \leq 1.$$

We let $Z=\{\frac{1}{2}\}\times[0,1]$. Now, by Lemma 6.9, for any $x\in\{\frac{1}{k^n}\}\times[0,\frac{1}{2k^n}]$, we have

$$P_x(\dot\sigma_{\gamma([0,1])}<\dot\sigma_{q_2})\geq p^3.$$

As a consequence, since h is harmonic in $K\setminus\big(\gamma([0,1])\cup Z\big)$, we have

$$h(x)=\mathbb{E}_x\big(f(\dot\sigma_{\gamma([0,1])\cup Z})\big)\geq\frac{p^3}{2},\quad\forall x\in\{\frac{1}{k^n}\}\times[0,\frac{1}{2k^n}],$$

where we use the fact that $h|_{Z\cap K}=0$ by symmetry. Clearly, by symmetry, we have $h(x)\leq -\frac{p^3}{2}$ for any $x\in\{\frac{k^n-1}{k^n}\}\times[0,\frac{1}{2k^n}]$.

Finally, for $A=\{1,2,\cdots,k\}^{n+1}\setminus\{1\cdots1,k\ldots k\}$, we have

$$\mathcal{E}(h)\geq\mathcal{E}_A(h)\geq\mathcal{E}_A(h'')=(k^{n+1}-2)r^{-(n+1)}\big(\frac{p^3}{k^{n+1}-2}\big)^2\mathcal{E}(h'),$$

where h'' is the unique function on A such that $h''|_{F_1^{n+1}L_2}=\frac{p^3}{2}$, $h''|_{F_k^{n+1}L_4}=-\frac{p^3}{2}$. □

6.5. **A trace theorem.** Now, for a Dirichlet form $(\mathcal{E},\mathcal{F})$ on K satisfying **(C1)** and **(C2)**, we proceed to study the trace of $\mathcal{F}$ on $L_i, i=1,2,3,4$. As an immediate application of Proposition 6.10, we have the following estimate, as a reverse direction of Lemma 6.5 (b).

Lemma 6.11. *There is $C>0$ depending only on k such that*
(a). $\mathcal{D}_{G_{B_K}}(f)\leq C\cdot\mathcal{E}_{B_K}(f)$ for any $f\in\mathcal{F}_{B_K}$,
(b). $\mathcal{D}_{G_K}(f)\leq C\cdot\mathcal{E}(f)$ for any $f\in\mathcal{F}$.

We will focus on L_1 for convenience. For a function u on L_1, we introduce the notation

$$D_n(u)=\sqrt{\sum_{l=0}^{k^n-1}\big(u(\frac{l}{k^n},0)-u(\frac{l+1}{k^n},0)\big)^2}.$$

Then, $[[u]]_{\Lambda_{2,2}^{\sigma(r)}(L_1)}=\big\|r^{-n/2}D_n(u)\big\|_{l^2}=\sqrt{\sum_{n=0}^\infty r^{-n}|D_n(u)|^2}$.

Also, for $f\in C\big(cl(V_K)\big)$, we write

$$\tilde D_n(f)=\sqrt{\sum_{w\in\{1,2,\cdots,k\}^n}\mathcal{D}_{G_{B_K}}(f\circ F_w)}.$$

Then, $\mathcal{D}_{G_K}(f)=\sum_{n=0}^\infty r^{-n}|\tilde D_n(f)|^2$.

Lemma 6.12. *Let $f\in C\big(cl(V_K)\big)$, then we have*

$$D_n(f|_{L_1})\leq 3\cdot\sum_{m=n}^\infty\tilde D_m(f),\quad\forall n\geq0.$$

Proof. Clearly, for $0\leq l\leq k^n$, we have

$$f(\frac{l}{k^n},0)=f(\frac{l}{k^n},\frac{1}{k^{n+1}})+\sum_{m=n}^\infty\big(f(\frac{l}{k^n},\frac{1}{k^{m+2}})-f(\frac{l}{k^n},\frac{1}{k^{m+1}})\big).$$

Thus,

$$\begin{aligned}
D_n(f|_{L_1}) &= \sqrt{\sum_{l=0}^{k^n-1}\big(f(\frac{l}{k^n},0)-f(\frac{l+1}{k^n},0)\big)^2}\\
&= \sqrt{\sum_{l=0}^{k^n-1}\Big(f(\frac{l}{k^n},\frac{1}{k^{n+1}})-f(\frac{l+1}{k^n},\frac{1}{k^{n+1}})+\sum_{l'=l,l+1}(-1)^{l'-l}\sum_{m=n}^{\infty}\big(f(\frac{l'}{k^n},\frac{1}{k^{m+2}})-f(\frac{l'}{k^n},\frac{1}{k^{m+1}})\big)\Big)^2}\\
&\le \sqrt{\sum_{l=0}^{k^n-1}\big(f(\frac{l}{k^n},\frac{1}{k^{n+1}})-f(\frac{l+1}{k^n},\frac{1}{k^{n+1}})\big)^2}+\sum_{m=n}^{\infty}\sum_{l'=0,1}\sqrt{\sum_{l=0}^{k^n-1}\big(f(\frac{l'+l}{k^n},\frac{1}{k^{m+2}})-f(\frac{l'+l}{k^n},\frac{1}{k^{m+1}})\big)^2}\\
&\le 3\cdot\sum_{m=n}^{\infty}\tilde{D}_m(f).
\end{aligned}$$

□

Combining Lemma 6.11 and 6.12, we can easily prove the following trace theorem.

Theorem 6.13. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on K satisfying **(C1)** and **(C2)**. There exists $C>0$ depending only on k such that*

$$[[f|_{L_1}]]^2_{\Lambda^{\sigma(r)}_{2,2}(L_1)}\le C\cdot\mathcal{E}(f),\quad \forall f\in\mathcal{F}.$$

Proof. The theorem follows from the following inequality,

$$\begin{aligned}
[[f|_{L_1}]]_{\Lambda^{\sigma(r)}_{2,2}(L_1)} &= \big\|r^{-n/2}D_n(f|_{L_1})\big\|_{l^2}\\
&\le 3\cdot\big\|\sum_{m=n}^{\infty}r^{-n/2}\tilde{D}_m(f)\big\|_{l^2}\\
&= 3\cdot\big\|\sum_{m=0}^{\infty}r^{m/2}r^{-(m+n)/2}\tilde{D}_{m+n}(f)\big\|_{l^2}\\
&\le 3\cdot\sum_{m=0}^{\infty}r^{m/2}\|r^{-(m+n)/2}\tilde{D}_{m+n}(f)\big\|_{l^2}\le\frac{3}{1-r^{1/2}}\mathcal{D}^{1/2}_{G_K}(f)\le C'\cdot\mathcal{E}^{1/2}(f),
\end{aligned}$$

where we use Lemma 6.12 in the second line and use Lemma 6.11 (b) in the last inequality. Here C' depends only on k noticing that $r\le\frac{N}{k^2}$. □

Finally, at the end of this section, we consider a trace result considering the whole square boundary $\partial_0K=\bigcup_{i=1}^4L_i$ of K. For short, we write

$$\partial_nK=\bigcup_{w\in W_n}F_w(\partial_0K),\quad\forall n\ge1.$$

In addition, for $u\in C(\partial_0K)$, we write

$$[[u]]_{\Lambda^{\sigma(r)}_{2,2}(\partial_0K)}=\sqrt{\sum_{i=1}^4[[u]]^2_{\Lambda^{\sigma(r)}_{2,2}(L_i)}},$$

and define $\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)=\big\{u\in C(\partial_0 K):[[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}<\infty\big\}$.

For $u\in C(\partial_n K)$, we write

$$[[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_n K)}=\sqrt{\sum_{w\in W_n} r^{-n}[[u\circ F_w]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}},$$

and define $\Lambda_{2,2}^{\sigma(r)}(\partial_n K)=\big\{u\in C(\partial_n K):[[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_n K)}<\infty\big\}$.

Theorem 6.14. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on K satisfying **(C1)** and **(C2)**. For $n\geq 0$, we define*

$$\mathcal{H}_n=\big\{h\in\mathcal{F}:\mathcal{E}(h)=\inf\{\mathcal{E}(f):f|_{\partial_n K}=h|_{\partial_n K},f\in\mathcal{F}\}\big\},$$

which is the space of functions that are harmonic in $K\setminus\partial_n K$.

Then, we have $\mathcal{H}_n|_{\partial_n K}=\Lambda_{2,2}^{\sigma(r)}(\partial_n K)$. In addition, there are $C_1,C_2>0$ depending only on k such that

$$C_1\cdot[[h|_{\partial_n K}]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_n K)}\leq\mathcal{E}(h)\leq C_2\cdot[[h|_{\partial_n K}]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_n K)},\quad\forall h\in\mathcal{H}_n.$$

Proof. We consider the $n=0$ case only, since the $n\geq 1$ case follows immediately from it.

First, by Theorem 6.13, one can see $\mathcal{H}_0|_{\partial_0 K}\subset\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)$, and there is $C_1>0$ depending only on k so that $\mathcal{E}(h)\geq C_1\cdot[[h|_{\partial_0 K}]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}$ for any $h\in\mathcal{H}_0$.

For the other direction, it suffices to show that for any $u\in\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)$, there is $f\in\mathcal{F}$ such that

$$\begin{cases} f|_{\partial_0 K}=u,\\ \mathcal{E}(f)\leq C_2\cdot[[u]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}\end{cases}$$

for some $C_2>0$ depending only on k. To achieve this, we apply the construction in Corollary 6.7. We take two steps to construct a desired f.

Step 1. We find $f'\in\mathcal{F}$ such that $f'|_{L_1}=u|_{L_1}$, $f'|_{L_3}=u|_{L_3}$, and

$$\mathcal{E}(f')\leq C_3\cdot[[u]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)},$$

for some $C_3>0$ depending only on k.

To complete *Step 1*, we first use Corollary 6.7 to construct f' on $T_{1,K}$ and $\Gamma_v T_{1,K}$ $\big($recall Γ_v in (2.1)$\big)$. For the middle part, we take

$$\begin{aligned}f'(x)=&u(q_1)\big(1-h(x)\big)h'(x)+u(q_2)h(x)h'(x)\\&+u(q_3)h(x)\big(1-h'(x)\big)+u(q_4)\big(1-h(x)\big)\big(1-h'(x)\big),\quad\forall x\in K\setminus\big(T_{1,K}\cup\Gamma_v T_{1,K}\big),\end{aligned}$$

where h is the unique function such that $h|_{L_4}=0,h|_{L_2}=1$ and h is harmonic in $K\setminus(L_2\cup L_4)$; h' is the unique function such that $h'|_{T_{1,K}}=1,h'|_{\Gamma_v T_{1,K}}=0$ and h' is harmonic in $K\setminus\big(T_{1,K}\cup\Gamma_v T_{1,K}\big)$. Clearly, $f'\in\mathcal{F}$ and satisfies the desired conditions.

Now, let $u'=u-f'|_{\partial_0 K}$. We have

$$[[u']]_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}\leq[[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}+[[f'|_{\partial_0 K}]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}\leq C_4\cdot[[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)}$$

for some $C_4 > 0$ depending only on k, where we use the estimate $\mathcal{E}(f') \geq C_1 \cdot [[f'|_{\partial_0 K}]]^2_{\Lambda^{\sigma(r)}_{2,2}(\partial_0 K)}$ and Step 1 in the last inequality. In additon, we see that

$$u'|_{L_1 \cup L_3} = 0.$$

Step 2. We construct $f'' \in \mathcal{F}$ such that $f''|_{\partial_0 K} = u'$ and $\mathcal{E}(f'') \leq C_3 \cdot ([[u']]_{\Lambda^{\sigma(r)}_{2,2}(\partial_0 K)})^2$.

This can be fulfilled with a same argument as *Step 1*. First, we construct f'' on $\Gamma_{d_1} T_{1,K}$ and $\Gamma_{d_2} T_{2,K}$ using Corollary 6.7, so that $f''|_{L_2 \cup L_4} = u'|_{L_2 \cup L_4}$ $\big($recall $\Gamma_{d_1}, \Gamma_{d_2}$ in (2.1)$\big)$. Then we extend f'' to K with 0. The energy estimate of f'' is clear, and $f''|_{L_1 \cup L_3} = 0$ is guaranteed by the construction.

Finally, we take $f = f' + f''$. We then have $f|_{\partial_0 K} = u$ and $\mathcal{E}(f) \leq C_2 \cdot [[u]]^2_{\Lambda^{\sigma(r)}_{2,2}(\partial_0 K)}$ for some $C_2 > 0$ depending only on k. □

7. A uniform resistance estimate

In this section, we provide a uniform estimate on the resistance metric $R(\cdot, \cdot)$ for a Dirichlet form on $\mathcal{USC}$ satisfying **(C1)**, **(C2)**. Same as the last section, here uniform means that the estimate only depends on k, does not depend on the choice of K.

To achieve a uniform estimate, we will adopt the geodesic metric on $\mathcal{USC}$, instead of the Euclidean metric. For a $\mathcal{USC}$ K, the *geodesic metric* d_G on K is defined as

$$d_G(x, y) = \inf \big\{ length(\gamma) : \gamma : [0, 1] \to K \text{ is a rectifiable curve }, \gamma(0) = x, \gamma(1) = y \big\},$$

for any $x \neq y \in K$. It is not so different compared with the Euclidean metric d on K.

