An introduction to self-avoiding walks

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Some basic notations, Hammersley-Welsh Theorem
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A Theorem of Duminil-Copin and Smirnov
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Let $d \geq 1$, consider the lattice $\mathbb{Z}^d (\subset \mathbb{R}^d)$. For $n \geq 0$, we define a walk

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n), \gamma_i \in \mathbb{Z}^d, |\gamma_{i-1} - \gamma_i| = 1.$$

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We define a self-avoiding walk

$$\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n), \gamma_i \in \mathbb{Z}^d, |\gamma_{i-1} - \gamma_i| = 1, \gamma_i \neq \gamma_j \text{ for any } i \neq j.$$ 

$SAW_n$ — the set of all self-avoiding walks with length $n$. 

**background**
how large is $c_n$?

Denote $c_n = \#\text{SAW}_n$.

**Question 1.** $c_n =$?
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$d = 2$, $c_1 = 4$, $c_2 = 12$, $c_3 = 36$, $c_4 = 100$, ..., $c_{71} \approx 4.2 \times 10^{30}$, ...

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A rough bound for $c_n$:

$$d^n \leq c_n \leq 2d(2d - 1)^{n-1}.$$
the connective constant

Observation:

Cutting any $\gamma \in \text{SAW}_{n+m}$ at step $n$ gives a $\gamma' \in \text{SAW}_n$ plus a $\gamma'' \in \text{SAW}_m$.

Proposition

$$c_{n+m} \leq c_n c_m.$$
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**Proposition**

$$c_{n+m} \leq c_n c_m.$$ 

Hence $\mu_c \equiv \lim_{n \to \infty} c_n^{1/n} = \inf_{n \geq 1} c_n^{1/n} \in [0, \infty)$ exists, called the **connective constant**.

Remark. This works for general lattice $(\mathbb{L}, \mathbb{E})$. 

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An introduction to self-avoiding walks
the connective constant

Some examples:

\[ \mathbb{Z}^1, \mu_c = 1. \]
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\( \ldots \ldots \)

for the ladder, \( \mu_c = \frac{\sqrt{5}+1}{2}. \) (easy!!)
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Some examples:

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for the ladder, $\mu_c = \frac{\sqrt{5}+1}{2}$. (easy!!)

for the hexagonal lattice, $\mu_c = \sqrt{2 + \sqrt{2}}$. (hard!!)
There exists $\kappa > 0$ such that for any $n \geq 1$,

$$\mu_c^n \leq c_n \leq e^{\kappa \sqrt{n}} \mu_c^n.$$  

(1)
half-space walks

Choose a direction $e_1$. $\gamma = (\gamma_0, \cdots, \gamma_n) \in SAW_n$ is said to be a half-space (self-avoiding) walk if

$$\gamma_0(e_1) < \gamma_i(e_1) \quad \text{for any } i = 1, \cdots, n.$$ 

Denote by $\text{HSAW}_n$ the set of all half-space walks with length $n$ and $h_n := \#\text{HSAW}_n$. 
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**Proposition**

$$c_n \leq \sum_{m=0}^{n} h_{n-m} h_{m+1}, \quad n \geq 1. \quad (2)$$
self-avoiding bridges

\( \gamma = (\gamma_0, \cdots, \gamma_n) \in \text{SAW}_n \) is said to be a \((\text{self-avoiding}) \) bridge if

\[ \gamma_0(e_1) < \gamma_i(e_1) \leq \gamma_n(e_1) \quad \text{for any } i = 1, \cdots, n - 1. \]

Denote by \( \text{SAB}_n \) the set of all self-avoiding bridges with length \( n \) and \( b_n := \#\text{SAB}_n \).
self-avoiding bridges

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Denote by \( \text{SAB}_n \) the set of all self-avoiding bridges with length \( n \) and \( b_n := \#\text{SAB}_n \).

**Proposition**

\[ b_{n+m} \geq b_n b_m. \]

Define \( \mu_b := \lim_{n \to \infty} b_n^{1/n} = \sup_{n \geq 1} b_n^{1/n} \).

\[ b_n \leq c_n \Rightarrow \mu_b \leq \mu_c. \quad (\mu_b = \mu_c \text{ indeed!!}) \]
an unfolding argument

For any given $\gamma \in HSAW_n$, we can find a sequence of integers $a_1 > a_2 > \cdots > a_k \geq 1$ and decompose $\gamma$ into SABs with widths $a_i$, $i = 1, \cdots, k$. Denote by $h_{n,[a_1,a_2,\ldots,a_k]}$ the cardinality of the set of all such HSAWs.
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$$h_{n,[a_1,a_2,\ldots,a_k]} \leq h_{n,[a_1+a_2,\ldots,a_k]} \leq \cdots \leq h_{n,[a_1+\cdots+a_k]} = b_{n,[a_1+\cdots+a_k]}.$$
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Hence

$$h_n = \sum_{k \geq 1} \sum_{a_1 > a_2 > \cdots > a_k} h_{n,[a_1,a_2,\ldots,a_k]} \leq \sum_{k \geq 1} \sum_{a_1 > a_2 > \cdots > a_k} b_{n,[a_1+\ldots+a_k]} = \sum_{A=1}^{n} P_D(A) b_{n,[A]} \leq P_D(n) b_n,$$

where $P_D(n) := \# \text{ partitions of } n \text{ into different positive integers.}$
a result of Hardy-Ramanujan

Denote by $P_D(n)$ the number of partitions of $n$ into different positive integers.

