DIRICHLET FORMS ON UNCONSTRAINED SIERPINSKI CARPETS

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Abstract. We construct symmetric self-similar Dirichlet forms on unconstrained Sierpinski carpets, which are natural extension of planar Sierpinski carpets by allowing the small cells to live off the $1/k$ grids. The intersection of two cells can be a line segment of irrational length, and we also drop the non-diagonal assumption in this recurrent setting. A uniqueness theorem is also provided. Moreover, the additional freedom of unconstrained Sierpinski carpets allows us to slide the cells around. In this situation, we view unconstrained Sierpinski carpets as moving fractals. We prove that the self-similar Dirichlet forms will vary continuously in a $\Gamma$-convergence sense, and the generated diffusion processes, viewed as processes in $\mathbb{R}^2$, will converge in distribution.

1. Introduction

In this paper, we study the self-similar Dirichlet forms on the unconstrained Sierpinski carpets ($\mathcal{USC}$ for short). See Figure 1 for some examples.

Figure 1. Unconstrained Sierpinski carpets ($\mathcal{USC}$).

To see the meaning of the adjective “unconstrained”, let’s first recall the definition of the classical Sierpinski carpets. By dividing the unit square into $k^2, k \geq 3$ identical small squares and deleting a few symmetrically (keeping all that bordering the boundary), we get an iterated function system (i.f.s. for short) consisting of similarities that map the unit square to a remaining small square. The attraction of the i.f.s. is a Sierpinski carpet, as long as the fractal is connected. See Figure 2 for the standard Sierpinski carpet, with $k = 3$.

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Figure 2. The standard Sierpinski carpet.

Clearly, all the squares are constrained on the 1/k grids in a classical Sierpinski carpet. The larger family of fractals, the unconstrained Sierpinski carpets, are exactly those self-similar sets generated in a similar manner, with the constraint removed by allowing the 1/k sized squares to move around. See Definition 2.1 for precise details. We will construct and study the diffusions on unconstrained Sierpinski carpets.

The study of diffusion processes on fractals emerged as an independent research field in the late 80’s. Initial interest in such processes came from mathematical physicists working in the theory of disordered media [1, 31, 51]. On self-similar sets, the pioneering works are the constructions of Brownian motions on the Sierpinski gasket [25, 42] originated by Goldstein and Kusuoka independently and later [10] by Barlow and Perkins. The method features the study of a compatible sequence of graphs, and is extended to post critically finite fractals [36, 37] by Kigami. See [12, 13, 15, 16, 23, 29, 30, 45, 46, 47, 49, 50, 52, 54] and books [2, 39, 53] for the Dirichlet forms on such fractals.

The Sierpinski carpets are typical self-similar sets that are not finitely ramified. The construction of diffusions on Sierpinski carpets [3], initiated by Barlow and Bass, is a milestone in analysis on fractals.

In the pioneering work [3], the diffusion is defined to be a weak limit of renormalized Brownian motions on domains in $\mathbb{R}^2$, and the method later extends to generalized Sierpinski carpets (carpets in $\mathbb{R}^d, d \geq 3$) [7]. An alternative approach, with the language of Dirichlet forms [24], was given by Kusuoka and Zhou [43]. It remained a difficult question whether the diffusions resulted from the two methods are the same until in 2010, Barlow, Bass, Kumagai and Teplyaev provided an affirmative answer [8]. See [4, 5, 6, 7] for more properties about the diffusions on Sierpinski carpets. Also see [27] for a recent work on the standard Sierpinski carpet based on the resistance estimate [5] and $\Gamma$-convergence [19, 48].

More or less, the constructions depend on delicate structure of the Sierpinski carpets. In [3], the non-diagonal condition is required since Brownian motion on $\mathbb{R}^2$ does not hit a single point. Also, the corner moves and slide moves [3, 7] depend on the exact way that cells intersect. In [27], the resistance estimate used transformations among different graphs, which essentially depends on the local structure of the fractal. A more general framework
is provided in [13]. However, some extra conditions (GB), (LS) and the “Knight moves” method were used to fulfill the story.

Although it may not be possible to define a good diffusion on an arbitrary fractal, it is still one of the central question in this field to see how far we can loosen the geometric restrictions [2]. One of our main goal in this paper is to extend the constructions to the more general unconstrained Sierpinski carpets. In particular, we break the conditions (GB) of [43]. Since we are dealing the recurrent case, the non-diagonal condition [3, 7, 8] is also dropped.

**Theorem 1.** Let $K$ be a USC and $\mu$ be the normalized Hausdorff measure on $K$. There is a unique (up to a constant multiplier) strongly local, regular, irreducible, symmetric, self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, that satisfies the Poincaré inequality: \[ \iint (f(x) - f(y))^2 \mu(dx)\mu(dy) \leq C \cdot \mathcal{E}(f, f) \] for some $C > 0$.

Our approach to existence is purely analytic. We are inspired by [43], and adopt a few results there. To overcome the essential difficulty from the worse geometry, we develop a trace theorem. Let $\partial_n K$ be the collection of boundary of cells of side length $k^{-n}$, we use the trace theorem to estimate the energy of functions that are harmonic in $K \setminus \partial_n K$,

\[ C_1 \cdot [h|_{\partial_n K}]^2_{A^{e^2}_2(\partial_n K)} \leq \mathcal{E}(h) \leq C_2 \cdot [h|_{\partial_n K}]^2_{A^{e^2}_2(\partial_n K)}, \quad \forall h \in \mathcal{H}_n. \]

See Section 4 and Section 6 for details. Compared with [33], our story is on the opposite direction, we deduce important results including a uniform resistance estimate and an estimate of the energy measure on the boundary from the trace theorem. With all the preparation derived, the uniqueness is routine following the celebrated work [8].

The second goal, which also inspires the authors to break the constraint, is to study the family of USC as a whole. For example, one can perturb the i.f.s. of the USC on the right hand side of Figure 1 and imagine this as squares sliding along edges. A natural question is whether or not the Dirichlet form and the diffusion vary continuously accordingly. We provide an answer to this in Section 9.

**Theorem 2.** Let $K$, $K_n$, $n \geq 1$ be USC, $R$, $R_n$, $n \geq 1$ be the resistance metrics on them according to Theorem 1, and $(X, \mathbb{P}_x)$, $(X^{(n)}, \mathbb{P}^{(n)}_x)$ be the associated diffusion processes. Then $R_n \to R$ and $\mathbb{P}^{(n)}_{x_n}((X^{(n)}_t)_{t \geq 0} \in \cdot) \Rightarrow \mathbb{P}_x((X_t)_{t \geq 0} \in \cdot), \forall x_n \to x$ if and only if the geodesic metrics $d_{G,n}$ on $K_n, n \geq 1$ are equicontinuous.

See Definition 9.1 and Theorem 9.2 for the exact meaning of $R_n \to R$ and $\mathbb{P}^{(n)}_{x_n}((X^{(n)}_t)_{t \geq 0} \in \cdot) \Rightarrow \mathbb{P}_x((X_t)_{t \geq 0} \in \cdot)$. The condition on the geodesic metric is essential since it prevents the creation of new shortest path for the diffusions (Proposition 9.3). More precisely, we can not allow $x_n, y_n \in K_n$ that are bounded away in $d_{G,n}$, but get arbitrarily close in the Euclidean metric $d$. See Example 9.6 and Figure 7 for an illustration. The proof of Theorem 2 relies also on the trace theorem, as well as the uniqueness theorem.

At the end, we briefly introduce the structure of the paper. First, as the preliminary part, we introduce the precise definition and geometric properties of USC in Section 2. Later sections can be divided into two parts.
In Section 3-5 and also Appendix A, we focus on the existence of the self-similar Dirichlet forms. In Section 3, we introduce the Poincare constants from [43], and provide a much shorter proof of the existence of a limit form of discrete forms, based on Γ-convergence. For convenience of readers, we slightly rewrite the proof of relationships between different Poincare constants in [43] in Appendix A. In Section 4, we use the idea of the trace theorem in the discrete situation to obtain an estimate of the discrete resistance metrics. We use a quite different approach compared with [43], which is purely analytic. Finally, we construct the self-similar Dirichlet form in Section 5.