Lemma 7.1. *There exists $C > 0$ depending only on K such that*

$$d(x, y) \leq d_G(x, y) \leq C \cdot d(x, y), \qquad \forall x, y \in K.$$

The following theorem is the main result in this section, which is a first application of the trace theorem, Theorem 6.14.

Theorem 7.2. *Let K be a $\mathcal{USC}$ and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form satisfying **(C1)**, **(C2)**. Let $\theta = -\frac{\log r}{\log k}$. Then there are $C_1, C_2 > 0$ depending only on k such that*

$$C_1 \cdot d_G(x, y)^\theta \leq R(x, y) \leq C_2 \cdot d_G(x, y)^\theta, \qquad \forall x, y \in K,$$
$$C_1 \cdot \rho^\theta \leq R\big(x, K \setminus B^{(G)}_\rho(x)\big) \leq C_2 \cdot \rho^\theta, \qquad \forall x \in K, 0 < \rho \leq 1,$$

where $B^{(G)}_\rho(x) = \{y \in K : d_G(x, y) < \rho\}$.

Proof. It suffices to show $R(x, y) \leq C_2 \cdot d_G(x, y)^\theta$ and $C_1 \cdot \rho^\theta \leq R\big(x, K \setminus B^{(G)}_\rho(x)\big)$ for some $C_1, C_2 > 0$ depending only on k.

"Proof of $R(x, y) \leq C_2 \cdot d_G(x, y)^\theta$". First, by Proposition 6.10 and a routine chain argument, we see that if $x', y' \in \partial_0 K$, then $R(x', y') \leq C_3$, for some $C_3 > 0$ depending only on k. Then, $R(x', y') \leq C_3 + C_3 \cdot r + C_3 \cdot r^2 + \cdots = \frac{C_3}{1-r} := C_4$ for any $x', y' \in K$. As a consequence, one can easily check that $R(x, y) \leq C_4 \cdot r^n$ if there is $w \in W_n$ such that $\{x, y\} \subset F_w K$.

Now, we consider x, y in the statement. We choose n such that $k^{-n-1} < d_G(x, y) \leq k^{-n}$, and $w, w' \in W_n$ such that $x \in F_w K, y \in F_{w'} K$. Noticing that $d(\cdot, \cdot) \leq d_G(\cdot, \cdot)$, so there are

at most $C_5 = [\pi(1+\sqrt{2})^2] + 1$ many $w'' \in W_n$ such that $d_G(x, F_{w''}K) \le k^{-n}$. Thus, we can find a chain $w = w^{(0)}, w^{(1)}, \cdots, w^{(l)} = w'$ in W_n such that

$$l \le C_5, \quad F_{w^{(l')}}K \cap F_{w^{(l'-1)}}K \neq \emptyset, \quad \forall 1 \le l' \le l.$$

Thus, we get $R(x,y) \le C_4C_5 \cdot r^n \le C_2 \cdot d_G(x,y)^\theta$ for some $C_2 > 0$ depending only on k.

"Proof of $C_1 \cdot \rho^\theta \le R\big(x, K \setminus B_\rho^{(G)}(x)\big)$*"*. The proof is based on the trace theorem, Theorem 6.14, and the observation that

$$d_G(x, \partial_0 K) \le C_6, \forall x \in K, \text{ for some } C_6 > 0 \text{ depending only on } k. \tag{7.1}$$

In fact, a very rough chain argument gives that $C_6 \le \frac{N}{k} \cdot (1 - \frac{1}{k})^{-1}$. We choose n so that $k^{-n} \le C_6^{-1} \cdot \frac{\rho}{4} < k^{-n+1}$. Define $u \in C(\partial_n K)$ to be

$$u(y) = \big(1 - \rho^{-1} d_G(x,y)\big) \vee 0, \quad \forall y \in \partial_n K.$$

Clearly, u is Lipschitz on each edge $F_w L_i$, $w \in W_n$, $i \in \{1,2,3,4\}$, i.e. $|u(y) - u(y')| \le \rho d(y,y')$, $\forall y, y' \in F_w L_i$.

Let $h \in \mathcal{H}_n$ such that $h|_{\partial_n K} = u$ as in Theorem 6.14. It is easy to check the following:

(1). $h(x) \ge \frac{3}{4}$ and $h|_{K \setminus B_\rho^{(G)}(x)} \le \frac{1}{4}$.

(2). $h|_{F_w K} = 0$ for any $w \in W_n$ such that $F_w K \cap B_\rho^{(G)}(x) = \emptyset$. In particular, $h|_{F_w K}$ is nonzero on at most $C_7 = \big[\pi(4C_6 k + \sqrt{2})^2\big] + 1$ many cells $F_w K$ with $w \in W_n$ (similar to C_5 in the proof of the upper bound estimate).

Thus, by (2) and Theorem 6.14,

$$\mathcal{E}(h) \le C_8 \cdot \sum_{w \in W_n} r^{-n} [[(h \circ F_w)|_{\partial_0 K}]]^2_{\Lambda_{2,2}^{\sigma(r)}(\partial_0 K)} \le C_9 \cdot r^{-n},$$

for some $C_8, C_9 > 0$ depending only on k. This together with (1) yields the desired estimate. □

8. Uniqueness

In this section, we prove the uniqueness of the self-similar Dirichlet form on $\mathcal{USC}$ under some assumptions.

Theorem 8.1. *Let K be a $\mathcal{USC}$ and μ be the normalized Hausdorff measure on K. There is a unique (up to a constant multiplier) strongly local, regular, irreducible, symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\mu)$ that is self-similar in the sense of Definition 5.1 and $\mathcal{F}$ is compactly embedded in $L^2(K,\mu)$.*

Let's first show that the conditions **(C1)**, **(C2)** are satisfied for any form under consideration in the above theorem.

Lemma 8.2. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, irreducible, symmetric self-similar Dirichlet form on $L^2(K,\mu)$. There is $C > 0$ such that $(C\mathcal{E}, \mathcal{F})$ satisfies **(C1)**.*

Proof. In this proof, all the functions are taken to be quasi-continuous, and we say $f = g$ on A if $f(x) = g(x)$ for quasi every $x \in A$.

First, it is clear that $Cap(\partial_0 K) > 0$. In fact, since $(\mathcal{E}, \mathcal{F})$ is irreducible and local, $P_x(\dot{\sigma}_{K\setminus F_1 K} < \infty) = P_x(\dot{\sigma}_{F_1(\partial_0 K)} < \infty) > 0$ for quasi every $x \in F_1 K$ by Theorem 4.7.1 and Theorem 4.5.1 in []. By self-similarity, we then have $P_x(\dot{\sigma}_{\partial_0 K} < \infty) > 0$ for quasi every $x \in K$. This implies $Cap(\partial_0 K) > 0$ by Theorem 4.2.1 in [].

In particular, we know $Cap(L_2) = Cap(L_4) \geq \frac{Cap(\partial_0 K)}{4} > 0$. Since $(\mathcal{E}, \mathcal{F})$ is regular, there is a function $f \in \mathcal{F}$ so that $f|_{L_2} = 1, f|_{L_4} = 0$. In addition, by Theorem 4.6.5. in [] (we also use the fact $\mathcal{F}_e \cap L^2(K, \mu) = \mathcal{F}$, where $\mathcal{F}_e$ is the extended Dirichlet space), there is $h \in \mathcal{F}$ such that $h|_{L_2 \cup L_4} = f|_{L_2 \cup L_4}$ and $\mathcal{E}(h) \leq \mathcal{E}(g)$ for any $g \in \mathcal{F}$ satisfying $g|_{L_2 \cup L_4} = f|_{L_2 \cup L_4}$. Since $(\mathcal{E}, \mathcal{F})$ is irreducible recurrent (as K is compact), we can see that $\mathcal{E}(h) > 0$, by Theorem 4.7.1 of []. So the effective resistance $R(L_2, L_4) = \mathcal{E}(h)^{-1}$ associated with $(\mathcal{E}, \mathcal{F})$ is positive and finite. This implies that we can find $C > 0$ so that $C^{-1} \cdot R(L_2, L_4) = 1$.

It remains to prove $0 < r \leq \frac{N}{k^2}$. We can assume that $r > \frac{1}{k}$ since otherwise there is nothing to prove. With this, the linear function $u : [0, 1] \to \mathbb{R}$ defined by $u(x) = x$ is in $\Lambda_{2,2}^{\sigma(r)}$ (see Definition 6.1), and by a same argument in Section 6.3, one can see that there is $h \in \mathcal{F}$ such that

$$h(x_1, x_2) = x_1, \quad \forall x = (x_1, x_2) \in \partial_0 K.$$

Without loss of generality, we assume h is harmonic in $K \setminus \partial_0 K$, i.e $\mathcal{E}(h) = \min\big\{\mathcal{E}(f) : f|_{\partial_0 K} = h|_{\partial_0 K} \text{ q.e.}\big\}$. By self-similarity, and by gluing dilated and translated copies of h together, we can obtain a $h_1 \in \mathcal{F}$ such that

$$h_1(x_1, x_2) = x_1, \quad \forall x = (x_1, x_2) \in \partial_1 K,$$

and

$$\mathcal{E}(h_1) = \frac{r^{-1} N}{k^2} \mathcal{E}(h).$$

On the other hand, we should have $\mathcal{E}(h_1) \geq \mathcal{E}(h)$. This gives that $r \leq \frac{N}{k^2}$. □

Lemma 8.3. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, irreducible, symmetric, self-similar Dirichlet form on $L^2(K, \mu)$, and assume $\mathcal{F}$ is compacted embedded in $L^2(K, \mu)$. Then **(C2)** holds.*

Proof. To verify **(C2)**, we use the additional assumption that $\mathcal{F}$ is compactly embedded in $L^2(K, \mu)$. As a simple consequence,

$$\big\|f - \int_K f d\mu\big\|^2_{L^2(K,\mu)} \leq C \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F},$$

for some constant $C > 0$. With this, and the property **(C1)** (following from Lemma 8.2), it is direct to prove

$$|f(x) - f(y)|^2 \leq C' \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F} \cap C(K) \tag{8.1}$$

for some $C' > 0$, for any $x, y \in K$, which can be fulfilled by a similar argument as Lemma 3.6. Then, by the regular property of $(\mathcal{E}, \mathcal{F})$, it follows that $\mathcal{F} \subset C(K)$ and (8.1) holds for any $f \in \mathcal{F}$. Thus, $Cap(\{x\}) > 0$ for any $x \in K$. This finishes the proof of **(C2)**. □

Thus, in the rest of this section, we can always assume **(C1)**, **(C2)**. The proof of Theorem 8.1 is an application of the heat kernel estimate. To be more precise, let $p(t, x, y)$ be the *heat*

kernel associated with the heat operator P_t of a Dirichlet form $(\mathcal{E},\mathcal{F})$, i.e. $\{P_t\}_{t>0}$ is the unique strongly continuous semigroup on $L^2(K,\mu)$ such that

$$\mathcal{E}(f)=\lim_{t\to 0}\frac{1}{t}<(1-P_t)f,f>_{L^2(K,\mu)},$$

where $<f,g>_{L^2(K,\mu)}=\int_K f(x)g(x)\mu(dx)$. We can prove the following *heat kernel estimate*,

$$\begin{aligned}c_1\cdot t^{-d_H/d_W}\exp\big(-c_2\cdot(\frac{d(x,y)^{d_W}}{t})^{\frac{1}{d_W-1}}\big)&\le p(t,x,y)\\ &\le c_3\cdot t^{-d_H/d_W}\exp\big(-c_4\cdot(\frac{d(x,y)^{d_W}}{t})^{\frac{1}{d_W-1}}\big),\quad \forall 0<t\le 1,x,y\in K,\end{aligned}\tag{8.2}$$

for some constants $c_1,c_2,c_3,c_4>0$, where $d_H=\frac{\log N}{\log k}$ is the Hausdorff dimension of K, $d_W=-\frac{\log r}{\log k}+d_H$ is named *walk dimension* in various contents []. For short, we will write $\theta=-\frac{\log r}{\log k}$.

In fact, with conditions **(C1)**, **(C2)**, a same method on affined nest fractals and on p.c.f. fractals [] applies. We briefly explain the proof in the follows.

First, using Theorem 7.2 (only the upper bound estimate for R is needed) and the self-similarity of the Dirichlet form $(\mathcal{E},\mathcal{F})$, we obtain the *Nash inequality* by a well known argument. See Theorem 8.1 in the book [] for details.

Proposition 8.4 (Nash inequality)**.** *Let $d_S=\frac{2d_H}{d_H+\theta}$ with $\theta=-\frac{\log r}{\log k}$ and $f\in\mathcal{F}$, we have*

$$\|f\|^{2+4/d_S}_{L^2(K,\mu)}\le C\cdot\big(\mathcal{E}(f)+\|f\|^2_{L^2(K,\mu)}\big)\cdot\|f\|^{4/d_S}_{L^1(K,\mu)},$$

for some constant $C>0$ depending only on K.

The Nash inequality provides an upper bound estimate $p(t,x,y)\le c_1\cdot t^{-d_S/2}$ for $0<t\le 1$. In addition, $|p(t,x,y)-p(t,x,y')|\le c_2\cdot t^{-1-d_S/2}R(y,y'),\forall 0<t\le 1,x,y,y'\in K$. See [**?**] or [] for a proof.