**Theorem (Hardy-Ramanujan 1917)**

$$\log P_D(n) \sim \pi \sqrt{n/3}$$ \hspace{1cm} (3)

as $n \to \infty$. 
completing the proof of Hammersley-Welsh

Using upper bound of $P_D(n)$, we obtain

$$h_n \leq e^{C\sqrt{n}} b_n.$$ 

Combining with (2),

$$c_n \leq \sum_{m=0}^{n} h_{n-m} h_{m+1} \leq \sum_{m=0}^{n} e^{C\sqrt{n-m}} b_{n-m} \cdot e^{C\sqrt{m+1}} b_{m+1}$$

$$\leq (n+1) e^{C'\sqrt{n}} b_{n+1} \leq (n+1) e^{C'\sqrt{n}} b_{n+1} \leq e^{\kappa\sqrt{n} \mu_c^n}.$$
the known and unknown

The predicted asymptotic of $c_n$ for $\mathbb{Z}^d$:

$$c_n \sim A n^{\gamma-1} \mu_c^n$$ as $n \to \infty$.

where

$$\gamma = \begin{cases} 
1, & d = 1 \quad \text{(trivial)} \\
\frac{43}{32}, & d = 2 \quad \text{(conjecture)} \\
1.16 \ldots, & d = 3 \quad \text{(unknown)} \\
1, & d = 4 \quad \text{(logarithmic corrections)} \\
1, & d \geq 5 \quad \text{(solved by Hara-Slade)}
\end{cases}$$
mean square displacement

for $\gamma = (\gamma_0 = 0, \gamma_1, \cdots, \gamma_n) \in SAW_n$, denote by $\| \cdot \|$ the Euclidean norm. The *mean square displacement* is defined as

$$E_n \| \gamma_n \|^2 = \frac{1}{\#SAW_n} \sum_{\gamma \in SAW_n} \| \gamma_n \|^2.$$
mean square displacement

for $\gamma = (\gamma_0 = 0, \gamma_1, \cdots, \gamma_n) \in SAW_n$, denote by $|| \cdot ||$ the Euclidean norm. The *mean square displacement* is defined as

$$E_n ||\gamma_n||^2 = \frac{1}{\#SAW_n} \sum_{\gamma \in SAW_n} ||\gamma_n||^2.$$

It is predicted as

$$E_n ||\gamma_n||^2 \sim Dn^{2\nu} \quad \text{as } n \to \infty,$$

where

$$\nu = \begin{cases} 
1, & d = 1 \quad \text{(trivial)} \\
\frac{3}{4}, & d = 2 \quad \text{(conjecture)} \\
0.588 \cdots, & d = 3 \quad \text{(unknown)} \\
\frac{1}{2}, & d = 4 \quad \text{(logarithmic corrections)} \\
\frac{1}{2}, & d \geq 5 \quad \text{(solved by Hara-Slade)}
\end{cases}$$
connective constant of the hexagonal lattice

Denote by \( \mathbb{H} \) the hexagonal lattice (or the honeycomb lattice).

**Theorem (Duminil-Copin & Smirnov ’12)**

\[
\mu_c(\mathbb{H}) = \sqrt{2 + \sqrt{2}}.
\]
strategy of the proof

By Hammersley-Welsh, $\mu_c = \mu_b$. Let $z_c = \frac{1}{\sqrt{2+\sqrt{2}}}$. It is sufficient to show the following two facts:

$$b_n \leq n \left(\sqrt{2 + \sqrt{2}}\right)^n$$

for all $n \geq 1$ \ \ \Rightarrow \ \ \mu_b \leq \sqrt{2 + \sqrt{2}};

\&

$$G(z_c) = \sum_{n=0}^{\infty} c_n z_c^n = \infty$$

\Rightarrow \ \ \mu_c \geq \sqrt{2 + \sqrt{2}}.$$
Let $\Omega$ be a simply connected bounded domain in $\mathbb{H}$.

Fix any $a \in \partial \Omega$ and $\sigma > 0$ and $z > 0$, Define $F_z(x) : \Omega \rightarrow \mathbb{C}$ by

$$F_z(x) = \sum_{\gamma \subset \Omega: \ a \rightarrow x} e^{-i\sigma W_\gamma z |\gamma|},$$

where $W_\gamma$ is the winding number of $\gamma$ i.e. $W_\gamma = \frac{\pi}{3} (\# \text{left turns} - \# \text{right turns})$ and $|\gamma|$ is the length of $\gamma$. 
Lemma

If $\sigma = \frac{5}{8}$ and $z = z_c$, then for every vertex $v \in \Omega$, $F_z(x)$ satisfies the following

$$(p - v)F_z(p) + (q - v)F_z(q) + (r - v)F_z(r) = 0,$$

where $p, q, r$ are the three mid-edges of the three edges adjacent to $v$.