In Section 6-9, we prove the uniqueness and the continuity. The main tool to provide useful estimates is the trace theorem, developed in Section 6. In Section 7, we provide a uniform estimate of the resistance metrics with respect to the geodesic metrics. In Section 8, we prove the uniqueness of the self-similar Dirichlet forms following [8]. In Section 9, we prove Theorem 2. The key technique is to use the Γ-convergence developed by one of the author [12] (summarized in Appendix B), and to prove the energy measure on the boundary is 0, which is also an application of the trace theorem. The convergence of diffusions is proved with the tightness property of the sequence by [17] (and we adopt a different proof towards the convergence of finite dimensional distributions, see Appendix C).

From time to time, we will write \( a \asymp b \) for two functions (forms, and sequences) \( a \) and \( b \), if there is a constant \( C > 0 \) so that \( C^{-1} \cdot b \leq a \leq C \cdot b \). We will always abbreviate that \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \).

2. The geometry of unconstrained Sierpinski carpets

In this section, we present a brief review of the geometric properties of USC. Similar as the classical Sierpinski carpets, the USC are self-similar sets generated by replacing the initial square with smaller squares and iterating.

In this paper, we consider fractals on \( \mathbb{R}^2 \). We will simply write \( x \in \mathbb{R}^2 \) for a point in \( \mathbb{R}^2 \), and from time to time we write \( x = (x_1, x_2) \) to specify the two coordinates of \( x \). For two points \( x, y \in \mathbb{R}^2 \), we will always write

\[
\overline{x, y} = \{(1 - t)x + ty : 0 \leq t \leq 1\},
\]

for the line segment connecting \( x, y \in \mathbb{R}^2 \). The metric \( d \), throughout this paper, will always be the Euclidean metric on \( \mathbb{R}^2 \). In addition, we will write

\[
d(A, B) = \inf_{x \in A, y \in B} d(x, y),
\]

for \( A, B \subset \mathbb{R}^2 \), which is always positive if \( A, B \) are disjoint compact sets. We also write \( d(x, A) = d(\{x\}, A) \) for short. Readers should distinguish this with the Hausdorff metric \( \delta \) which will only appear in Section 9 and Appendix C.

Let \( \square \) be a unit square in \( \mathbb{R}^2 \). We let

\[
q_1 = (0, 0), \quad q_2 = (1, 0), \quad q_3 = (1, 1), \quad q_4 = (0, 1)
\]

be the four vertices of \( \square \). For convenience, we denote the group of self-isometries on \( \square \) by

\[
\mathcal{G} = \{\Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}, id, \Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3}\},
\]
where \( \Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2} \) are reflections (\( v \) for vertical, \( h \) for horizontal, \( d_1, d_2 \) for two diagonals),
\[
\begin{align*}
\Gamma_v(x_1, x_2) &= (x_1, 1 - x_2), & \Gamma_h(x_1, x_2) &= (1 - x_1, x_2), \\
\Gamma_{d_1}(x_1, x_2) &= (x_2, x_1), & \Gamma_{d_2}(x_1, x_2) &= (1 - x_2, 1 - x_1),
\end{align*}
\]
(2.1)
for \( x = (x_1, x_2) \in \square \), \( id \) is the identity mapping, and \( \Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3} \) are rotations,
\[
\begin{align*}
\Gamma_{r_1}(x_1, x_2) &= (1 - x_2, x_1), & \Gamma_{r_2} &= (\Gamma_{r_1})^2, & \Gamma_{r_3} &= (\Gamma_{r_1})^3,
\end{align*}
\]
(2.2)
around the center of \( \square \) counterclockwisely with angle \( \frac{j\pi}{2} \), \( j = 1, 2, 3 \). In this paper, we will focus on structures with \( G \)-symmetry.

**Definition 2.1 (Unconstrained Sierpinski carpets).**

Let \( k \geq 3 \) and \( 4(k - 1) \leq N \leq k^2 - 1 \). Let \( \{\Psi_i\}_{1 \leq i \leq N} \) be a finite set of similarities with the form \( \Psi_i(x) = \frac{x}{k} + c_i, c_i \in \mathbb{R}^2 \). Assume the following holds:

- (Non-overlapping). \( \Psi_i(\square) \cap \Psi_j(\square) \) is either a line segment, or a point, or empty, \( i \neq j \).
- (Connectivity). \( \bigcup_{i=1}^{N} \Psi_i(\square) \) is connected.
- (Symmetry). \( \Gamma(\bigcup_{i=1}^{N} \Psi_i(\square)) = \bigcup_{i=1}^{N} \Psi_i(\square) \) for any \( \Gamma \in \mathcal{G} \).
- (Boundary included). \( \overline{q_1 q_2} \subset \bigcup_{i=1}^{N} \Psi_i(\square) \subset \square \).

Then, call the unique compact subset \( K \subset \square \) such that
\[
K = \bigcup_{i=1}^{N} \Psi_i K
\]

an unconstrained Sierpinski carpet (USC).

**Remark.** To reduce the number of brackets, we write \( \Psi_i K \) instead of \( \Psi_i(K) \), following the notations of [53]. Similarly, we also use the notations like \( \Psi_{w L_i} \).

Here the condition \( k \geq 3 \) is to avoid trivial set by the symmetry condition. The condition \( N \geq 4(k - 1) \) is a requirement of the boundary included condition, and the condition \( N \leq k^2 - 1 \) ensures that we are dealing with a non-trivial planar self-similar set. Note that when \( k = 3, N = 8 \), \( K \) is the standard Sierpinski carpet. See Figure 3 for more examples of USC.

![Figure 3. More unconstrained Sierpinski carpets (USC).](image)

Below we highlight some basic settings.
Basic settings and notations.

(1) Throughout the paper, \( k, N \) are fixed numbers. Most constants appear in Section 6 and 7 only depend on \( k \).

(2) The squares \( \Psi_i(\square) \) along the boundary of \( \square \) is useful, and we number them in the following way. For \( 0 \leq j \leq 3 \) and \( 1 \leq i \leq k - 1 \), we stipulate that

\[
\Psi_{(k-1)j+i}(x) = \frac{1}{k} (x - q_j + 1) + q_{j+1} + \frac{i-1}{k} (q_{j+2} - q_{j+1})
\]  

with cyclic notation \( q_5 = q_1 \).

(3) For convenience, we denote the four sides of \( \square \) by

\[
L_1 = \bar{q}_1, \quad L_2 = \bar{q}_2, \quad L_3 = \bar{q}_3, \quad L_4 = \bar{q}_4.
\]

(4) We write \( \partial_0 K = \bigcup_{i=1}^{4} L_i \) for the square boundary of \( K \).

In the rest of this section, we prove some geometric properties of \( \mathcal{USC} \). Let’s first introduce some more notations.

Definition 2.2. Let \( W_0 = \{\emptyset\} \), \( W_1 = \{1, 2, \ldots, N\} \) be the alphabet associated with \( K \), and for \( n \geq 1 \), \( W_n = W_1^n := \{w = w_1 w_2 \cdots w_n : w_i \in W_1, \forall 1 \leq i \leq n\} \) be the collection of words of length \( n \). Also, write \( W_* = \bigcup_{n=0}^{\infty} W_n \) for the collection of finite words.

(a) For each \( w = w_1 \cdots w_n \in W_n \), we denote \( |w| = n \) the length of \( w \), and write

\[
\Psi_w = \Psi_{w_1} \circ \cdots \circ \Psi_{w_n}.
\]

So each \( w \in W_n \) represents a \( n \)-cell \( \Psi_w K \) in \( K \).

(b) For \( w, w' \in W_n \) with \( n \geq 1 \), we denote \( w \sim w' \) if \( \Psi_w K \cap \Psi_{w'} K \neq \emptyset \), and say \( w, w' \) are neighbouring if in addition \( w \neq w' \). For \( A, B \subset W_n \), we write \( A \sim_n B \) if \( w \sim w' \) for some \( w \in A \) and \( w' \in B \). In particular, we write \( w \sim A \) if \( \{w\} \sim A \).