Furthermore, by the resistance metric estimate (Theorem 7.2) and the Ahlfors regularity of the measure μ, there is a lower bound estimate $p(t,x,y)\ge c_3\cdot t^{-d_S/2}$ and an estimate of the hitting time $c_4\cdot\rho^{d_W}\le\mathbb{E}^x\tau(x,\rho)\le c_5\cdot\rho^{d_W}$, where $\tau(x,s)=\inf\big\{t\ge 0:X_t\notin B_\rho(x)\big\}$ and $B_\rho(x)=\{y\in x:d(x,y)<\rho\}$. See [] for details.

Finally, by Theorem 3.1.1 of Barlow's book [] or by following [], noticing that $d_G\asymp d$, we can obtain (8.2).

Proposition 8.5. *Let K be a $\mathcal{USC}$ and $(\mathcal{E},\mathcal{F})$ be a Dirichlet form satisfying **(C1)**, **(C2)**. Let $\theta=-\frac{\log r}{\log k}$, $d_H=\frac{\log N}{\log k}$ and $d_W=\theta+d_H$. We have (8.2) holds for some $c_1,c_2,c_3,c_4>0$ depending only on K.*

The rest of this subsection is just the same as that in []. First, we refer to [] for a characterization of $\mathcal{F}$ as a Besov space $B^{\sigma(r)}_{2,\infty}(K)$.

For $\sigma>0$, define the Besov space $B^{\sigma}_{2,\infty}(K)$ on K as

$$B^{\sigma}_{2,\infty}(K)=\big\{f\in L^2(K,\mu):[[f]]_{B^{\sigma}_{2,\infty}(K)}<\infty\big\},$$

where

$$[[f]]_{B^{\sigma}_{2,\infty}(K)} = \sqrt{\sup_{0<\rho<1} \rho^{-2\sigma-d_H} \int_K \int_{B_\rho(x)} |f(x)-f(y)|^2 \mu(dy)\mu(dx)}.$$

Notice that the Hausdorff dimension d_H does not depend on the form $(\mathcal{E},\mathcal{F})$.

Proposition 8.6 ([]). *Let K be a $\mathcal{USC}$ and $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on K satisfying (8.2).*

(a). $\mathcal{F} = B^{d_W/2}_{2,\infty}(K)$, and there are $C_1, C_2 > 0$ depending only on c_1, c_2, c_3, c_4 such that

$$C_1 \cdot [[f]]_{B^{d_W/2}_{2,\infty}(K)} \le \mathcal{E}(f) \le C_2 \cdot [[f]]_{B^{d_W/2}_{2,\infty}(K)}, \quad \forall f \in \mathcal{F}.$$

(b). $d_W/2 = \inf\big\{\sigma > 0 : B^{\sigma}_{2,\infty}(K) = Constants\big\}$.

As an immediately consequence of Proposition 8.5 and 8.6, we see any Dirichlet forms satisfying **(C1)**, **(C2)** are not very different.

Corollary 8.7. *Let K be a $\mathcal{USC}$, and $(\mathcal{E},\mathcal{F})$, $(\mathcal{E}',\mathcal{F}')$ be two Dirichlet forms on $L^2(K,\mu)$ satisfying **(C1)**, **(C2)**. Then, we have $\mathcal{F}' = \mathcal{F}$, and there is a constant $C > 0$ depending only on K so that*

$$C^{-1} \cdot \mathcal{E}'(f) \le \mathcal{E}(f) \le C \cdot \mathcal{E}'(f), \qquad \forall f \in \mathcal{F} = \mathcal{F}'.$$

We finalize the proof with an argument of []. The following Proposition 8.8 is essentially the same as Theorem 2.1 in [].

Proposition 8.8 ([]). *Let K be a $\mathcal{USC}$, and $(\mathcal{E},\mathcal{F}), (\mathcal{E}',\mathcal{F})$ be two Dirichlet forms on $L^2(K,\mu)$ satisfying **(C1)**, **(C2)**, with a renormalization factor $0 < r \le \frac{N}{k^2}$. If*

$$\mathcal{E}(f) \le \mathcal{E}'(f), \quad \forall f \in \mathcal{F},$$

*then for $\eta > 0$, $\big((1+\eta)\mathcal{E}' - \mathcal{E}, \mathcal{F}\big)$ is also a Dirichlet form on $L^2(K,\mu)$ satisfying **(C1)**, **(C2)** with the renormalization factor r.*

proof of Theorem 8.1. By Lemma 8.2, 8.3, it suffices to consider Dirichlet forms satisfying **(C1)**, **(C2)**.

The existence follows from Theorem 5.2. We prove the uniqueness by contradiction. Assume $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}',\mathcal{F}')$ be two different Dirichlet forms satisfying **(C1)**, **(C2)**. By Corollary 8.7, we know $\mathcal{F} = \mathcal{F}'$. In addition, letting

$$\sup(\mathcal{E}'|\mathcal{E}) = \sup\big\{\frac{\mathcal{E}'(f)}{\mathcal{E}(f)} : f \in \mathcal{F}\setminus Constants\big\}, \quad \inf(\mathcal{E}'|\mathcal{E}) = \inf\big\{\frac{\mathcal{E}'(f)}{\mathcal{E}(f)} : f \in \mathcal{F}\setminus Constants\big\},$$

we know that

$$\frac{\sup(\mathcal{E}'|\mathcal{E})}{\inf(\mathcal{E}'|\mathcal{E})} \le C^2, \tag{8.3}$$

where C is the same constant in Corollary 8.7. Now for $\eta > 0$, let $\mathcal{E}''(f) = (1+\eta)\mathcal{E}'(f) - \inf(\mathcal{E}'|\mathcal{E})\mathcal{E}(f)$, for any $f \in \mathcal{F}$. By Proposition 8.8, and by choosing a suitable constant $C' > 0$, one can see that $(C'\mathcal{E}'', \mathcal{F})$ is also a Dirichlet form satisfying **(C1)**, **(C2)**. In addition,

$$\frac{\sup(C'\mathcal{E}''|\mathcal{E})}{\inf(C'\mathcal{E}''|\mathcal{E})} = \frac{\sup(\mathcal{E}''|\mathcal{E})}{\inf(\mathcal{E}''|\mathcal{E})} = \frac{(1+\eta)\sup(\mathcal{E}'|\mathcal{E}) - \inf(\mathcal{E}'|\mathcal{E})}{\eta \inf(\mathcal{E}'|\mathcal{E})} > \frac{1}{\eta} \cdot \big(\frac{\sup(\mathcal{E}'|\mathcal{E})}{\inf(\mathcal{E}'|\mathcal{E})} - 1\big).$$

When η is small, this contradicts Corollary 8.7, since a same bound estimate as (8.3) with $\mathcal{E}'$ replaced by $C'\mathcal{E}''$ should hold by Corollary 8.7. □

9. Continuity: sliding the $\mathcal{USC}$

By removing the constrain that squares living on grids, we are able to view the $\mathcal{USC}$ family (with same k, N) as a collection of moving fractals, instead of isolated ones in the $\mathcal{GSC}$ setting. In this section, we no longer focus on one $\mathcal{USC}$, but let the $\mathcal{USC}$ slide around. This situation is essentially new compared with the old setting.

Recall the *Hausdorff metric* on compact subsets of $\mathbb{R}^s$ is defined as

$$\delta(A, A') = \inf\big\{\rho : A \subset U_\rho(A'), A' \subset U_\rho(A)\big\}, \quad \forall \text{ compact } A, A' \subset \mathbb{R}^s,$$

where $U_\rho(A) = \bigcup_{x\in A} B_\rho(x)$ and $B_\rho(x) = \{y \in \mathbb{R}^s : d(x, y) < \rho\}$. We say A_n converges to A with respect to the Hausdorff metric if $\delta(A_n, A) \to 0$ as $n \to \infty$, and we simply write $A_n \to A$ for short. Clearly $A_n \to A$ implies $A_n \times A_n \to A \times A$ as well.

Definition 9.1. *For $n \geq 1$, let compact $A_n \subset \mathbb{R}^s$ and $f_n \in C(A_n)$.*

(a). We say $f_n, n \geq 1$ are equicontinuous if and only if

$$\lim_{\rho\to 0} \big(\sup\{|f_n(x) - f_n(y)| : n \geq 1, x, y \in A_n, d(x, y) < \rho\}\big) = 0.$$

(b). Let $A_n \to A$ and $f \in C(A)$ for a compact $A \in \mathbb{R}^s$. We write $f_n \rightarrowtail f$ if $f_n(x_n) \to f(x)$ for any $x_n \to x$ with $x_n \in A_n, n \geq 1$ and $x \in A$.

In this section, we will always let K be a $\mathcal{USC}$, $K_n, n \geq 1$ be a sequence of $\mathcal{USC}$ with same k, N, and assume $K_n \to K$. We will always write R for the resistance metric associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K satisfying **(C1)**, **(C2)**, and write R_n, $(\mathcal{E}_n, \mathcal{F}_n)$ for the according metric and form on $K_n, n \geq 1$. Recall that Theorem 8.1 ensures the existence and uniqueness of such forms and metrics.

The following is the main result in this section.

Theorem 9.2. *Let K, K_n, $n \geq 1$ be as above. Then $R_n \rightarrowtail R$ if and only if the geodesic metrics $d_{G,n}$ on K_n, $n \geq 1$ are equicontinuous.*

There are some consequences of Theorem 9.2. One can consider the convergence of resolvent kernels. Let u_α, $\alpha > 0$ be the *resolvent kernel* associated with $(\mathcal{E}, \mathcal{F})$, i.e. $u_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y)dt$, where $p(t, x, y)$ is the heat kernel. Let $u_{\alpha,n}$, $n \geq 1$ be the according ones on K_n.

Theorem 9.3. *Assume the geodesic metrics $d_{G,n}$ on K_n, $n \geq 1$ are equicontinuous. Then for each $\alpha > 0$, the resolvent kernels $u_{\alpha,n}$ on K_n converges to the resolvent kernel u_α on K in the sense of $u_{\alpha,n} \rightarrowtail u_\alpha$.*

See [] for a proof. Readers can read [] for more consequences of Theorem 9.3.

In [], one of the author studied the similar problem on fractals satisfying the finitely ramified property by using the technique of Γ-convergence (see Appendix C for necessary results that will be used in the $\mathcal{USC}$ setting) and the uniqueness theorem. For the $\mathcal{USC}$, additional difficulty arises on the boundaries of cells, so that we can not obtain the self-similarity of the limit form directly. This will be resolved with the trace theorem, Theorem 6.14 and an estimate of energy measure.

9.1. **Equicontinuity of the geodesic metrics.** Before proving Theorem 9.2, we would like to explain the condition about the geodesic metrics a little more. One explanation comes from Theorem 7.2, where one can obviously see that $R_n, n \geq 1$ fail to be equicontinuous if the condition does not hold. However, it is still not clear when the condition holds. For this, we would like to provide some equivalent characterizations of the condition with Proposition 9.4, 9.5.

Proposition 9.4. *$d_{G,n}, n \geq 1$ are equicontinuous if and only if $d_{G,n}(x_n, y_n) \to 0$ for any $x_n, y_n \in K_n$ such that $x_n \to x$ and $y_n \to x$ for some $x \in \square$.*

Proof. The "only if" part is obvious. We prove the "if" part by contradiction. Notice that $d_{G,n}, n \geq 1$ are equicontinuous if and only if

$$\lim_{\rho \to 0} \big(\sup\{d_{G,n}(x, y) : n \geq 1, x, y \in K_n, d(x, y) < \rho\} \big) = 0,$$

by using the triangle inequality. Assume that $d_{G,n}$ are not equicontinuous, then there are $\varepsilon > 0$, a subsequence n_l, and $x_{n_l}, y_{n_l} \in K_{n_l}, l \geq 1$ so that

$$\inf_{l \geq 1} d_{G,n_l}(x_{n_l}, y_{n_l}) \geq \varepsilon, \quad \lim_{l \to \infty} d(x_{n_l}, y_{n_l}) = 0.$$

By choosing a further subsequence if necessary, we can additionally assume $x_{n_l} \to x$, $y_{n_l} \to x$ for some $x \in \square$. This contradicts the fact that $d_{G,n}(x_n, y_n) \to 0$ for any x_n, y_n such that $x_n \to x$ and $y_n \to x$ for some $x \in \square$. □

The $\mathcal{USC}$ are complicated, it is convenient to consider their first level approximation instead.

Proposition 9.5. *Let $\{F_i\}_{i=1}^N$, $\{F_i^{(n)}\}_{i=1}^N$ be the associated i.f.s. of K, K_n, $n \geq 1$.*

(a). $K_n \to K$ if and only if $\bigcup_{i=1}^N F_i^{(n)} \square \to \bigcup_{i=1}^N F_i \square$ as $n \to \infty$.

(b). The following three conditions are equivalent:

(b1). $d_{G,n}, n \geq 1$ are equicontinuous.

(b2). $\tilde{d}_{G,n}, n \geq 1$ are equicontinuous, where $\tilde{d}_{G_n}$ is the geodesic metric on $\bigcup_{i=1}^N F_i^{(n)} \square$.