Remark. Property (6) means $F$ is holomorphic:

$$\oint_C F(\zeta)d\zeta = 0.$$
the upper bound for $\mu_b$

Consider the strip-like domain $S_{T,L}$ with height $T$ and width $L$. Let

$$A_{T,L}(z) = \sum_{\gamma \subset S_{T,L}, \gamma : a \rightarrow \alpha \setminus \{a\}} z^{|\gamma|},$$

$$B_{T,L}(z) = \sum_{\gamma \subset S_{T,L}, \gamma : a \rightarrow \beta} z^{|\gamma|},$$

$$C_{T,L}(z) = \sum_{\gamma \subset S_{T,L}, \gamma : a \rightarrow \epsilon \cup \epsilon'} z^{|\gamma|}.$$ 

**Lemma**

$$1 = \cos \left( \frac{3\pi}{8} \right) A_{T,L}(z_c) + B_{T,L}(z_c) + \cos \left( \frac{\pi}{4} \right) C_{T,L}(z_c). \quad (7)$$
Define $B_T(z_c) = \lim_{L \to \infty} B_{T,L}(z_c)$, we have from above that

$$B_T(z_c) \leq 1.$$  

Hence

$$b_n z_c^n \leq \sum_{T=0}^{n} B_T(z_c) \leq n,$$

which implies $b_n \leq n \left(\sqrt{2 + \sqrt{2}}\right)^n$. 

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the lower bound for $\mu_c$

Consider another region $\Omega_T$ (a partial hexagonal region), one may show that

$$G(z_c) \geq \sum_{T=1}^{\infty} \sum_{\gamma \subset \Omega_T: a \rightarrow \partial \Omega_T} z_c^{\gamma} \geq \sum_{T=1}^{\infty} 1 = \infty.$$
Let $F_0$ be the unit triangle with vertices \( \{O, a_0, a_1\} \) in \( \mathbb{R}^2 \), where \( O = (0, 0), a_0 = (1/2, \sqrt{3}/2), b_0 = (1, 0) \).

Define a sequence of graphs $F_n$ inductively by

\[
F_{n+1} = \frac{1}{2} \{ F_n \cup (F_n + a_0) \cup (F_n + b_0) \}.
\]

Let $G_n = 2^n F_n$ and $G = \bigcup_{n=0}^{\infty} G_n$. Let $a_n = 2^n a_0$, $b_n = 2^n b_0$.

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ a self-avoiding path on $G$, denote by $|\gamma|$ the length of $\gamma$.

Define

\[
\mathcal{W}^{(n)} := \{ \gamma \subset G : \gamma_0 = O, \gamma_{|\gamma|} = a_n, \gamma_i \neq b_n, i \geq 0 \}.
\]
self-avoiding walks on the pre-Sierpinski gasket

Let $Z_n(\beta) = \sum_{\gamma \in \mathcal{W}(n)} e^{-\beta|\gamma|}$.

**Theorem (Hattori-Hattori-Kusuoka ’90)**

There exists $\beta_c > 0$ such that

(i) if $\beta < \beta_c$, then $\lim_{n \to \infty} 3^{-n} \log Z_n(\beta)$ exists and is positive,

(ii) if $\beta > \beta_c$, then $\lim_{n \to \infty} 2^{-n} \log Z_n(\beta)$ exists and is negative,

and

(iii) $\lim_{n \to \infty} Z_n(\beta_c) = \frac{\sqrt{5}-1}{2}$.

**Remark.** $\exp(-\beta_c) \approx 0.437057.$
Let $\mathbb{P}_n$ be the uniform probability measure on the set all the self-avoiding paths with length $n$. Let $\| \cdot \|$ denote the Euclidean norm. Then

**Theorem (Hattori-Kusuoka '92)**

*For any $s > 0$,*

$$
\lim_{n \to \infty} \frac{1}{\log n} \mathbb{E}_{\mathbb{P}_n}[\| \gamma(n) \|^s] = \nu s,
$$

*where $\nu = \log 2 / \log \left( \frac{7-\sqrt{5}}{2} \right) \approx 0.79862$.*

Remark. the exponent $\nu = \log 2 / \log \left( \frac{7-\sqrt{5}}{2} \right) > \nu_{SRW} = \log 2 / \log 5$, where $\nu_{SRW}$ is the exponent of mean displacement of simple random walks on $G$.
how about SAWs on other fractals?

Question:
What happens on the graphs generated by other (connected) fractals?

- Does there exist some $\beta_c$ such that

$$\sum_{\gamma \subset G: \gamma_0=0} e^{-\beta \gamma} = \infty$$

while

$$\sum_{\gamma \subset G: \gamma_0=0} e^{-\beta \gamma} < \infty$$

for any $\beta > \beta_c$?

- Does there exist $\nu$ such that for any $s > 0$,

$$\lim_{n \to \infty} \frac{1}{\log n} \mathbb{E}^n[||\gamma(n)||^s] = \nu s?$$
references


Thank You!!