(c) For \( w = w_1 \cdots w_n \in W_n, w' = w'_1 \cdots w'_m \in W_m \), we write

\[
w \cdot w' = w_1 \cdots w_n w'_1 \cdots w'_m \in W_{n+m}.
\]

For \( A \subset W_n, B \subset W_m \), we denote \( A \cdot B = \{w \cdot w' : w \in A, w' \in B\} \). In particular, we abbreviate \( \{w\} \cdot B \) to \( w \cdot B \).

(d) For \( A \subset W_n \), we write \( l(A) \) the collection of real functions on \( A \), and define a symmetric bilinear form \( \mathcal{D}_{n,A} \) on \( l(A) \) as,

\[
\mathcal{D}_{n,A}(f, g) = \sum_{w, w' \in A, w \sim w'} (f(w) - f(w')) (g(w) - g(w')), \quad \forall f, g \in l(A).
\]

In particular, we write \( \mathcal{D}_{n,A}(f) := \mathcal{D}_{n,A}(f, f) \), and \( \mathcal{D}_n := \mathcal{D}_{n,W_n} \) for short.

(e) For \( n \geq 0 \), we define

\[
\partial W_n = \{w \in W_n : \Psi_w K \cap \partial_0 K \neq \emptyset\}.
\]

Also, we write

\[
W_{n,i} = \{w \in W_n : \Psi_w K \cap L_i \neq \emptyset\}, \quad \text{for } i = 1, 2, 3, 4,
\]

and \( W_{*,i} = \bigcup_{n=0}^{\infty} W_{n,i} \).
In the following, we show that a $\mathcal{USC}$ $K$ has the following geometric properties (A1)-(A4). In particular, (A1), (A2), (A4) are the same as in [43], Section 2, while (A3) is relaxed for $\mathcal{USC}$ (we fix the constant $c_0$ in (A3) for later use).

(A1). The open set condition: there exists a non-empty open set $U \subset \mathbb{R}^2$ such that
$$\bigcup_{i=1}^{N} \Psi_i U \subset U \quad \text{and} \quad \Psi_i U \cap \Psi_j U = \emptyset, \forall i \neq j \in \{1, \cdots, N\}.$$

(A2). For $f \in l(W_n)$, $\mathcal{D}_n(f) = 0$ if and only if $f$ is a constant function.

(A3). There is a constant $0 < c_0 < 1$ satisfying the following.

(1). If $x, y \in K$ and $d(x, y) < c_0 k^{-n}$, then there exist $w, w', w'' \in W_n$ such that $x \in \Psi_w K$, $y \in \Psi_{w'} K$ and $\Psi_w K \cap \Psi_{w''} K \neq \emptyset, \Psi_{w'} K \cap \Psi_{w''} K \neq \emptyset$.

(2). If $w, w' \in W_n$ and there is no $w'' \in W_n$ so that
$$\Psi_w K \cap \Psi_{w''} K \neq \emptyset, \quad \Psi_{w'} K \cap \Psi_{w''} K \neq \emptyset,$$
then $d(x, y) \geq c_0 k^{-n}$ for any $x \in \Psi_w K$ and $y \in \Psi_{w'} K$.

(A4). $\partial W_n \neq W_n$ for $n \geq 2$.

Remark 1. By (A1), the Hausdorff dimension of $K$ is $d_H = \frac{\log N}{\log k}$, the unique solution to the equation $\sum_{i=1}^{N} (\frac{1}{k})^{\alpha_i} = 1$. Throughout the paper, we will always choose $\mu$ to be the normalized $d_H$-dimensional Hausdorff measure on $K$. In other words, $\mu$ is the unique self-similar probability measure on $K$ such that $\mu = \frac{1}{N} \sum_{i=1}^{N} \mu \circ \Psi_i^{-1}$.

Remark 2. In [43], (A3)-(2) was stated as:
If $w, w' \in W_n$ and $\Psi_w K \cap \Psi_{w'} K = \emptyset$, then $d(x, y) \geq c_0 k^{-n}$ for any $x \in \Psi_w K$ and $y \in \Psi_{w'} K$.

One can check that this does not hold if we let two cells $\Psi_i K, \Psi_j K, i \neq j \in \{1, \cdots, N\}$ to have intersection length
$$|\Psi_i K \cap \Psi_j K| = \sum_{l=1}^{\infty} k^{-\frac{l(l+1)}{2}},$$
where $|\Psi_i K \cap \Psi_j K|$ denotes the length of $\Psi_i K \cap \Psi_j K$.

Remark 3. The condition (A3) is exactly the condition that “the Euclidean metric is 2-adapted” introduced in [11] by Kigami.

Proof of (A1)-(A4).

(A1), (A2), (A4) are obvious. It remains to verify (A3).

Let
$$c_0 = \min \left\{ \frac{1}{2}, k \cdot \min \left\{ d(\Psi_i K, \Psi_j K) : \Psi_i K \cap \Psi_j K = \emptyset, 1 \leq i, j \leq N \right\} \right\}.$$

(A3)-(1). Let $x, y \in K$ with $d(x, y) < c_0 k^{-n}$ for some $n \geq 1$. Let
$$m_0 = 1 + \max \left\{ m \geq 0 : \text{there is} \tau \in W_m \text{such that} x, y \in \Psi_\tau K \right\}.$$

Case 1. $m_0 > n$.
In this case, there is $w \in W_n$ such that $x, y \in \Psi_w K$, so (A3)-(1) holds immediately.

Case 2. $m_0 \leq n$.
In this case, we show that there are $w \neq w' \in W_{m_0}$ and $i, i' \in \{1, 2, 3, 4\}$ such that
$$\Psi_w K \cap \Psi_{w'} K \neq \emptyset,$$
and
\[ x \in \bigcup_{\tilde{w} \in W_{n-m_0,i}} \Psi_{w',\tilde{w}} K, \quad y \in \bigcup_{\tilde{w} \in W_{n-m_0,i'}} \Psi_{w'',\tilde{w}} K, \quad |i - i'| = 2. \]

This makes the geometry clear, and (A3)-(1) follows easily. We prove the observation below.

First, by the definition of \( m_0 \), there is \( \tau \in W_{m_0-1} \) such that \( x \in \Psi_{\tau} K, \ y \in \Psi_{\tau} K \). Next, noticing that \( d(\Psi^{-1}_{\tau} x, \Psi^{-1}_{\tau} y) < c_0 k^{-n+(m_0-1)} \leq c_0 k^{-1} \), by the definition of \( c_0 \), we have \( 1 \leq j \neq j' \leq N \) so that \( \Psi^{-1}_{\tau} x \in \Psi_{j} K, \ \Psi^{-1}_{\tau} y \in \Psi_{j'} K \) and \( \Psi_{j} K \cap \Psi_{j'} K \neq \emptyset \). We let \( w = \tau \cdot j \) and \( w' = \tau \cdot j' \). Finally, choose \( 1 \leq i, i' \leq 4 \) such that
\[ \Psi_{w} K \cap \Psi_{w'} K \subset \Psi_{w'} L_i, \quad \Psi_{w} K \cap \Psi_{w'} K \subset \Psi_{w'} L_{i'}, \quad |i - i'| = 2. \]

If \( x \notin \bigcup_{\tilde{w} \in W_{n-m_0,i}} \Psi_{w'\tilde{w}} K \), then \( d(x, y) \geq d(x, \Psi_{w'} K) \geq k^{-n} \), which contradicts the fact \( d(x, y) < c_0 k^{-n} \). So \( x \in \bigcup_{\tilde{w} \in W_{n-m_0,i}} \Psi_{w'\tilde{w}} K \), and by a same argument \( y \in \bigcup_{\tilde{w} \in W_{n-m_0,i'}} \Psi_{w'\tilde{w}} K \).