(b3). $\tilde{d}_{G,n}(x_n, y_n) \to 0$ for any $x_n, y_n \in \bigcup_{i=1}^N F_i^{(n)} \square$ such that $x_n \to x$ and $y_n \to x$ for some $x \in \square$.

Proof. (a). The "only if" part is trivial. The "if" part follows from a same proof as Corollary 3.3 in [].

(b). "(b1)⇒ (b2)". This is due to $\tilde{d}_{G,n}(x, y) \leq d_{G,n}(x, y)$, $\forall x, y \in K_n$.

"(b2)⇒ (b1)". By (7.1), $d_{G,n}(x, y) \leq C \cdot \tilde{d}_{G,n}(x, y)$ if x, y do not belong to a same 1-cell of K_n, where $C > 0$ depends only on k, N. As a consequence, by the self-similarity of K_n, one can see that for any $x, y \in K_n$,

$$d_{G,n}(x, y) \leq C \cdot k^{-l_{x,y}} \tilde{d}_{G,n}\big((F_w^{(n)})^{-1} x, (F_w^{(n)})^{-1} y\big)$$

where $w \in W_*$, $l_{x,y}$ are chosen such that $x, y \in F_w^{(n)} K_n$, $|w| = l_{x,y}$ and $l_{x,y} = \max\{l' :$ there is $w' \in W_{l'}$ so that $x, y \in F_{w'}^{(n)} K_n\}$.

As a consequence, for any $\varepsilon > 0$, there are $m > 0$ and $0 < c \leq \sqrt{2} k^{-m}$ such that

$$d_{G,n}(x, y) \leq C \cdot \big(k^{-m} \vee \sup_{1 \leq l \leq m \wedge l_{x,y}} k^{-l} \psi(k^l \big(d(x, y)\big)) \big) \leq \varepsilon,$$

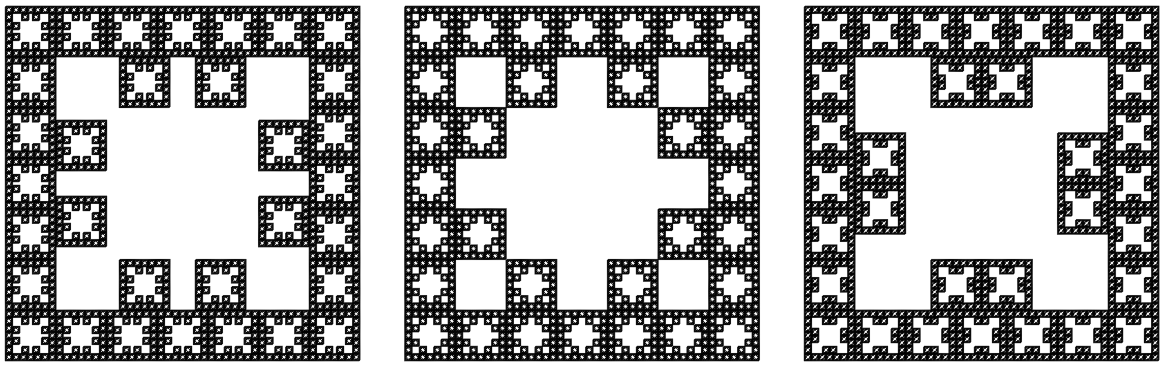
holds for any $n \geq 1, x, y \in K_n$ and $d(x,y) < c$, where $\psi \in C_+[0,\sqrt{2}]$ is chosen such that $\psi(0) = 0$ and $\tilde{d}_{G,n}(x,y) \leq \psi\big(d(x,y)\big)$ for any $x, y \in \bigcup_{i=1}^N F_i^{(n)}\square$ and $n \geq 1$.

"(b2)$\Leftrightarrow$ (b3)" follows by a same argument as Proposition 8.4. □

Example 9.6. *Let $k = 7$ and $N = 4(k-1)+8 = 32$. For $1 \leq i \leq 24$, F_i are defined as usual. Define $F_{25}(x) = \frac{1}{7}x + (z + \frac{2}{7}, \frac{1}{7})$, with $0 \leq z \leq \frac{1}{14}$. To satisfy the symmetry conditions, the contraction maps F_i for $25 \leq i \leq 32$ are uniquely determined. We write $K(z)$ for the unique compact K such that $K = \bigcup_{i=1}^{32} F_i K$, see Figure 7. We can see there are two cases.*

Case 1. *$z \neq 0$, $z \neq \frac{1}{14}$. In this case, if $z_n \to z$, then the geodesic metric $d_{G,n}$ of $K(z_n)$, $n \geq 1$ are equicontinuous.*

Case 2. *$z = 0$ or $z = \frac{1}{14}$. In this case, if $z_n \to z$, then the geodesic metric $d_{G,n}$ of $K(z_n)$, $n \geq 1$ are not equicontinuous (unless $z_n = z$ for all $n \geq n_0$ for some n_0).*



$z = \frac{1}{28}$ $\qquad$ $z = 0$ $\qquad$ $z = \frac{1}{14}$

FIGURE 7. The $\mathcal{USC}$ $K(z)$.

Finally, it is more natural to let the $\mathcal{USC}$ sliding continuously and consider whether the associated resistance metric or form changes continuously. It is obvious that Theorem 9.2 applies to the following continuously (admissible) sliding situation.

Definition 9.7 (Sliding paths of $\mathcal{USC}$)**.** *Let $\Gamma : [0,1] \to \{$the family of $\mathcal{USC}$ with same $k, N\}$.*

(a). Say Γ is a sliding path of $\mathcal{USC}$ if Γ is continuous in the Hausdorff metric.

(b). Say Γ is an admissible sliding path if in addition

$$d_{G,n}(x_n, y_n) \to 0, \text{ for any } t_n \to t \in [0,1],\ x_n, y_n \in \Gamma(t_n), \text{ and } d(x_n, y_n) \to 0.$$

Then there is a continuous version of Theorem 9.2 as follows.

Theorem 9.8. *Let Γ be an admissible sliding path, then $R(t,x,y)$ is continuous on $\mathcal{K} := \big\{(t,x,y) : x, y \in \Gamma(t), 0 \leq t \leq 1\big\}$, where $R(t,\cdot,\cdot)$ is the resistance metric on $\Gamma(t)$ (associated with the unique form satisfying **(C1)**, **(C2)**).*

A natural intuition is to split the space of $\mathcal{USC}$ with same k, N into "orbits", where any two $\mathcal{USC}$ in a same orbit are linked by an admissible sliding path. A topology on the $\mathcal{USC}$ family may be induced according to these orbits. However, it seems quite complicated, so we postpone the orbits of $\mathcal{USC}$ to further studies. See Section 10 for more details.

9.2. **The limit form.** Let K, K_n, R, R_n, $(\mathcal{E},\mathcal{F})$, $(\mathcal{E}_n,\mathcal{F}_n)$, $\{F_i\}_{i=1}^N$, $\{F_i^{(n)}\}_{i=1}^N$ be the same as above. We will adopt the following Γ-convergence on $C(\square)$. See Definition C.4 and [] for more details.

Definition 9.9. *Let $(\mathcal{E}_n,\mathcal{F}_n)$ be quadratic forms defined on $C(K_n)$, and $(\mathcal{E},\mathcal{F})$ be a quadratic form defined on $C(K)$. (set $\mathcal{E}(f)=\infty$ if $f\notin\mathcal{F}$, and the same for $\mathcal{F}_n$.) We say $(\mathcal{E}_n,\mathcal{F}_n)$ Γ-converges to $(\mathcal{E},\mathcal{F})$ on $C(\square)$ if and only if (a),(b) hold:*

(a). If $f_n\rightarrowtail f$, where $f_n\in C(K_n), n\geq 1$ and $f\in C(K)$, then

$$\mathcal{E}(f)\leq\liminf_{n\to\infty}\mathcal{E}_n(f_n).$$

(b). For each $f\in C(K)$, there exists a sequence $f_n\in C(K_n), n\geq 1$ such that $f_n\rightarrowtail f$ and

$$\mathcal{E}(f)=\lim_{n\to\infty}\mathcal{E}_n(f_n).$$

We begin with an easy observation by Theorem 7.2, Theorem C.3, Theorem C.5 and Theorem C.6.

Lemma 9.10. *Assume the geodesic metrics $d_{G,n}$ on K_n are equicontinuous, then there is a subsequence n_l and a resistance metric R_0 on K so that $R_{n_l}\rightarrowtail R_0$. In particular, $(\mathcal{E}_{n_l},\mathcal{F}_{n_l})$ Γ-converges on $C(\square)$ to $(\mathcal{E}_0,\mathcal{F}_0)$, where $(\mathcal{E}_0,\mathcal{F}_0)$ is the form associated with R_0.*

In addition, $(\mathcal{E}_0,\mathcal{F}_0)$ is strongly local and regular. Also, there is $0<r_0<1$ so that $r_{n_l}\to r_0$, where r_n is the renormalization factor of $(\mathcal{E}_n,\mathcal{F}_n)$.

To prove Theorem 9.2, we need to verify $R_0=R$. We will apply the uniqueness theorem, Theorem 8.1, but the main difficulty is to verify that $(\mathcal{E}_0,\mathcal{F}_0)$ is a self-similar form. Nevertheless, it is not hard to see a partial result. For short, let's assume $R_n\rightarrowtail R_0$ in the following lemmas.

Lemma 9.11. *Let $f\in\mathcal{F}_0$ and assume $f|_{\partial_1 K}=0$, then we have $f\circ F_i\in\mathcal{F}_0$, $1\leq i\leq N$ and $\mathcal{E}_0(f)=\sum_{i=1}^N r_0^{-1}\mathcal{E}_0(f\circ F_i)$.*

Proof. There is a sequence $f_n\in\mathcal{F}_n$ such that $f_n\rightarrowtail f$ and $\mathcal{E}_0(f)=\lim_{n\to\infty}\mathcal{E}_n(f_n)$. Clearly, for each $1\leq i\leq N$, $f_n\circ F_i^{(n)}\rightarrowtail f\circ F_i$, so by Lemma 9.10,

$$\mathcal{E}_0(f)=\lim_{n\to\infty}\mathcal{E}_n(f_n)=\lim_{n\to\infty}\sum_{i=1}^N r_n^{-1}\mathcal{E}_n(f_n\circ F_i^{(n)})\geq\sum_{i=1}^N r_0^{-1}\mathcal{E}_0(f\circ F_i).$$

This also gives that $f\circ F_i\in\mathcal{F}_0$, $1\leq i\leq N$.

To see the other direction, we will use the condition $f|_{\partial_1 K}=0$. For each $1\leq i\leq N$, we pick a sequence $g_{i,n}\in C(K_n)$, such that $g_{i,n}\rightarrowtail f\circ F_i$ and $\mathcal{E}_0(f\circ F_i)=\lim_{n\to\infty}\mathcal{E}_n(g_{i,n})$. In addition, we may assume $g_{i,n}|_{\partial_0 K}=0$, since otherwise we only need to take $g_{i,n}=\big(|g_{i,n}|-\varepsilon\big)^+\cdot sgn(g_{i,n})$, where $\varepsilon>0$ is a small enough depending on $g_{i,n}$. Then, for each n, we glue $g_{i,n}$ together to obtain a function $g_n\in C(K_n)$, so that $g_{i,n}=g_n\circ F_i^{(n)}$, $1\leq i\leq N$. Then,

$$\sum_{i=1}^N r_0^{-1}\mathcal{E}_0(f\circ F_i)=\lim_{n\to\infty}\sum_{i=1}^N r_n^{-1}\mathcal{E}_n(g_{i,n})=\lim_{n\to\infty}\mathcal{E}_n(g_n)\geq\mathcal{E}_0(f).$$

□

Since the form $(\mathcal{E}_0, \mathcal{F}_0)$ is strongly local and regular, we can apply the tool of energy measure. For each function $f \in \mathcal{F}_0$, noticing that $f \in C(K)$ in our setting, we can define the *energy measure* ν_f associated with f to be the unique Radon measure on K such that

$$\int_K g(x)\nu_f(dx) = 2\mathcal{E}_0(fg, f) - \mathcal{E}_0(f^2, g), \quad \forall g \in \mathcal{F}_0.$$

In particular, it is well known that $\nu_f(K) = 2\mathcal{E}_0(f)$ (see Lemma 3.2.3 of []). A useful fact is that the measure of ν_f on a Borel subset $A \subset K$ only depends on the value of f on A.

Lemma 9.12. *Let $f, g \in \mathcal{F}_0$ and assume $f = g$ on a subset A in K, then $\nu_f(A) = \nu_g(A)$.*

Proof. We have the triangle inequality for the energy measures $\sqrt{\nu_f(A)} \le \sqrt{\nu_g(A)} + \sqrt{\nu_{f-g}(A)}$ (see [], page 123 for a proof). Also, by [], $\nu_f\big(\{x \in K : f(x) = 0\}\big) = 0$ for any $f \in \mathcal{F}_0$. □

Let's make Lemma 9.11 a little stronger with the language of the energy measure.