(A3)-(2). We can prove (A3)-(2) by contradiction. Let \( w, w' \in W_n \), and assume there is no \( w'' \in W_n \) so that \( \Psi_{w} K \cap \Psi_{w''} K \neq \emptyset, \Psi_{w'} K \cap \Psi_{w''} K \neq \emptyset \). If there are \( x \in \Psi_{w} K, y \in \Psi_{w'} K \) so that \( d(x, y) < c_0 k^{-n} \), then we can find \( x' \in \Psi_{w}(K \setminus \partial K) \) and \( y' \in \Psi_{w'}(K \setminus \partial K) \) so that \( d(x', y') < c_0 k^{-n} \). In particular, \( \Psi_{w} K \) and \( \Psi_{w'} K \) are the only \( n \)-cells containing \( x' \) and \( y' \) respectively. By (A3)-(1) with respect to \( x' \) and \( y' \), we know there is \( w'' \in W_n \) satisfying \( \Psi_{w} K \cap \Psi_{w''} K \neq \emptyset, \Psi_{w'} K \cap \Psi_{w''} K \neq \emptyset \). A contradiction.

\[ \square \]

3. A LIMIT FORM OF KUSUOKA AND ZHOU

In this section, we will introduce the celebrated results of Kusuoka and Zhou [43]. First, we define the Poincare constants \( \lambda_n, R_n, \sigma_n \), which were introduced in [43]. In particular, we will provide a short proof of Theorem 3.4 (a statement combining Theorem 5.4 and Theorem 7.2 of [43] in the recurrent case) based on the method of \( \Gamma \)-convergence. The method of \( \Gamma \)-convergence in the construction of Dirichlet forms on self-similar sets was also used by Grigor’yan and Yang in [27]. Our story will not involve the constant \( \lambda_n^{(D)} \) in [43]. Also, we modify the definition of \( R_n \) for the compatibility with the weaker version of (A3)-(2).

**Definition 3.1** (Poincare constants [43]). For \( n \geq 0, A \subset W_n \) and \( f \in l(A) \), we write
\[ [f]_A = (#A)^{-1} \sum_{w \in A} f(w). \]

(a). For \( n \geq 1 \), define
\[ \lambda_n = \sup \left\{ \sum_{w \in W_n} (f(w) - [f]_{W_n})^2 : f \in l(W_n), D_n(f) = 1 \right\}. \]

(b). For \( A, B \subset W_n \) with \( A \cap B = \emptyset \), define
\[ R_n(A, B) = \max \left\{ (D_n(f))^{-1} : f \in l(W_n), f|_A = 0, f|_B = 1 \right\}, \]
the effective resistance between \( A \) and \( B \).
For \( w \in W_n \), we write
\[ \mathcal{N}_w = \bigcup \{ w' \in W_{|w|} : \text{there is } w'' \in W_{|w|} \text{ so that } \Psi_{w} K \cap \Psi_{w''} K \neq \emptyset, \Psi_{w'} K \cap \Psi_{w''} K \neq \emptyset \}. \]
call it the 2-adapted neighbourhood of \( w \) (see Chapter 2 of [41] for the adaptedness), and abbreviate \( W_{|w|} \setminus N_w \) to \( N^c_w \).

For \( m \geq 1 \), define

\[
R_m = \inf \left\{ R_{|w|+m}(w \cdot W_m, N^c_w \cdot W_m) : w \in W_\ast \setminus \{ \emptyset \} \right\}.
\]

(c). For \( m \geq 1 \) and \( w \overset{n}{\sim} w' \), define

\[
\sigma_m(w, w') = \sup \left\{ N^m([f]_w \cdot W_m - [f]_{w'} \cdot W_m)^2 : f \in l(W_{|w|+m}), D_{|w|+m, {w, w'}} \cdot W_m(f) = 1 \right\}.
\]

For \( m \geq 1 \), define

\[
\sigma_m = \sup \left\{ \sigma_m(w, w') : n \geq 1, w, w' \in W_n, w \overset{n}{\sim} w' \right\}.
\]

Remark. Proposition 3.2 consists of (2.3), (2.4) and (2.6) of Theorem 2.1 of [43], and (2.2) there will not be involved in this paper.

Properties of \( R_n \) were extensively explored in [41] by Kigami for compact metric spaces. See Lemma 4.6.15 of [41] for a generalized version of (3.2). In particular, (3.2) implies the process is recurrent on an infinite Sierpinski graph since \( N < k^2 \). It is not hard to see from (3.1) that

\[
C^{-1} \cdot N^m R_m \leq \lambda_n \leq C \cdot \sigma_m, \quad \forall n \geq 1, m \geq 1,
\]

and,

\[
R_n \geq C \cdot (k^2 N^{-1})^n, \quad \forall n \geq 1.
\]

In addition, all the constants \( \lambda_n, R_n \) and \( \sigma_n, n \geq 1 \) are positive and finite.

Proposition 3.2 (43, Theorem 2.1). For a fixed USC \( K \), there is a constant \( C > 0 \) such that

\[
C^{-1} \cdot \lambda_n N^m R_m \leq \lambda_{n+m} \leq C \cdot \lambda_n \sigma_m, \quad \forall n \geq 1, m \geq 1,
\]

and, (3.3)

\[
R_n \geq C \cdot (k^2 N^{-1})^n, \quad \forall n \geq 1.
\]

In addition, all the constants \( \lambda_n, R_n \) and \( \sigma_n, n \geq 1 \) are positive and finite.

Remark. Proposition 3.2 consists of (2.3), (2.4) and (2.6) of Theorem 2.1 of [43], and (2.2) there will not be involved in this paper.

Properties of \( R_n \) were extensively explored in [41] by Kigami for compact metric spaces. See Lemma 4.6.15 of [41] for a generalized version of (3.2). In particular, (3.2) implies the process is recurrent on an infinite Sierpinski graph since \( N < k^2 \). It is not hard to see from (3.1) that

\[
C^{-1} \cdot N^m R_m \leq \lambda_m \leq C \cdot \sigma_m,
\]

for some \( C > 0 \) independent of \( m \). In fact, by letting \( m = 1 \) in (3.1), one immediately see \( \lambda_n \approx \lambda_{n+1} \). Then, by letting \( n = 1 \) in (3.1), we get (3.3). To see that \( N^m R_m \approx \lambda_m \approx \sigma_m \), we need to verify another inequality, stated in the following condition (B).

(B). There is a constant \( C > 0 \) such that

\[
\sigma_n \leq C \cdot N^n R_n, \quad \forall n \geq 1.
\]

Lemma 3.3. Assume (A1)-(A4) and (B), then there is \( r > 0 \) such that

\[
N^{-n} \lambda_n \approx r^{-n}.
\]

In addition, \( N^n R_n \approx \lambda_n \approx \sigma_n \) and \( r \leq \frac{N}{k^2} \).

Proof. The estimate \( N^n R_n \approx \lambda_n \approx \sigma_n \) is an immediate consequence of (3.3) and (B). Then, by (3.1), there are constants \( C_1, C_2 > 0 \) so that

\[
C_1 \cdot \lambda_n \lambda_m \leq \lambda_{n+m} \leq C_2 \cdot \lambda_n \lambda_m, \quad \forall n, m \geq 1.
\]
It follows by a routine argument (see [5, 43]) that there exists \( r > 0 \) such that \( \lambda_n \asymp N^n r^{-n} \).

Finally, by (3.2), \( N^{-n} \lambda_n \asymp R_n \geq C \cdot (k^2 N^{-1})^n \), so \( r \leq \frac{N}{k^2} \).

We call \( r \) the renormalization factor in this paper. For USC, we always have \( r \leq 1 - \frac{1}{k^2} \).

With some abuse of notations, we write

\[
D_n(f) = D_n(P_n f), \quad \forall f \in L^1(K, \mu),
\]

where \( P_n : L^1(K, \mu) \to l(W_n) \) is defined as

\[
(P_n f)(w) = N^n \int_{\Psi_w K} f(x) \mu(dx), \quad \forall w \in W_n.
\]

Then, \( D_n, n \geq 1 \) can be viewed as quadratic forms on \( L^2(K, \mu) \).

In [43], it is shown that a limit form of the discrete energies \( D \) can be well defined under slightly different conditions (see (B1), (B2) of [43]). The same result still holds here. We combine Theorem 5.4 and Theorem 7.2 in [43] together into the following Theorem 3.4.