Lemma 9.13. *Let $f \in \mathcal{F}_0$, then for any $A \subset K \setminus \partial_1 K$, we have*

$$\nu_f(A) = \sum_{i=1}^{N} r_0^{-1} \nu_{f\circ F_i}\big(F_i^{-1}(A)\big).$$

Proof. First, if $f|_{\partial_1 K} = 0$, we can see from Lemma 9.11 that

$$\begin{aligned}\int_K g(x)\nu_f(dx) &= 2\mathcal{E}_0(fg, f) - \mathcal{E}_0(f^2, g)\\ &= \sum_{i=1}^{N} r_0^{-1}\big(2\mathcal{E}_0((f\circ F_i)(g\circ F_i), f\circ F_i) - \mathcal{E}_0(f^2\circ F_i, g\circ F_i)\big)\\ &= \sum_{i=1}^{N} r_0^{-1}\int_K g\circ F_i(x)\nu_{f\circ F_i}(dx),\end{aligned}$$

for any $g \in \mathcal{F}_0$ such that $g|_{\partial_1 K} = 0$. Clearly, $\{g \in \mathcal{F}_0 : g|_{\partial_1 K} = 0\}$ is dense in $\{g \in C(K) : g|_{\partial_1 K} = 0\}$ in our setting (see Theorem 2.3.3 in [] for a result on general Dirichlet forms). So the lemma holds for any $f \in \mathcal{F}_0$ with $f|_{\partial_1 K} = 0$.

For general $f \in \mathcal{F}_0$, we first consider compact $A \subset K \setminus \partial_1 K$. Since $(\mathcal{E}_0, \mathcal{F}_0)$ is regular, we can find $\psi \in \mathcal{F}_0$ so that $\psi|_A = 1$ and $\psi|_{\partial_1 K} = 0$. So by Lemma 9.12,

$$\nu_f(A) = \nu_{f\psi}(A) = \sum_{i=1}^{N} r_0^{-1}\nu_{(f\psi)\circ F_i}\big(F_i^{-1}(A)\big) = \sum_{i=1}^{N} r_0^{-1}\nu_{f\circ F_i}\big(F_i^{-1}(A)\big).$$

Since ν_f is a Radon measure, the lemma follows immediately. □

9.3. **Energy on the boundary.** We now show that $\nu_f(\partial_1 K) = 0$ for any $f \in \mathcal{F}_0$, which by Lemma 9.13, will immediately yield that $(\mathcal{E}_0, \mathcal{F}_0)$ is indeed a self-similar form. We will fulfill this by an elegant application of the trace theorem, Theorem 6.14.

Lemma 9.14. *Let*

$$\mathcal{H}' = \big\{h \in \mathcal{F}_0 : \mathcal{E}_0(h) = \inf\{\mathcal{E}_0(f) : f|_{\partial_1 K} = h|_{\partial_1 K}, f \in \mathcal{F}_0\}\big\},$$

which is the space of functions that are harmonic in $K \setminus \partial_1 K$ with respect to $(\mathcal{E}_0, \mathcal{F}_0)$.

Then, we have $\mathcal{H}'|_{\partial_1 K} = \Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)$. *In addition, there are* $C_1, C_2 > 0$ *depending only on* k *such that*

$$C_1 \cdot [[h|_{\partial_1 K}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)} \leq \mathcal{E}_0(h) \leq C_2 \cdot [[h|_{\partial_1 K}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)}, \quad \forall h \in \mathcal{H}'.$$

Proof. First, we show the upper bound estimate. For a $u \in \Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)$, we can find a subsequence n_l, $l \geq 1$ and for each l, a function $u_{n_l} \in \Lambda_{2,2}^{\sigma(r_{n_l})}(\partial_1 K_{n_l})$ such that $u_{n_l} \rightarrowtail u$ (in fact, we can choose u_{n_l} to be piecewise constant). Let g_{n_l} be the harmonic extension of u_{n_l} on K_{n_l}, $l \geq 1$. Clearly g_{n_l} are equicontinuous, so we can find a limit $g \in C(K)$ and a subsequence, still denoted by n_l, $l \geq 1$, such that $g_{n_l} \rightarrowtail g$. By Theorem 6.14, we see that

$$\mathcal{E}_0(h) \leq \mathcal{E}_0(g) \leq \liminf_{l\to\infty} \mathcal{E}_{n_l}(g_{n_l}) \leq C_2 \cdot \liminf_{l\to\infty} [[u_{n_l}]]_{\Lambda_{2,2}^{\sigma(r_{n_l})}(\partial_1 K_{n_l})} \leq C_2 \cdot [[u]]_{\Lambda_{2,2}^{\sigma(r)}(\partial_1 K)},$$

where $C_2 > 0$ is the same constant in Theorem 6.14, and $h \in \mathcal{H}'$ such that $h|_{\partial_1 K} = u$.

Next, we show the lower bound estimate. For $h \in \mathcal{H}'$, we can find a sequence of functions $g_n \in \mathcal{F}_n$ such that $g_n \rightarrowtail h$ and $\mathcal{E}_0(h) = \lim_{n\to\infty} \mathcal{E}_n(g_n)$. Then by Theorem 6.14,

$$\mathcal{E}_0(h) = \lim_{n\to\infty} \mathcal{E}_n(g_n) \geq C_1 \cdot \liminf_{n\to\infty} [[g_n|_{\partial_1 K_n}]]_{\Lambda_{2,2}^{\sigma(r_n)}(\partial_1 K_n)} \geq C_1 \cdot [[h|_{\partial_1 K}]]_{\Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)},$$

where $C_1 > 0$ is the same constant in Theorem 6.14, and in the last inequality we use Fatou's lemma. □

Lemma 9.15. *For any* $f \in \mathcal{F}_0$, *we have* $\nu_f(\partial_1 K) = 0$.

Proof. For each $m \geq 1$, we divide $\partial_1 K$ into $4Nk^m$ pieces, denoted by $\{I_{l,m}\}_{l=1}^{4Nk^m}$, where each piece is of the form $F_w L_j$ with $w \in W_{m+1}$, $j \in \{1, 2, 3, 4\}$ and $F_w L_j \subset \partial_1 K$. Let $u = f|_{\partial_1 K}$, then by Lemma 9.14, $u \in \Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)$. We have the following two observations.

Observation 1. $\lim_{m\to\infty} \sum_{l=1}^{4Nk^m} [[u]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(I_{l,m})} = 0$.

Observation 2. Let $supp(u)$ *be the support of* u, *and assume* $\max_{x,y\in supp(u)} d(x, y) \leq k^{-m-1}$ *for some* $m \geq 1$, *then*

$$[[u]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)} \leq C_1 \cdot \sum_{l=1}^{4Nk^m} [[u]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(I_{l,m})},$$

for some $C_1 > 0$ *depending only on* r_0.

The first observation is obvious, so we only consider the second one. It is more convenient to consider $\tilde{u}$ define on $[0, 1]$, which is supported on a subinterval of length at most k^{-m}. It is easy to see

$$\begin{aligned}
[[\tilde{u}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}} &= \sum_{m'=0}^{\infty} r_0^{-m'} \sum_{i=1}^{k^{m'}} \big(\tilde{u}(\frac{i}{k^{m'}}) - \tilde{u}(\frac{i-1}{k^{m'}})\big)^2 \\
&\leq \sum_{i=1}^{k^m} [[\tilde{u}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}([\frac{i-1}{k^m}, \frac{i}{k^m}])} + 2 \cdot \sum_{m'=0}^{m-1} r_0^{-m'} \|\tilde{u}\|^2_{L^\infty[0,1]},
\end{aligned}$$

and by a routine chain argument, $\|\tilde{u}\|_{L^\infty[0,1]} \le C_2 \cdot r_0^{m/2} [[\tilde{u}]]_{\Lambda_{2,2}^{\sigma(r_0)}([\frac{i-1}{k^m}, \frac{i}{k^m}])}$ for some $1 \le i \le k^m$ and $C_2 > 0$ depending only on r_0. Thus $[[\tilde{u}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}} \le C_1 \cdot \sum_{i=1}^{k^m} [[\tilde{u}]]^2_{\Lambda_{2,2}^{\sigma(r_0)}([\frac{i-1}{k^m}, \frac{i}{k^m}])}$ for some $C_1 > 0$ depending only on r_0. Observation 2 then follows immediately.

Now, fix $m \ge 1$, by Observation 2 (noticing that the observation can be generalized to the case that u is constant outside a small area of diameter k^{-m}), by a routine construction of this paper, for each $1 \le l \le 4Nk^m$, we can find $u_{l,m} \in \Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)$ such that

(1). $u_{l,m}|_{I_{l,m}} = u|_{I_{l,m}}$.

(2). $[[u_{l,m}]]_{\Lambda_{2,2}^{\sigma(r_0)}(\partial_1 K)} \le C_3 \cdot [[u]]_{\Lambda_{2,2}^{\sigma(r_0)}(I_{l,m})}$, for some $C_3 > 0$ depending only on r_0.

For each l, let $h_{l,m} \in \mathcal{H}'$ such that $h_{l,m}|_{\partial_1 K} = u_{l,m}$. Then we have

$$\nu_f(\partial_1 K) = \sum_{l=1}^{4Nk^m} \nu_f(I_{l,m}) = \sum_{l=1}^{4Nk^m} \nu_{h_{l,m}}(I_{l,m}) \le 2 \cdot \sum_{l=1}^{4Nk^m} \mathcal{E}_0(h_{l,m}) \le C_4 \cdot \sum_{l=1}^{4Nk^m} [[u]]^2_{\Lambda_{2,2}^{\sigma(r_0)}(I_{l,m})},$$

for some $C_4 > 0$ depending only k, r_0, where we use Lemma 9.12 in the second equality, and use Lemma 9.14 in the last inequality. By letting $m \to \infty$, and using Observation 1, we obtain that $\nu_f(\partial_1 K) = 0$. □

Proof of Theorem 9.2. Let $(\mathcal{E}_0, \mathcal{F}_0)$ be the limit form as in Lemma 9.10. Then $\mathcal{F}_0 \subset C(K)$, and by Lemma 9.11 and Lemma 9.14, it is not hard to see that $f \in \mathcal{F}_0$ if and only if $f \circ F_i \in \mathcal{F}_0, 1 \le i \le N$. It follows from Lemma 9.13 and Lemma 9.15 that $(\mathcal{E}_0, \mathcal{F}_0)$ is a self-similar form. It is also direct to check $R_0(L_1, L_2) = 1$ by using Γ-convergence.

So we can see that $(\mathcal{E}_0, \mathcal{F}_0)$ satisfies **(C1)**, **(C2)**. By the uniqueness theorem, Theorem 8.1, it follows that $R_0 = R$, where R is the resistance metric associated with the unique form $(\mathcal{E}, \mathcal{F})$ on K satisfying **(C1)**, **(C2)**.

As a consequence of Lemma 9.10, there is a subsequence $R_{n_l}, l \ge 1$ such that $R_{n_l} \rightarrowtail R$. We have the same result holds for any subsequence of R_n, so we claim that $R_n \rightarrowtail R$. (Otherwise, we can find $\varepsilon > 0$, a subsequence n_l' and $x_l \to x, y_l \to y$ with $x_l, y_l \in K_{n_l'}$ and $x, y \in K$ such that $\left|R_{n_l'}(x_l, y_l) - R(x, y)\right| > \varepsilon$.) This finishes the proof. □

10. Further questions

Appendix A. Proof of Proposition 3.2

In this appendix, we prove Proposition 3.2. The proof is essentially the same as that in [](Theorem 2.1), with slightly adjustment due to the change of condition **(A3)**. We reproduce it here for the convenience of readers.

In the following context, all the constants $C > 0$ appeared depends only on $\mathcal{USC}$.

For $l \ge 1$ and $w \in W_n, n \ge 1$, let's define

$$\mathcal{N}_{l,w} = \big\{w' \in W_{|w|} : \text{there exist } w(i), 0 \le i \le l, w(0) = w, w(l) = w', \\ F_{w(i)}K \cap F_{w(i+1)}K \ne \emptyset, \forall 0 \le i < l\big\}.$$

In particular, note that $\mathcal{N}_{2,w} = \mathcal{N}_w$ in Definition 3.1 (c).

We first prove the inequality $N^m R_m \lambda_n \le C \cdot \lambda_{n+m}$ in Proposition 3.2, which might be affected by the change of **(A3)**.

Lemma A.1. *Let $m, n \geq 1$, and $\{\psi_w\}_{w\in W_n}$ be a collection of nonnegative functions in $l(W_{n+m})$ such that $\sum_{w\in W_n}\psi_w = 1$ and $\psi_w(v) = 0$, $\forall w \in W_n, v \in (W_n \setminus \mathcal{N}_w)\cdot W_m$.*

Then for each $f \in l(W_n)$, we have

$$\mathcal{D}_{n+m}(\tilde{f}) \leq C \cdot \big(\max_{w\in W_n} \mathcal{D}_{n+m}(\psi_w)\big)\mathcal{D}_n(f),$$

for some constant $C > 0$, where $\tilde{f} = \sum_{w\in W_n} f(w)\psi_w \in l(W_{n+m})$.