**Theorem 3.4.** Assume (A1)-(A4), (B) and \( 0 < r < 1 \). Let

\[
F = \{ f \in L^2(K, \mu) : \sup_{n \geq 1} r^{-n} D_n(f) < \infty \}.
\]

(a) Let \( \theta = -\frac{\log r}{\log k} \). Then there is a constant \( C > 0 \) such that

\[
|f(x) - f(y)|^2 \leq C \cdot d(x,y)^\theta \sup_{n \geq 1} r^{-n} D_n(f), \quad \forall x, y \in K, \forall f \in F.
\]

In particular, \( F \subset C(K) \).

(b) There are a regular symmetric Dirichlet form \((\tilde{\mathcal{E}}, \mathcal{F})\), \( C_1, C_2 > 0 \) such that

\[
C_1 \cdot \sup_{n \geq 1} r^{-n} D_n(f) \leq \tilde{\mathcal{E}}(f) \leq C_2 \cdot \liminf_{n \to \infty} r^{-n} D_n(f), \quad \forall f \in \mathcal{F}.
\]

Theorem 3.4 was proved with a probabilistic approach in [43]. In the rest of this section, we provide an alternative approach to prove Theorem 3.4 by using the method of \( \Gamma \)-convergence, but postpone the difficult verification of the condition (B) to the next section.

The alternative short proof (especially the proof of regularity of \((\tilde{\mathcal{E}}, \mathcal{F})\)) only works in the recurrent situation. In other words, we need to use \( r < 1 \). Barlow-Bass’s work [7] is the only successful case of construction of non-recurrent Dirichlet forms on self-similar sets. In particular, we will use the following observation, which comes from Theorem 7.2 of [43].

**Lemma 3.5.** There is \( C > 0 \) such that

\[
|f(w \cdot w') - [f]_w \cdot W_n|^2 \leq C \cdot N^{-n} \lambda_n D_{m+n, w \cdot W_n}(f),
\]

for any \( m \geq 0, w \in W_m, n \geq 1, w' \in W_n \) and \( f \in l(W_{m+n}) \).

**Proof.** For short, we define \( g \in l(W_n) \) by \( g(v) = f(w \cdot v), \forall v \in W_n \). Let \( w' = w'_1 \cdots w'_n \) be the word appears in the lemma. Let \( B_n = W_n \) and \( B_l = w'_1 \cdots w'_{n-l} \cdot W_l \) for \( 0 \leq l \leq n - 1 \). By applying the definition of \( \lambda_l, l \geq 1 \), we have

\[
|g|_{B_{l-1}} - [g]_{B_l}|^2 \leq N^{-l+1} \sum_{v \in B_{l-1}} (g(v) - [g]_{B_l})^2 \leq N^{-l+1} \lambda_l D_n(g).
\]
On the other hand, by using the left side of (3.1) and (3.2), we have
\[ N^{-l} \lambda_l \leq N^{-l} \cdot C_1 \frac{\lambda_n}{N^{n-l} R_{n-l}} = C_1 \cdot \frac{N^{-n} \lambda_n}{R_{n-l}} \leq C_2 \cdot \left( \frac{N}{k^2} \right)^{n-l} (N^{-n} \lambda_n), \]
for some \( C_1, C_2 > 0 \). Thus by taking \( C = \frac{C_2 N}{(1-k^{-1}N^{1/2})^2} \), we have
\[ |g(w') - [g]_{W_n}| \leq \sum_{l=1}^{n} |[g]_{B_{l-1}} - [g]_{B_l}| \leq \sum_{l=1}^{n} \sqrt{N^{-l+1} \lambda_l} \cdot \sqrt{D_n(g)} \leq \sqrt{C} \cdot \sqrt{N^{-n} \lambda_n} \cdot \sqrt{D_n(g)}. \]
The lemma then follows because \( |f(w \cdot w') - [f]_{W_{m+w}}| = |g(w') - [g]_{W_n}| \) and \( D_n(g) = D_{m+n,w-W_n}(f) \). \( \square \)

We also need the following easy observation.

**Lemma 3.6.** Let \( n, m \geq 1 \) and \( f \in l(W_{n+m}) \). Let \( f' \in l(W_n) \) such that \( f'(w) = [f]_{w-W_n}, \forall w \in W_n \). Then we have
\[ D_n(f') \leq 8 \cdot N^{-m} \sigma_m D_{n+m}(f). \]

**Proof.** By the definition of \( \sigma_m \), for any \( w \sim w' \), we see that
\[ |f'(w) - f'(w')|^2 \leq N^{-m} \sigma_m D_{n+m,\{w,w'\}}(f). \]
Taking the summation over all \( w \sim w', w, w' \in W_n \), noticing that each \( w \) has at most 8 neighbours, we get the desired inequality. \( \square \)

**Remark.** This lemma is also useful in proving Proposition 8.3, see Appendix A.

Before we prove Theorem 3.4, now we briefly recall the concept of \( \Gamma \)-convergence. To see general results on \( \Gamma \)-convergence, please refer to the book [19].

The concept of \( \Gamma \)-convergence can be defined on general topological spaces (see [19], Definition 4.1). In particular, on topological spaces with the first axiom of countability, there is an equivalent characterization that is easier to use (see [19], Proposition 8.1). We state it below in Definition 3.7.

**Definition 3.7.** Let \((X, \tau)\) be a topological space satisfying the first axiom of countability. For functions \( f_n, n \geq 1 \) and \( f \) from \( X \) to \( \mathbb{R} \cup \{-\infty, \infty\} \), we say \( f_n \) \( \Gamma \)–converges to \( f \) if the following holds.

(a) For any \( x \in X \) and for any sequence \( x_n \) converging to \( x \) in \( X \),
\[ f(x) \leq \liminf_{n \to \infty} f_n(x_n). \]

(b) For any \( x \in X \), there is a sequence \( x_n \) converging to \( x \) in \( X \) such that
\[ f(x) = \lim_{n \to \infty} f_n(x_n). \]

In this paper, we are interested in non-negative symmetric bilinear forms \((\mathcal{E}, \mathcal{F})\) on \( L^2(K, \mu) \), where the linear subspace \( \mathcal{F} \subset L^2(K, \mu) \) is called the domain, \( \mathcal{E} \) maps from \( \mathcal{F} \times \mathcal{F} := \{(f, g) : f, g \in \mathcal{F}\} \) to \( \mathbb{R} \). We say that the bilinear form \((\mathcal{E}, \mathcal{F})\) is closed if \( \mathcal{F} \) is a Hilbert space equipped with the inner product \( \mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \int_K f(x)g(x)\mu(dx) \).
To apply Γ-convergence, it’s more natural to consider the quadratic form (with extended real values) associated with the symmetric bilinear form \((E,F)\) (see Definition 11.7 of [19]):

\[
E(f) = \begin{cases} 
E(f,f), & \text{if } f \in F, \\
\infty, & \text{if } f \notin F,
\end{cases}
\]

where we still use \(E\) to denote the quadratic form \(L^2(K,\mu) \to [0,\infty]\) by a little abuse of the notation. Conversely, a functional \(E : L^2(K,\mu) \to [0,\infty]\) is a non-negative quadratic form (with extended real values) if and only if

(a). \(E(0) = 0\);
(b). \(E(tf) = t^2E(f)\), for any \(f \in L^2(K,\mu)\) and any \(t \in \mathbb{R}\);
(c). \(E(f + g) + E(f - g) = 2E(f) + 2E(g)\) for any \(f, g \in L^2(K,\mu)\).

Given such a functional, one can define a bilinear form \((E,F)\) by \(F = \{f \in L^2(K,\mu) : E(f) < \infty\}\) and \(E(f,g) = \frac{1}{2}(E(f + g) - E(f - g))\). So there is a simple one-to-one correspondence between quadratic forms and symmetric bilinear forms. For a proof of this fact, please refer to Proposition 11.9 of [19] (where a stronger result is proved).

A crucial observation is that \((E,F)\) is closed if and only if the associated quadratic form \(E\) is lower-semicontinuous. See [19], Proposition 12.16 for a proof.

By some well known properties of Γ-convergence (See [19], Proposition 6.8, Theorem 8.5 and Theorem 11.10), we know that Γ-convergence on \(L^2(K,\mu)\) is weak in the following sense.

**Proposition 3.8.** Let \(E_n, n \geq 1\) be a sequence of non-negative lower-semicontinuous quadratic forms on \(L^2(K,\mu)\) (with extended real values). There are a subsequence \(n_l, l \geq 1\) and a lower-semicontinuous non-negative quadratic form (with extended real values) \(E\) on \(L^2(K,\mu)\) such that \(E_{n_l} \Gamma\)-converges to \(E\).

**Proof of Theorem 3.4.** First, we introduce the operators \(Q_n, n \geq 0\), which will be useful for proving both (a) and (b). For \(n \geq 0\), let

\[
M_n = \{f \in L^2(K,\mu) : f(x) = P_nf(w), \text{ for any } w \in W_n \text{ and a.e. } x \in \Psi_wK\},
\]

which is the space of functions whose restriction on each \(n\)-cell is a.e. a constant. Define \(Q_n\) to be the orthogonal projection from \(L^2(K,\mu)\) to \(M_n\), and we choose a nice version (for the proof of (a)) as follows:

\[
Q_nf(x) = \max_{w \in W_n : x \in \Psi_wK} P_nf(w), \quad \forall x \in K.
\]

It is routine to see that \(Q_nf\) converges to \(f\) in \(L^2(K,\mu)\). In fact, if \(f \in C(K)\), then \(Q_nf\) converges uniformly to \(f\), hence converges to \(f\) in \(L^2(K,\mu)\). For general \(f \in L^2(K,\mu)\), for any \(\varepsilon > 0\), we choose \(f' \in C(K)\) so that \(\|f' - f\|_{L^2(K,\mu)} < \frac{\varepsilon}{3}\), and then for \(n\) large enough

\[
\|f - Q_nf\|_{L^2(K,\mu)} \leq \|f - f'\|_{L^2(K,\mu)} + \|f' - Q_nf'\|_{L^2(K,\mu)} + \|Q_nf - Q_nf'\|_{L^2(K,\mu)} < \varepsilon.
\]
(a). For \( f \in \mathcal{F}, \) we can see that \( Q_n f \) converges uniformly. In fact, for each \( x \in K \) and \( m \geq 0, \ n \geq 1, \) by Lemma \[ 3.5 \] and \[ 3.3 \] we have

\[
|Q_{m+n}f(x) - Q_m f(x)|^2 \leq \sup_{w \in W_n, w' \in W_n:x \in \Psi_{w, w'}K} |P_{m+n}f(w \cdot w') - P_m f(w)|^2 \\
\leq C_1 \cdot N^{-n} \lambda_n \mathcal{D}_{m+n}(f) \leq C_2 \cdot r^{-n} \mathcal{D}_{m+n}(f) \\
\leq C_2 \cdot r^m \sup_{n' \geq 1} r^{-n'} \mathcal{D}_{n'}(f),
\]

for some constant \( C_1, C_2 > 0. \) Since \( Q_n f \) converges to \( f \) in \( L^2(K, \mu), \) the uniform limit of \( Q_n f \) is a version of \( f. \) So in the following we assume \( f(x) = \lim_{n \to \infty} Q_n f(x), \forall x \in K. \) By letting \( n \to \infty \) in \[ 3.5 \], we get

\[
|f(x) - Q_m f(x)|^2 \leq C_2 \cdot r^m \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f), \quad \forall x \in K.
\]

Now, consider \( x \neq y \in K \) such that \( d(x, y) < c_0 k^{-1} \) (Recall the constant \( c_0 \) in \( (A3) \)). Choose \( m \geq 1 \) so that \( c_0 k^{-m-1} < d(x, y) < c_0 k^{-m}, \) and \( w, w' \in W_m \) so that \( x \in \Psi_{w, K}, Q_m f(x) = P_m f(w) \) and \( y \in \Psi_{w', K}, Q_m f(y) = P_m f(w'). \) Then, by \( (A3) \), there is \( w'' \in W_m \) such that \( w'' \sim w, w'' \sim w', \) whence

\[
|Q_m f(x) - Q_m f(y)| = |P_m f(w) - P_m f(w')| \\
\leq |P_m f(w) - P_m f(w'')| + |P_m f(w'') - P_m f(w')| \\
\leq 2 \sqrt{D_m(f)} \leq 2 \cdot \sqrt{r^m \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f)}.
\]

Combining \[ 3.6 \] and \[ 3.7 \] and the fact \( d(x, y) \geq c_0 k^{-m-1}, \) one immediately has

\[
|f(x) - f(y)|^2 \leq C_3 \cdot r^m \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f) \leq C_4 \cdot d(x, y)^\frac{\log r}{\log k} \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f)
\]

for some constant \( C_3, C_4 > 0 \) as desired. Finally, if \( d(x, y) > c_0 k^{-1}, \) we have \( |f(x) - f(y)|^2 \leq C_5 \cdot \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f) \) for some \( C_5 > 0 \) by simply using \[ 3.6 \] with \( m = 0. \)

(b). By Proposition \[ 3.8 \] there is a subsequence \( n_l, l \geq 1 \) and a limit form \( \tilde{E} \) on \( L^2(K, \mu) \) so that \( r^{-n_l} \mathcal{D}_{n_l} \Gamma \)-converges to \( \tilde{E}. \)

We now show that

\[
C_6 \cdot \sup_{n \geq 1} r^{-n} \mathcal{D}_n(f) \leq \tilde{E}(f) \leq C_7 \cdot \liminf_{n \to \infty} r^{-n} \mathcal{D}_n(f), \quad \forall f \in L^2(K, \mu),
\]

for some constants \( C_6, C_7 > 0 \) independent of \( f. \) This will rely on the fact that \( N^{-m} \sigma_m \asymp r^{-m}, \) which is a consequence of Lemma \[ 3.3 \] Clearly, by the definition of \( \Gamma \)-convergence, we have

\[
\tilde{E}(f) \leq \liminf_{l \to \infty} r^{-n_l} \mathcal{D}_{n_l}(f) \leq 8C_8 \cdot \liminf_{n \to \infty} r^{-n} \mathcal{D}_n(f),
\]

where the second inequality is due to Lemma \[ 3.6 \] and the fact that \( r^{-n_l} \leq C_8 \cdot r^{-n} (N^{-n_l} \sigma_{n-l}^{-1}) \) for some \( C_8 > 0 \) and all \( n > n_l. \) For the other direction, we pick a sequence \( f_l, l \geq 1 \) that
converges to \( f \) in \( L^2(K, \mu) \), and \( \mathcal{D}(f) = \lim_{l \to \infty} r^{-n_l}D_{n_l}(f_l) \). Then,

\[
\mathcal{E}(f) = \lim_{l \to \infty} r^{-n_l}D_{n_l}(f_l) \geq \frac{1}{8} C_9 \cdot \lim_{l \to \infty} r^{-n}D_{n}(f) = \frac{1}{8} C_9 \cdot r^{-n}D_{n}(f), \quad \forall n \geq 1,
\]

where the inequality is due to Lemma 3.6 and the fact that \( r^{-n_l} \geq C_9 \cdot r^{-n}N^{-n_l+n} \sigma_{n_l-n} \) for some \( C_9 > 0 \) and all \( n_l > n \). Thus, we get (3.8).

Clearly \( \mathcal{F} = \{ f \in L^2(K, \mu) : \mathcal{E}(f) < \infty \} \) by (3.8), and \( \mathcal{E} \) induces a non-negative symmetric bilinear form \( (\mathcal{E}, \mathcal{F}) \) on \( L^2(K, \mu) \). In addition, \( (\mathcal{E}, \mathcal{F}) \) is closed as \( \mathcal{E} \) is lower-semicontinuous. It remains to show the Markov property and the regular property of \( (\mathcal{E}, \mathcal{F}) \).