Proof. The proof is same as that of [], Lemma 2.8. For any $v \overset{n+m}{\sim} v'$ in W_{n+m}, we write $S(v, v') = \big\{w \in W_n : \psi_w(v) + \psi_w(v') > 0\big\}$ and $a(v, v') = [u]_{S_{v,v'}}$. Note that $\#S(v, v')$ is uniformly bounded depending only on the $\mathcal{USC}$. Then,

$$\begin{aligned}
\mathcal{D}_{n+m}(\tilde{f}) &= \sum_{v\overset{n+m}{\sim}v'} \big(\tilde{f}(v) - \tilde{f}(v')\big)^2\\
&= \sum_{v\overset{n+m}{\sim}v'} \Big(\sum_{w\in S_{v,v'}} \big((f(w) - a(v, v')\big)\big(\psi_w(v) - \psi_w(v')\big)\Big)^2\\
&\leq C_1 \cdot \sum_{v\overset{n+m}{\sim}v'} \sum_{w\in S_{v,v'}} \big((f(w) - a(v, v')\big)^2\big(\psi_w(v) - \psi_w(v')\big)^2\\
&= C_1 \cdot \sum_{w\in W_n} \sum_{v,v':w\in S(v,v')} \big((f(w) - a(v, v')\big)^2\big(\psi_w(v) - \psi_w(v')\big)^2
\end{aligned}$$

where $C_1 = \max_{v\overset{n+m}{\sim}v'} \#S(v, v')$. To continue the estimate, we notice that

$$\big((f(w) - a(v, v')\big)^2 \leq \max_{w'\in\mathcal{N}_{5,w}} \big(f(w) - f(w')\big)^2 \leq 5 \cdot \mathcal{D}_{n,\mathcal{N}_{5,w}}(f).$$

Thus, we have

$$\begin{aligned}
\mathcal{D}_{n+m}(\tilde{f}) &\leq 5C_1 \cdot \sum_{w\in W_n} \mathcal{D}_{n,\mathcal{N}_{5,w}}(f) \sum_{v,v':w\in S(v,v')} \big(\psi_w(v) - \psi_w(v')\big)^2\\
&\leq 5C_1 \cdot \sum_{w\in W_n} \mathcal{D}_{n,\mathcal{N}_{5,w}}(f)\big(\max_{w\in W_n} \mathcal{D}_{n+m}(\psi_w)\big).
\end{aligned}$$

The lemma follows immediately, noticing that $\sum_{w\in W_n} \mathcal{D}_{n,N_{5,w}}(f) \leq C_2 \cdot \mathcal{D}_n(f)$ for some $C_2 > 0$. □

Lemma A.2. *Let $m, n \geq 1$. There exists a collection of nonnegative functions $\{\psi_w\}_{w\in W_n}$ in $l(W_{n+m})$ such that $\sum_{w\in W_n}\psi_w = 1$, $\psi_w(v) = 0, \forall w \in W_n, v \in (W_n \setminus \mathcal{N}_w)\cdot W_m$, and*

$$\mathcal{D}_{n+m}(\psi_w) \leq C \cdot R_m^{-1}, \quad \forall w \in W_n,$$

for some constant $C > 0$.

Proof. For each $w \in W_n$, let ϕ_w be the unique function supported on $\mathcal{N}_w \cdot W_m$ such that

$$\mathcal{D}_{n+m}(\phi_w) = R_{n+m}^{-1}\big(w \cdot W_m, (W_n \setminus \mathcal{N}_w)\cdot W_m\big) \leq R_m^{-1}, \quad \phi_w|_{w\cdot W_m} = 1.$$

Let $\Phi = \sum_{w\in W_n}\phi_w$ and $\psi_w = \frac{\phi_w}{\Phi}$, noticing that $\Phi \geq 1$. We immediately have $\sum_{w\in W_n}\psi_w = 1$ and $\psi_w(v) = 0, \forall w \in W_n, v \in (W_n \setminus \mathcal{N}_w)\cdot W_m$. It remains to estimate the energy of each ψ_w.

First, we notice that $\Phi \geq 1$, so $\left|\frac{1}{\Phi(x)} - \frac{1}{\Phi(y)}\right| \leq |\Phi(x) - \Phi(y)|$. Thus, we have

$$\begin{aligned}\mathcal{D}_{n+m,\mathcal{N}_{3,w}}(1/\Phi) \leq \mathcal{D}_{n+m,\mathcal{N}_{3,w}}(\Phi) &\leq \mathcal{D}_{n+m}\Big(\sum_{w':\mathcal{N}_{2,w'}\cap\mathcal{N}_{3,w}\neq\emptyset} \phi_{w'}\Big)\\ &\leq C_1 \cdot \sum_{w':\mathcal{N}_{2,w'}\cap\mathcal{N}_{3,w}\neq\emptyset} \mathcal{D}_{n+m}(\phi_{w'}) \leq C_2 \cdot R_m^{-1},\end{aligned}$$

for some $C_1, C_2 > 0$. As ψ_w is supported on $\mathcal{N}_w = \mathcal{N}_{2,w}$, we then have

$$\begin{aligned}\sqrt{\mathcal{D}_{n+m}(\psi_w)} &= \sqrt{\mathcal{D}_{n+m,\mathcal{N}_{3,w}}(\phi_w/\Phi)}\\ &\leq \sqrt{\mathcal{D}_{n+m,\mathcal{N}_{3,w}}(\phi_w)} + \sqrt{\mathcal{D}_{n+m,\mathcal{N}_{3,w}}(1/\Phi)} \leq (C_2^{1/2}+1)\cdot R_m^{-1/2},\end{aligned}$$

noticing that $0 \leq \phi_w \leq 1, \Phi \geq 1$. □

Proposition A.3. *There is a constant $C > 0$ such that*

$$N^m R_m \lambda_n \leq C \cdot \lambda_{n+m+2}, \qquad \forall n, m \geq 1.$$

Proof. Let $f \in l(W_n)$ so that $\mathcal{D}_n(f) = 1$ and $\sum_{w\in W_n}\big(f(w) - [f]_{W_n}\big)^2 = \lambda_n$. Let $\tilde{f} \in l(W_{n+2})$ be defined as $\tilde{f}(w\cdot\tau) = f(w)$, $\forall w \in W_n, \tau \in W_2$. Clearly,

$$\mathcal{D}_{n+2}(\tilde{f}) \leq 3k^2\mathcal{D}_n(f) = 3k^2.$$

Next, we let $\{\psi_w\}_{w\in W_{n+2}}$ be defined as in Lemma A.2, and define $\tilde{\tilde{f}} = \sum_{w\in W_{n+2}} \tilde{f}(w)\psi_w$. By Lemma A.1 and A.2, we know there is a $C_1 > 0$ such that

$$\mathcal{D}_{n+m+2}(\tilde{\tilde{f}}) \leq C_1 \cdot R_m^{-1}\mathcal{D}_{n+2}(\tilde{f}) \leq 3k^2C_1 \cdot R_m^{-1}.$$

On the other hand, let $I = W_2 \setminus \big(\partial W_2 \cup \{w \in W_2 : w \overset{2}{\sim} \partial W_2\}\big)$. We have $I \neq \emptyset$, and

$$\tilde{\tilde{f}}(w\cdot\tau) = f(w), \quad \forall w \in W_n, \tau \in I\cdot W_m.$$

As a consequence, we have,

$$\sum_{w\in W_{n+m+2}} \big(\tilde{\tilde{f}}(w) - [\tilde{\tilde{f}}]_{W_{n+m+2}}\big)^2 \geq N^m \sum_{w\in W_n}\big(f(w) - [f]_{W_n}\big)^2 = N^m\lambda_n.$$

Thus, $\lambda_{n+m+2} \geq \Big(\sum_{w\in W_{n+m+2}}\big(\tilde{\tilde{f}}(w) - [\tilde{\tilde{f}}]_{W_{n+m+2}}\big)^2\Big)/\mathcal{D}_{n+m+2}(\tilde{\tilde{f}}) \geq C_2 \cdot N^m R_m\lambda_n$ for some $C_2 > 0$. □

The inequality $N^m R_m\lambda_n \leq C\cdot\lambda_{n+m}, \forall n, m \geq 1$ follows immediately if we have $\lambda_{n+m+2} \leq C'\cdot\lambda_{n+m}\sigma_2$.

In the following, we focus on the other inequalities of Proposition 3.2. First, we introduce the following easy lemma, which is also useful in Appendix B.

Lemma A.4. *Let $n, m \geq 1$, $A \subset W_n$ and $f \in l(A\cdot W_m)$. Let $f' \in l(A)$ such that $f'(w) = [f]_{w\cdot W_m}, \forall w \in A$. Then we have*

$$\mathcal{D}_{n,A}(f') \leq 8\cdot N^{-m}\sigma_m\mathcal{D}_{n+m,A\cdot W_m}(f).$$

Proof. By the definition of σ_m, we see that $\big|f'(w)-f'(w')\big|^2 \le N^{-m}\sigma_m\mathcal{D}_{n+m,\{w,w'\}\cdot W_m}(f)$. Taking the summation over all $w \overset{n}{\sim} w'$, $w,w' \in A$, noticing that each w has at most 8 neighbours, we get the desired inequality. □

Proof of Proposition 3.2. We use (1)-(5) to denote the inequalities in Proposition 3.2 in sequence.

(5). "$\big(k^2N^{-1}\big)^n \le C\cdot R_n$". For each $w \in W_*\setminus\{\emptyset\}$, we let $f(x)=\max\{0,1-c_0^{-1}d(x,F_wK)\}\in C(K)$, where c_0 is the constant in **(A3)**. Then, $f|_{F_wK}=1$ and $f|_{\bigcup_{w'\in W_{|w|}\setminus\mathcal{N}_w}}=0$. We define $f'=P_{n+|w|}f$, i.e.

$$f'(v)=N^{|v|}\int_{F_vK} f(x)\mu(dx), \forall v\in W_{|w|+n}.$$

One can easily check that $\mathcal{D}_{|w|+n}(f')\le C_1\cdot c_0^{-2}k^{-2n}N^n$, for some $C_1>0$. Thus,

$$C_1\cdot R_n\big(w\cdot W_n,(W_{|w|}\setminus\mathcal{N}_w)\cdot W_n\big)\ge c_0^2(k^2N^{-1})^n.$$

This gives the desired estimate.

(3). "$\lambda_{n+m}\le C\cdot\lambda_n\sigma_m$". Let $f\in l(W_{n+m})$ so that $\mathcal{D}_{n+m}(f)=1$ and $\sum_{v\in W_{n+m}}\big(f(v)-[f]_{W_{n+m}}\big)^2=\lambda_{n+m}$. Let $f'\in l(W_n)$ be defined as $f'(w)=[f]_{w\cdot W_m},\forall w\in W_n$. Then, by Lemma A.4, we know that

$$\mathcal{D}_n(f')\le C_2\cdot N^{-m}\sigma_m\mathcal{D}_{n+m}(f)=C_2\cdot N^{-m}\sigma_m,$$

for some constant $C_2>0$. On the other hand, we have

$$\begin{aligned}\lambda_{n+m}&=\sum_{v\in W_{n+m}}\big(f(v)-[f]_{W_{n+m}}\big)^2\\&=\sum_{w\in W_n}\sum_{v\in w\cdot W_m}\big(f(v)-f'(w)\big)^2+N^m\sum_{w\in W_n}\big(f'(w)-[f']_{W_n}\big)^2\\&\le\lambda_m\mathcal{D}_{n+m}(f)+N^m\lambda_n\mathcal{D}_n(f')\le\lambda_m+C_2\cdot\lambda_n\sigma_m.\end{aligned}$$

In addition, by Proposition A.3 and (5), we know that $\lambda_{n+m}\ge 2\cdot\lambda_m,\forall n\ge n'$ for some $n'\ge 1$. Thus, the desired inequality holds for large n. Clearly, for large $m\ge n'$, the inequality also holds. This finishes the proof.

(4). "$\sigma_{n+m}\le C\cdot\sigma_n\sigma_m$". This follows from Lemma A.4 and a same proof as (3).

(1). "$\lambda_n^{(D)}\le C\cdot\lambda_n$". Let $f\in l(W_n)$, $f|_{\partial W_n}=0$, $\mathcal{D}_n(f)=1$ and $N^n[f]^2_{W_n}=\lambda_n^{(D)}$. Let $\tilde f\in W_{n+1}$ be given by $\tilde f(1\cdot w)=f(w)$, $\tilde f(2\cdot w)=-f(w)$ and $\tilde f(i\cdot w)=0,\forall i\ne 1,2,w\in W_n$. Clearly, $\mathcal{D}_{n+1}(\tilde f)=2$, $[\tilde f]_{W_{n+1}}=0$ and

$$\sum_{v\in W_{n+1}}\big(\tilde f(v)-[\tilde f]_{W_{n+1}}\big)^2=2\Big(\sum_{w\in W_n}f^2(w)\Big)\ge 2N^{-n}\Big(\sum_{w\in W_n}f(w)\Big)^2=2\lambda_n^{(D)}.$$

So $\lambda_n^{(D)}\le\lambda_{n+1}$. The inequality follows from (3).

(2). "$N^mR_m\lambda_n\le C\cdot\lambda_{n+m}$". This follows from Proposition A.3 and (3). □

Appendix B. Γ-convergence and a short proof of Theorem 3.5

In this appendix, we briefly review the concept of Γ-convergence (with the first axiom of countability), and provide a short proof of Theorem 3.5. Readers can find general results on Γ-convergence in the book [].