**The Markov property.** Let \( f \in L^2(K, \mu) \), we need to show \( \mathcal{E}(\tilde{f}) \leq \mathcal{E}(f) \) for \( \tilde{f} = (f \lor 0) \land 1 \).

First, we will find a sequence \( f_l \in M_{n_l}, l \geq 1 \) such that \( f_l \to f \) in \( L^2(K, \mu) \) and \( \mathcal{E}(f_l) = \lim_{l \to \infty} r^{-n_l}D_{n_l}(f_l) \). By \( \Gamma \)-convergence, there is a sequence \( g_l \in L^2(K, \mu) \) so that \( g_l \to f \) in \( L^2(K, \mu) \) and \( \mathcal{E}(f_l) = \lim_{l \to \infty} r^{-n_l}D_{n_l}(g_l) \). It suffices to choose \( f_l = Q_{n_l}g_l, l \geq 1 \). In fact, we have \( D_{n_l}(f_l) = D_{n_l}(g_l) \) immediately, and in addition, \( f_l \to f \) in \( L^2(K, \mu) \) is the consequence of the estimate

\[
\| f_l - f \|_{L^2(K, \mu)} = \| Q_{n_l}g_l - f \|_{L^2(K, \mu)} \leq \| Q_{n_l}g_l - Q_{n_l}f \|_{L^2(K, \mu)} + \| Q_{n_l}f - f \|_{L^2(K, \mu)} \leq \| g_l - f \|_{L^2(K, \mu)} + \| Q_{n_l}f - f \|_{L^2(K, \mu)},
\]

and the fact that \( Q_{n_l}f \to f \) in \( L^2(K, \mu) \) as \( n \to \infty \).

Then, letting \( f_l = (f_l \lor 0) \land 1 \), we have \( \tilde{f}_l \to \tilde{f} \) in \( L^2(K, \mu) \) as \( \| \tilde{f}_l - \tilde{f} \|_{L^2(K, \mu)} \leq \| f_l - f \|_{L^2(K, \mu)}, \forall l \geq 1 \), and hence

\[
\mathcal{E}(\tilde{f}) \leq \liminf_{l \to \infty} r^{-n_l}D_{n_l}(\tilde{f}_l) \leq \liminf_{l \to \infty} r^{-n_l}D_{n_l}(f_l) = \mathcal{E}(f),
\]

where we use the Markov property of \( D_{n_l} \) on \( M_{n_l} \).

**The regular property.** Since \( \mathcal{F} \subset C(K) \) by (a), it remains to show \( \mathcal{F} \) is dense in \( C(K) \). We use the Stone-Weierstrass theorem of real valued functions on compact Hausdorff spaces (see Theorem 2.4.11 of [20] for a rudimentary proof). We need to verify that \( \mathcal{F} \) is an algebra contained in \( C(K) \) such that \( \mathcal{F} \) contains constants and separates points in \( K \).

First, \( \mathcal{F} \) is an algebra. \( \mathcal{F} \) is a linear space, so we only need to show \( f \in \mathcal{F}, g \in \mathcal{F} \) implies \( f \cdot g \in \mathcal{F} \). This is a consequence of the fact that \( \mathcal{F} \subset C(K) \), which implies that \( Q_{n_l}(f) \cdot Q_{n_l}(g) \to f \cdot g \) in \( L^2(K, \mu) \) (the convergence is also uniform indeed) and \( D_{n_l}(Q_{n_l}(f) \cdot Q_{n_l}(g)) \leq 2\| f \|_{L^2(K, \mu)} \| g \|_{L^2(K, \mu)}D_{n_l}(f) \), where in the second formula we use the observation \( \| Q_{n_l}f \|_{L^2(K, \mu)} \leq \| f \|_{L^2(K, \mu)} \) and \( \| Q_{n_l}g \|_{L^2(K, \mu)} \leq \| g \|_{L^2(K, \mu)} \). Then, by using \( \Gamma \)-convergence, one has \( \mathcal{E}(f \cdot g) \leq 2\| f \|_{L^2(K, \mu)} \mathcal{E}(g) + 2\| g \|_{L^2(K, \mu)} \mathcal{E}(f) \), hence \( f \cdot g \in \mathcal{F} \).

Second, \( \mathcal{F} \) clearly contains constant functions.

Finally, we show that for any \( x \neq y \in K \), there is \( f \in \mathcal{F} \) such that \( f(x) \neq f(y) \). We choose \( w \in W \) so that \( x \in \Psi_wK \) and \( y \in \bigcup_{w' \in N_w K} \Psi_{w'}K \). For any \( n > |w| \), we can find \( f_n \in L^2(K, \mu) \) such that \( f_n|\Psi_w K = 1, f_n|_{\bigcup_{w' \in N_w K} \Psi_{w'} K = 0} \) and \( D_{n}(f) \leq R^{-1}_{n-|w|} \). Let \( f \) be a weak limit of \( f_n \), then \( f \) separates \( x \) and \( y \). In addition, \( f \in \mathcal{F} \), since for any \( m \geq 1 \) we have \( r^{-m}D_m(f) = \lim_{n \to \infty} r^{-m}D_m(f_n) \leq C_{10} \cdot \liminf_{n \to \infty} r^{-n}D_n(f_n) \) for some \( C_{10} > 0 \) by a same reason as before.
Clearly $\mathcal{F}$ is dense in $L^2(K, \mu)$ by the regular property of $(\mathcal{E}, \mathcal{F})$. \hfill $\square$

4. An extension algorithm and its application to (B)

The essential difficulty of (B) for USC is caused by the unconstrained style of neighbourhoods $\mathcal{N}_w$, $w \in W_*$, which may be of infinitely many different types. Recall that in [43], the slightly different condition (B2) there is verified by the “Knight moves” method due to Barlow and Bass [3] for a limited range of local symmetric fractals including the classical Sierpinski carpets. Char, there are very limited results about slide moves and corner moves in the USC setting, when the process travels near $\partial_0 K$. Also, we can not construct functions of controllable energy simply by only using symmetry as before.

We will take two steps to overcome the difficulty. The argument will be purely analytic.

Firstly, we will provide a lower bound estimate of $R_n(W_{n,2}, W_{n,4})$ in terms of $\sigma_n$, i.e. $R_n(W_{n,2}, W_{n,4}) \geq C \cdot N^{-n} \sigma_n$ for some $C > 0$ independent of $n$. We will take advantage of the strongly recurrent setting in this step. This will be done in Lemma 4.11.

Secondly, we will estimate $R_n$ from below by constructing a function $g_w$ for any $w \in W_*$, which belongs to $l(W_{|w|+n})$, takes value 1 on $w \cdot W_n$, 0 on $N_w \cdot W_n$, and has energy controlled by $N^n \sigma_n^{-1}$. To fulfill this, first we will develop an extension algorithm by gluing affine copies of functions in $l(W_m)$ with energy controlled by $R_m^{-1}(W_{m,2}, W_{m,4})$, $m \leq n$ (supplied by the first step), to construct functions in $l(W_n)$ which have nice boundary conditions on $\partial W_n$ and energies controlled by $N^n \sigma_n^{-1}$; then again by gluing the affine copies of the obtained functions, and using cutting without increasing energy, we will get the desired function $g_w$ in $l(W_{|w|+n})$. This step will be accomplished in Proposition 4.8.

The condition (B) will then follow by combining these two steps.

We start this section with an introduction to the extension algorithm as well as its application to the condition (B), and leave the estimate of $R_n(W_{n,2}, W_{n,4})$ to the last part of this section.

4.1. Building bricks. We will break the fractal $K$ near boundary into infinitely many pieces (building bricks), and glue functions on them together to arrive desired boundary values.

Since all the four boundary lines $L_i$, $1 \leq i \leq 4$ are the same by $\mathcal{G}$-symmetry, we will focus on the bottom line segment $L_1 = \overline{q_1, q_2}$. Recall that $W_{m,1} \subset W_m$ stands for the collection of $m$-cells that have non-empty intersection with $L_1$, i.e. $W_{m,1} = \{w \in W_m : \Psi_w K \cap L_1 \neq \emptyset\} = \{1, 2, \ldots, k\}^m$. In addition, the cells $\Psi_w K$ is arranged so that $\Psi^-_1 K$ is on the left of $\Psi^1 K$ for $2 \leq i \leq k$.