Definition B.1. *Let (X,τ) be a topological space satisfying the first axiom of countability. For functions $f_n, f, n\geq 1$ on X, we say f_n Γ-converges to f if the following holds.*

(1). For any $x\in X$ and for any sequence x_n converging to x in X,

$$f(x)\leq \liminf_{n\to\infty} f_n(x_n).$$

(2). For any $x\in X$, there is a sequence x_n converging to x in X such that

$$f(x)=\lim_{n\to\infty} f_n(x_n).$$

The concept of Γ-convergence can be defined on general topological spaces. In particular, for topological spaces with the first axiom of countability, Definition B.1 is equivalent to the one for general case. See [], Proposition 8.1.

Here we consider (non-negative) quadratic forms on $L^2(K,\mu)$ (with the norm topology), where K is a $\mathcal{USC}$ and μ is the normalized Hausdorff measure on K. Recall that a quadratic form $\mathcal{E}$ is *closed* if and only if $(\mathcal{F},\|\bullet\|_{\mathcal{E}})$ is a Hilbert space, where

$$\|f\|_{\mathcal{E}}=\sqrt{\mathcal{E}(f)+\|f\|^2_{L^2(K,\mu)}},$$

and $\mathcal{F}=\big\{f\in L^2(K,\mu):\mathcal{E}(f)<\infty\big\}$. Note that we do not require $\mathcal{F}$ to be dense in $L^2(K,\mu)$ in general.

By some well known properties of Γ-convergence (See [] Proposition 6.8, Theorem 8.5, Theorem 11.10, Proposition 12.16), we know that Γ-convergence on $L^2(K,\mu)$ is weak in the following sense.

Theorem B.2. *Let $\mathcal{E}_n$ be a sequence of (nonnegative) closed quadractic forms on $L^2(K,\mu)$. There are a subsequence n_l and a closed form $\mathcal{E}$ on $L^2(K,\mu)$ such that $\mathcal{E}_{n_l}$ Γ-converges to $\mathcal{E}$.*

proof of Theorem 3.5. (a) is an easy consequence of Lemma 3.6 and **(A3)**.

(b). Recall that $\mathcal{D}_n$ can be naturally viewed as closed forms on $L^2(K,\mu)$. In fact, with some abuse of notations, for $f\in L^2(K,\mu)$, we write $\mathcal{D}_n(f)=\mathcal{D}_n(P_nf)$, where $P_n: L^1(K,\mu)\to l(W_n)$ is defined as $(P_nf)(w)=N^n\int_{F_wK} f(x)\mu(dx)$, $\forall w\in W_n$. Also, recall

$$N^{-n}\sigma_n\asymp N^{-n}\lambda_n\asymp r^{-n},$$

as a consequence of Lemma 3.4 and Proposition 3.2.

By Theorem B.2, there is a subsequence n_l and a limit form $\bar{\mathcal{E}}$ on $L^2(K,\mu)$ so that $r^{-n_l}\mathcal{D}_{n_l}$ Γ-converges to $\bar{\mathcal{E}}$.

First, we show that

$$C_1\cdot\sup_{n\geq 1} r^{-n}\mathcal{D}_n(f)\leq\bar{\mathcal{E}}(f)\leq C_2\cdot\liminf_{n\to\infty} r^{-n}\mathcal{D}_n(f),\quad\forall f\in L^2(K,\mu),\tag{B.1}$$

for some constants $C_1,C_2>0$ independent of f. Clearly, by the definition of Γ-convergence, we have

$$\bar{\mathcal{E}}(f)\leq\liminf_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(f)\leq C_2\cdot\liminf_{n\to\infty} r^{-n}\mathcal{D}_n(f),$$

where the second inequality is due to the fact $r^{-n-m} \leq C_2 \cdot r^{-n}N^{-m}\sigma_m$ for some $C_2 > 0$ and Lemma A.4. For the other direction, we pick a sequence f_l converges to f strongly in $L^2(K,\mu)$, and $\bar{\mathcal{E}}(f) = \lim_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(f_l)$. Then, clearly

$$\bar{\mathcal{E}}(f) = \lim_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(f_l) \geq C_1 \cdot \lim_{l\to\infty} r^{-n}\mathcal{D}_n(f_l) = C_1 \cdot r^{-n}\mathcal{D}_n(f), \quad \forall n \geq 1,$$

where the inequality is due to fact $r^{-n-m} \geq C_1 \cdot r^{-n}N^{-m}\sigma_m$ for some $C_1 > 0$ and Lemma A.4.

Thus, we get (B.1). Clearly $\mathcal{F} = \{f \in L^2(K,\mu) : \bar{\mathcal{E}}(f) < \infty\}$, and $(\bar{\mathcal{E}}, \mathcal{F})$ is symmetric. It remains to show the Markov property and the regular property of $(\bar{\mathcal{E}}, \mathcal{F})$.

The Markov property. Let $f \in L^2(K,\mu)$, we need to show $\bar{\mathcal{E}}(\bar{f}) \leq \bar{\mathcal{E}}(f)$, where $\bar{f} = (f\vee 0)\wedge 1$. To see this, we let $f_l \to f$ in $L^2(K,\mu)$ such that $\bar{\mathcal{E}}(f) = \lim_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(f_l)$. Without loss of generality, we can assume $(P_{n_l}f_l)(w) = f_l(x)$ for any $w \in W_{n_l}$ and $x \in F_wK$. Then, let $\bar{f}_l = (f_l \vee 0) \wedge 1$, we have $\bar{f}_l \to \bar{f}$ in $L^2(K,\mu)$, and

$$\bar{\mathcal{E}}(\bar{f}) \leq \liminf_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(\bar{f}_l) \leq \liminf_{l\to\infty} r^{-n_l}\mathcal{D}_{n_l}(f_l) = \bar{\mathcal{E}}(f),$$

where we use the Markov property of $\mathcal{D}_{n_l}$ on the space of piecewise constant functions $\big\{f \in L^2(K,\mu) : (P_{n_l}f)(w) = f(x), \forall w \in W_{n_l}, x \in F_wK\big\}$.

The regular property. Since $\mathcal{F} \subset C(K)$ by (a), it remains to show $\mathcal{F}$ is dense in $C(K)$. It suffices to show that for any $x \neq y \in K$, there is $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. We choose $w \in W_*$ so that $x \in F_wK$ and $y \notin \bigcup_{w'\in\mathcal{N}_w} F_{w'}K$. For any $n \geq |w|$, we can find $f_n \in l(W_n)$ such that $f_n|_{F_wK} = 1$, $f_n|_{\bigcup_{w'\notin\mathcal{N}_w} F_{w'}K} = 0$ and $\mathcal{D}_n(f) \leq R^{-1}_{n-|w|}$. Let f be a weak limit of f_n, one can see that $f \in \mathcal{F}$ and f seperates x, y.

Clearly, $\mathcal{F}$ is dense in $L^2(K,\mu)$ by the regular property of $(\bar{\mathcal{E}}, \mathcal{F})$. □

Appendix C. Convergence of resistance metrics

This last appendix is based on one of the author's previous work [].

We consider another setting of Γ-convergence in this section, which is related to the convergence of resistance metrics. A *resistance metric*, which is associated with a resistance form (in particular, such form satisfies **(C1)**, **(C2)** in this paper), can be defined as follows [].

Definition C.1. *Let X be a set. $R \in l(X \times X)$ is called a resistance metric on X if for any finite subset V of X, there exists a resistance form $\mathcal{E}_V$ on V such that*

$$R(x,y) = \sup\{\frac{|f(x)-f(y)|^2}{\mathcal{E}_V(f)} : f \in l(V) \setminus Constants\}, \quad \forall x, y \in V.$$

Let $\mathcal{RM}(X)$ denote the collection of all resistance metrics on X.

We state the basic setting of this section as follows.

(Basic setting). Let (B,d) be a compact metric space with a compact subset A, and let $\{A_n\}_{n\geq 1}$ *be a sequence of compact subsets in B such that*

$$\lim_{n\to\infty} \delta(A_n, A) \to 0,$$

where δ is the Hausdorff metric on compact subsets of B.

We list some easy facts below without proof. Readers can find details in [], Section 2.1.

Lemma C.2. *Let $f_n \in C(A_n)$ for $n \geq 1$.*

(a). If $\{f_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous, then there is a subsequence $\{f_{n_l}\}_{l\geq 1}$ and $f \in C(A)$ such that $f_{n_l} \rightarrowtail f$.

(b) If $f_n \rightarrowtail f$ for some $f \in l(A)$, then $\{f_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous, and $f \in C(A)$.

(c). Let $f \in C(A)$. $f_n \rightarrowtail f$ if and only if there exists $\{g_n\}_{n\geq 1} \cup \{g\} \subset C(B)$ such that $g_n|_{A_n} = f_n, \forall n \geq 1$, $g|_A = f$ and $g_n \rightrightarrows g$. Here $g_n \rightrightarrows g$ means g_n converges to g uniformly.

An important consequence of the lemma is that we can create resistance metric at the limit. The proof is rudimentary, see [], Section 2.2 for details.

Theorem C.3. *Let $R_n \in \mathcal{RM}(A_n)$ for $n \geq 1$, and assume*

$$\psi_1\big(d(x,y)\big) \leq R_n(x,y) \leq \psi_2\big(d(x,y)\big), \quad \forall n \geq 1, \forall x, y \in A_n$$

for some $\psi_1, \psi_2 \in C[0,\infty)$ with $\psi_1(0) = \psi_2(0) = 0$ and $\psi_2(t) \geq \psi_1(t) > 0$ for $t > 0$.

Then, there exists $R \in \mathcal{RM}(A)$ and a subsequence $\{n_l\}_{l\geq 1}$ such that $R_{n_l} \rightarrowtail R$. In addition,

$$\psi_1\big(d(x,y)\big) \leq R(x,y) \leq \psi_2\big(d(x,y)\big), \quad \forall x, y \in A.$$

It is well known (see the book [] by Kigami) that if R is a resistance metric on X so that (X, R) is separable metric space, there is a unique form, named *resistance form*, $(\mathcal{E}, \mathcal{F})$ on (X, R) such that $\mathcal{F} \subset C(X, R)$, and

$$R(x,y)^{-1} = \inf\big\{\mathcal{E}(f) : f(x) = 0, f(y) = 1, f \in \mathcal{F}\big\}, \quad \forall x, y \in X.$$

The form is called *regular* if $\mathcal{F}$ is dense in $C(X, R)$. If (X, R) is compact, then $(\mathcal{E}, \mathcal{F})$ is always regular. In particular, if μ is a Radon measure on X with full support, then $(\mathcal{E}, \mathcal{F})$ becomes a regular Dirichlet form on $L^2(X, \mu)$. See [] for details.

Definition C.4. *Let $(\mathcal{E}_n, \mathcal{F}_n)$ be quadratic forms defined on $C(A_n)$, and $(\mathcal{E}, \mathcal{F})$ be a quadratic form defined on $C(A)$. We say $(\mathcal{E}_n, \mathcal{F}_n)$ Γ-converges to $(\mathcal{E}, \mathcal{F})$ on $C(B)$ if and only if (a),(b) hold:*

(a). If $f_n \rightrightarrows f$, where $\{f_n\}_{n\geq 1} \cup \{f\} \subset C(B)$, then

$$\mathcal{E}(f|_A) \leq \liminf_{n\to\infty} \mathcal{E}_n(f_n|_{A_n}).$$

(b). For each $f \in C(B)$, there exists a sequence $\{f_n\}_{n\geq 1} \subset C(B)$ such that $f_n \rightrightarrows f$ and

$$\mathcal{E}(f|_A) = \lim_{n\to\infty} \mathcal{E}_n(f_n|_{A_n}).$$

Remark. In the definition above, we always set $\mathcal{E}(f) = \infty$ if $f \notin \mathcal{F}$ and similarly for $\mathcal{E}_n$. $f_n \rightrightarrows f$ can be replaced with $f_n|_{A_n} \rightarrowtail f|_A$ by Lemma C.2 (c). See [] for details.

The main result in this appendix is the following theorem stating that the convergence of resistance metrics will result in the Γ-convergence of the resistance forms. We copy the proof from [] for convenience of readers.

Theorem C.5. *Let $R_n \in \mathcal{RM}(A_n) \cap C(A_n^2)$ for $n \geq 1$ and $R \in \mathcal{M}(A)$. Assume that $R_n \rightarrowtail R$, then we also have $R \in \mathcal{RM}(A) \cap C(A^2)$. Let $(\mathcal{E}_n, \mathcal{F}_n)$ be the resistance form associated with R_n for $n \geq 1$, and let $(\mathcal{E}, \mathcal{F})$ be the resistance form associated with R. Then, we have $(\mathcal{E}_n, \mathcal{F}_n)$ Γ-converges to $(\mathcal{E}, \mathcal{F})$ on $C(B)$.*

Proof. That $R \in \mathcal{RM}(A) \cap C(A^2)$ follows from Lemma C.2. We need to prove (a),(b) in Definition C.4. By the remark above, we only need to let $f_n \in C(A_n)$, $f \in C(A)$ and $f_n \rightarrowtail f$ in the proof.

(a). Notice that R induces the same topology as d on A, so (A, R) is separable. Let $V_1 \subset V_2 \subset \cdots$ be a nested sequence of finite subsets of A, and assume that $V_* = \bigcup_{m\geq 1} V_m$ is dense in A. Then, by Theorem 2.3.7 in [], we have

$$\mathcal{E}(f) = \sup_{m\geq 1}[\mathcal{E}]_{V_m}(f) = \lim_{m\to\infty}[\mathcal{E}]_{V_m}(f), \quad \forall f \in C(A),$$

where $[\mathcal{E}]_{V_m}$ denotes the trace of $\mathcal{E}$ on V_m. For each $n \geq 1$, we can find a nested sequence of finite set $V_{1,n} \subset V_{2,n} \subset \cdots$ such that $\delta(V_m, V_{m,n}) \to 0$ as $n \to \infty$. In addition, for each $m \geq 1$, we require that $\#V_m = \#V_{m,n}$ for n large enough. To achieve this, we choose a converging sequence $x_n \to x$ with $x_n \in A_n$ for each $x \in V_*$. Then, one can see that (by Lemma 2.8 of [])

$$[\mathcal{E}]_{V_m}(f) = \lim_{n\to\infty}[\mathcal{E}_n]_{V_{m,n}}(f_n),$$

where f_n, f are the functions in the statement of (a). This implies

$$\mathcal{E}(f) = \lim_{m\to\infty}[\mathcal{E}]_{V_m}(f) \leq \liminf_{n\to\infty}\sup_{m\geq 1}[\mathcal{E}_n]_{V_{m,n}}(f_n) = \liminf_{n\to\infty}\mathcal{E}_n(f_n).$$

(b). It suffices to consider $f \in \mathcal{F}$. Let V_m and $V_{m,n}$, $m \geq 1$ be the same nest sequences in the proof of (a), and let $f'_n \in C(A_n), n \geq 1$ so that $f'_n \rightarrowtail f$. There exists a sequence $\{m_n\}_{n\geq 1}$ with $m_n \to \infty$ that grows slowly enough so that

$$\mathcal{E}(f) = \lim_{m\to\infty}[\mathcal{E}]_{V_m}(f) = \lim_{n\to\infty}[\mathcal{E}_n]_{V_{m_n,n}}(f'_n) = \lim_{n\to\infty}\mathcal{E}_n(f_n),$$

where f_n is the harmonic extension of $f'_n|_{V_{m_n,n}}$ on A_n with respect to $(\mathcal{E}_n, \mathcal{F}_n)$ for each $n \geq 1$. Since $\{R_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous and $\{\mathcal{E}_n(f_n)\}_{n\geq 1}$ is uniformly bounded, we can see that $\{f_n\}_{n\geq 1}$ is also uniformly bounded and equicontinuous. By applying Lemma C.2(a) and by a routine argument, we have $f_n \rightarrowtail f$. □

We end this appendix with a result concerning the local property.

Theorem C.6. *For each $n \geq 1$, let $(\mathcal{E}_n, \mathcal{F}_n)$ be a resistance form on A_n with $\mathcal{F}_n \subset C(A_n)$; let $(\mathcal{E}, \mathcal{F})$ be a resistance form on A with $\mathcal{F} \subset C(A)$. Assume $(\mathcal{E}_n, \mathcal{F}_n)$ Γ-converges to $(\mathcal{E}, \mathcal{F})$ on $C(B)$, and $(\mathcal{E}_n, \mathcal{F}_n)$ is local for each $n \geq 1$, then $(\mathcal{E}, \mathcal{F})$ is local.*

Proof. Let $f \in C(B)$ such that $f|_A \in \mathcal{F}$, we will show that $\mathcal{E}(f_+|_A, f_-|_A) = 0$, where $f_+ = f \vee 0$ and $f_- = f \wedge 0$. Then, by a linear combination, it follows immediately that $\mathcal{E}(g, h) = 0$ for any $g, h \in \mathcal{F}$ with $gh = 0$.

First, $\mathcal{E}(f|_A) \geq \mathcal{E}(|f|\big|_A)$ due to the Markov property, so $\mathcal{E}(f_+|_A, f_-|_A) \geq 0$.

Next, we prove $\mathcal{E}(f_+|_A, f_-|_A) \leq 0$. By Definition C.4 (b), we can choose $f'_{+,n} \in C(B)$ for each $n \geq 1$ so that

$$f'_{+,n} \rightrightarrows f_+ \text{ and } \mathcal{E}(f_+|_A) = \lim_{n\to\infty}\mathcal{E}_n(f'_{+,n}|_{A_n}).$$

Also, define $f'_{-,n}$ in a same way. Without loss of generality, we can assume that $f'_{+,n}|_{A_n} \in \mathcal{F}_n, f'_{-,n}|_{A_n} \in \mathcal{F}_n$ for all $n \geq 1$. There is a decaying sequence of positive numbers $\{\varepsilon_n\}_{n\geq 1}, \varepsilon_n \to$

0 such that $f_{+,n} = (f'_{+,n} - \varepsilon_n) \vee 0$ and $f_{-,n} = (f'_{-,n} + \varepsilon_n) \wedge 0$ have disjoint supports for each $n \geq 1$. Clearly, $f_{+,n} \rightrightarrows f_+$ and $f_{-,n} \rightrightarrows f_-$. Then,

$$\mathcal{E}(f|_A) \leq \liminf_{n\to\infty} \mathcal{E}_n\big((f_{+,n} + f_{-,n})|_{A_n}\big) = \liminf_{n\to\infty} \big(\mathcal{E}_n(f_{+,n}|_{A_n}) + \mathcal{E}_n(f_{-,n}|_{A_n})\big)$$
$$\leq \lim_{n\to\infty} \big(\mathcal{E}_n(f'_{+,n}|_{A_n}) + \mathcal{E}_n(f'_{-,n}|_{A_n})\big) = \mathcal{E}(f_+|_A) + \mathcal{E}(f_-|_A),$$

where we use Definition C.4 (a) in the first inequality, the local property of $(\mathcal{E}_n, \mathcal{F}_n)$ in the first equality and the Markov property in the second inequality. This implies that $\mathcal{E}(f_+|_A, f_-|_A) \leq 0$. □

References

1. S. Alexander and R. Orbach, *Density of states on fractals: "fractons",* J. Physique (Paris) Lett. 43 (1982), 625–631.
2. M.T. Barlow, *Diffusions on fractals.* Lectures on probability theory and statistics (Saint-Flour, 1995), 1–121, Lecture Notes in Math., 1690, Springer, Berlin, 1998.
3. M.T. Barlow and R.F. Bass, *The construction of Brownian motion on the Sierpinski carpet*, Ann. Inst. Henri Poincaré 25 (1989), no. 3, 225–257.
4. M.T. Barlow and R.F. Bass, *Local times for Brownian motion on the Sierpiński carpet,* Probab. Theory Related Fields 85 (1990), no. 1, 91–104.
5. M.T. Barlow and R.F. Bass, *On the resistance of the Sierpiński carpet. Proc. Roy. Soc. London Ser. A,* 431 (1990), no. 1882, 345–360.
6. M.T. Barlow and R.F. Bass, *Transition densities for Brownian motion on the Sierpinski carpet,* Probab. Theory Related Fields 91 (1992), 307–330.
7. M.T. Barlow and R.F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets,* Canad. J. Math. 51 (1999), no. 4, 673–744.
8. M.T. Barlow, R.F. Bass, T. Kumagai and A. Teplyaev, *Uniqueness of Brownian motion on Sierpinski carpets*, J. Eur. Math. Soc. 12 (2010), no. 3, 655–701.
9. M.T. Barlow and E.A. Perkins, *Brownian motion on the Sierpiński gasket,* Probab. Theory Related Fields 79 (1988), no. 4, 543–623.
10. S. Cao, *Convergence of energy forms on Sierpinski gaskets with added rotated triangle*, ArXiv: 2011.11149.
11. S. Cao, M. Hassler, H. Qiu, E. Sandine and R.S. Strichartz, *Existence and uniqueness of diffusions on the Julia sets of Misiurewicz-Sierpinski maps*, ArXiv:2008.07065.
12. S. Cao, S. Li, R.S. Strichartz and P. Talwai, *A trace theorem for Sobolev spaces on the Sierpinski gasket,* Commun. Pure Appl. Anal. 19 (2020), no. 7, 3901–3916.
13. S. Cao and H. Qiu, *Resistance forms on self-similar sets with finite ramification of finite type*, Potential Anal (2020). https://doi.org/10.1007/s11118-020-09840-w.
14. S. Cao and H. Qiu, *Brownian motion on the golden ratio Sierpinski gasket*, ArXiv:2010.05181.
15. G. Dal Maso, *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.
16. P.J. Fitzsimmons, B.M. Hambly and T. Kumagai, *Transition density estimates for Brownian motion on affine nested fractals,* Comm. Math. Phys. 165 (1994), no. 3, 595–620.
17. T.C. Flock and R.C. Strichartz, *Laplacians on a family of quadratic Julia sets I,* Trans. Amer. Math. Soc. 364 (2012), no. 8, 3915–3965.
18. M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes.* Second revised and extended edition. De Gruyter Studies in Mathematics, 19. Walter de Gruyter & Co., Berlin, 2011.
19. S. Goldstein, *Random walks and diffusions on fractals*, Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), 121–129, IMA Vol. Math. Appl., 8, Springer, New York, 1987.
20. A. Grigor'yan and M. Yang, *Local and non-local Dirichlet forms on the Sierpiński carpet,* Trans. Amer. Math. Soc. 372 (2019), no. 6, 3985–4030.

21. B.M. Hambly and T. Kumagai, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proc. London Math. Soc. (3) 78 (1999), no. 2, 431–458.
22. B.M. Hambly and S.O.G. Nyberg, *Finitely ramified graph-directed fractals, spectral asymptotics and the multidimensional renewal theorem,* Proc. Edinb. Math. Soc. (2) 46 (2003), no. 1, 1–34.
23. B. M. Hambly, V. Metz and A. Teplyaev, *Self-similar energies on post-critically finite self-similar fractals,* J. London Math. Soc., 74 (2006), 93–112.
24. S. Havlin and D. Ben-Avarham, *Diffusion in disordered media,* Adv. Phys. 36 (1987), 695-798.
25. M. Hino, *On singularity of energy measures on self-similar sets,* Probab. Theory Related Fields 132 (2005), no. 2, 265–290.
26. M. Hino and T. Kumagai, *A trace theorem for Dirichlet forms on fractals,* J. Funct. Anal. 238 (2006), no. 2, 578–611.
27. A. Jonsson, *A trace theorem for the Dirichlet form on the Sierpinski gasket,* Math. Z., 250 (2005), 599–609.
28. J. Kigami, *A harmonic calculus on the Sierpinski spaces,* Japan J. Appl. Math. 6 (1989), no. 2, 259–290.
29. J. Kigami, *A harmonic calculus on p.c.f. self-similar sets,* Trans. Amer. Math. Soc. 335 (1993), no. 2, 721–755.
30. J. Kigami, *Analysis on Fractals.* Cambridge Tracts in Mathematics, 143. Cambridge University Press, Cambridge, 2001.
31. J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates,* Mem. Amer. Math. Soc. 216 (2012), no. 1015, vi+132 pp.
32. T. Kumagai, *Regularity, closedness and spectral dimensions of the Dirichlet forms on P.C.F. self-similar sets.* J. Math. Kyoto Univ. 33 (1993), no. 3, 765–786.
33. S. Kusuoka, *A diffusion process on a fractal,* in "Probabilistic Methods in Mathematical Physics, Pro. Taniguchi Intern. Symp. (Katata/Kyoto, 1985)", Ito, K., Ikeda, N. (eds.). pp. 251-274, Academic Press, Boston, 1987.
34. S. Kusuoka and X.Y. Zhou, *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields 93 (1992), no. 2, 169–196.
35. K. Kuwae and T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. 11 (2003), no. 4, 599–673.
36. T. Lindstrøm, *Brownian motion on nested fractals,* Mem. Amer. Math. Soc. 83 (1990), no. 420, iv+128 pp.
37. V. Metz, *Hilbert's projective metric on cones of Dirichlet forms,* J. Funct. Anal. 127 (1995), no. 2, 438–455.
38. V. Metz, *Renormalization contracts on nested fractals,* J. Reine Angew. Math. 480, (1996), 161–175.
39. U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. 123 (1994), no. 2, 368–421.
40. R. Peirone, *Convergence and uniqueness problems for Dirichlet forms on fractals,* Boll. Unione Mat. Ital. Sez. B (8) 3, (2000), 431–460.
41. L.G. Rogers and A. Teplyaev, *Laplacians on the basilica Julia sets,* Comm. Pure Appl. Anal. 9 (2010), no. 1, 211–231.
42. R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters,* J. Physique Lettres, 44 (1983), pp.13–22.
43. C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals (English, French summary),* Ann. Sci. École Norm. Sup. 30 (1997), no. 5, 605–673.
44. R.S. Strichartz, *Differential Equations on Fractals: A Tutorial.* Princeton University Press, Princeton, NJ, 2006.
45. A. Teplyaev, *Harmonic coordinates on fractals with finitely ramified cell structure,* Canad. J. Math. 60 (2008), no. 2, 457–480.

Department of Mathematics, Cornell University, Ithaca 14853, USA
Email address: sc2873@cornell.edu

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China.
Email address: huaqiu@nju.edu.cn