**Definition 4.1.** Let $K$ be a USC.

(a). For $m \geq 1$, define $T_{m,K} = \bigcup_{w \in W_{m,1}} \Psi_w K$, and call it a level-$m$ lower wall of $K$.

(b). Let $B_K = \text{cl}(T_{1,K} \setminus T_{2,K})$. Here “cl” means closure (with respect to the metric $d$).

(c). Write $W_{*,1} = \bigcup_{m=0}^\infty W_{m,1}$. For each $w \in W_{*,1}$, call $\Psi_w B_K$ a level-$|w|$ building brick.

We can decompose $T_{1,K}$ into infinitely many building bricks, or into finitely many building bricks together with a lower wall $T_{n,K}$,

$T_{1,K} = \bigcup_{w \in W_{*,1}} \Psi_w B_K \cup L_1 = \bigcup_{m=0}^{n-2} \bigcup_{w \in W_{m,1}} \Psi_w B_K \cup T_{n,K}, \quad \text{for } n \geq 2. \quad (4.1)$
See Figure 4 for an illustration. In this section, since we are considering the discrete energies $B_{K,1}$, we focus on the finite decomposition of $T_{1,K}$. We will involve the infinite decomposition of $T_{1,K}$ in the continuous case in Section 6.

We introduce some new notations for cell graph case.

**Definition 4.2.** Let $K$ be a USC.

(a). For $m \geq 1$ and $n \geq m$, we define $\bar{T}_{m,n} = \{ w \in W_n : \Psi_w K \subset T_{m,K} \}$.

(b). For $n \geq 2$, we define $\bar{B}_n = \{ w \in W_n : \Psi_w K \subset B_K \}$.

In particular, $\bar{T}_{n,n} = W_{n,1}$.

We can rewrite the second decomposition in (4.1) as

$$\bar{T}_{1,n} = \bigcup_{m=0}^{n-2} \bigcup_{w \in W_{m,1}} w \cdot \bar{B}_{n-m} \cup \bar{T}_{n,n}, \quad n \geq 2,$$

where we use the notation $\bigcup$ to emphasize that the unions under consideration are disjoint.

**4.2. Functions with linear boundary values.** We will use the decomposition (4.2) to construct functions in $l(W_n)$, with good boundary condition on $\partial W_n$, $n \geq 2$. For convenience, in the remaining of this section, for any $w \in W_n, n \geq 1$ and $\Gamma \in \mathcal{G}$, by a little abuse of notation, we will denote $\Gamma(w)$ to be the unique element in $W_n$ so that $\Psi_{\Gamma(w)} K = \Gamma(\Psi_w K)$. We will say a function $f \in l(W_n)$ is $\Gamma$-symmetric if $f(w) = f(\Gamma(w))$ for all $w \in W_n$.

To start with, let’s define a building brick function $b_n$ on $\bar{B}_n, n \geq 2$. This takes several steps.

**Step b.1.** For $n \geq 1$, let $h_n$ be the unique function on $W_n$ such that

$$\begin{cases} 
   h_n|_{W_{n,2}} = 0, & h_n|_{W_{n,1}} = 1, \\
   \mathcal{D}_n(h_n) = \min \{ \mathcal{D}_n(f) : f|_{W_{n,2}} = 0, f|_{W_{n,1}} = 1, f \in l(W_n) \}.
\end{cases}$$
For $n \geq 1$, let $A_n = \{ w \in W_n : w \sim \tilde{T}_{1,n} \}$, and define $h'_n$ be the unique function on $W_n$ such that

$$
\begin{cases}
    h'_n|_{W_{n,3}} = 1, & h'_n|_{A_n} = 0, \\
    \mathcal{D}_n(h'_n) = \min \{ \mathcal{D}_n(f) : f|_{W_{n,3}} = 1, f|_{A_n} = 0, f \in l(W_n) \}.
\end{cases}
$$

We will return to the energy estimate of $h_n$ later in Subsection 4.3. Currently we will estimate energies of other functions in terms of $\mathcal{D}_n(h_n)$. In particular, in Step b.1, one has the energy of $h'_n$ well controlled by the energy of $h_{n-1}$ for $n \geq 2$.

**Lemma 4.3.** For $n \geq 2$, $\mathcal{D}_n(h'_n) \leq 2k \cdot \mathcal{D}_{n-1}(h_{n-1})$.

**Proof.** We define a function $\tilde{h}'_n$ in $l(W_n)$ as follows:

$$
\tilde{h}'_n(i \cdot w) = \begin{cases}
    h_{n-1} \circ \Gamma_{r_3}(w), & \text{if } i \in W_{1,3}, w \in W_{n-1}, \\
    0, & \text{if } i \notin W_{1,3}, w \in W_{n-1}.
\end{cases}
$$

Then $\tilde{h}'_n|_{W_{n,3}} = 1$ and $\tilde{h}'_n|_{A_n} = 0$. In addition, by the $\Gamma_v$-symmetry of $h_{n-1}$, one can check directly that $\tilde{h}'_n(i \cdot w) = h'_n(j \cdot \Gamma_{h}(w))$ for $i, j \in W_{1,3}$ with $i - j = 1$ and $w \in W_{n-1,2}$. As a consequence, the cross energy $\mathcal{D}_{n,i \cdot W_{n-1,2} \cap W_{n-1,4}}(h'_n) \leq \mathcal{D}_n(h_{n-1})$, and thus $\mathcal{D}_n(h'_n) \leq \mathcal{D}_n(\tilde{h}'_n) = k\mathcal{D}_{n-1}(h_{n-1}) + \text{sum of cross energies} \leq 2k \cdot \mathcal{D}_{n-1}(h_{n-1})$. □

**Step b.2.** Next, for $n \geq 2$, we define $b_n^{(1)} \in l(\bar{B}_n)$ and $b_n^{(2)} \in l(B_n)$ as

$$
b_n^{(1)} = h_n|_{B_n},
$$

and

$$
b_n^{(2)}(i \cdot w) = \frac{i - 1}{k} + \frac{1}{k} h_{n-1}(w), \quad \forall 1 \leq i \leq k, w \in W_{n-1} \setminus \bar{T}_{1,n-1}.
$$

**Lemma 4.4.** (a). For $n \geq 2$, we have $b_n^{(1)}|_{W_{n,4} \setminus \bar{B}_n} \equiv 0$ and $b_n^{(1)}|_{W_{n,2} \cap \bar{B}_n} \equiv 1$.

(b). For $n \geq 2$ and $1 \leq i \leq k$, we have

$$
b_n^{(2)}|_{(i \cdot W_{n-1,2}) \cap \bar{B}_n} \equiv \frac{i - 1}{k}, \quad b_n^{(2)}|_{(i \cdot W_{n-1,4}) \cap \bar{B}_n} \equiv \frac{i}{k}.
$$

(c). For $n \geq 2$, $\mathcal{D}_{n,\bar{B}_n}(b_n^{(1)}) \leq \mathcal{D}_n(h_n)$ and $\mathcal{D}_{n,\bar{B}_n}(b_n^{(2)}) \leq \frac{1}{k} \cdot \mathcal{D}_{n-1}(h_{n-1})$.

**Proof.** It is straightforward to check (a) and (b), noticing that $h_n|_{W_{n,4}} = 0$ and $h_n|_{W_{n,2}} = 1$.

(c). There is nothing to say about $\mathcal{D}_{n,\bar{B}_n}(b_n^{(1)})$. For $b_n^{(2)}$, by using (b), we can see that $b_n^{(2)}(v) = b_n^{(2)}(v')$ for all $v, v' \in \bar{B}_n$ such that $v \sim v'$ and $v_1 \neq v'_1$, and thus

$$
\mathcal{D}_{n,\bar{B}_n}(b_n^{(2)}) = \sum_{i=1}^{k} \mathcal{D}_{n-1,\bar{W}_{n-1} \setminus \bar{T}_{1,n-1}}(\frac{i - 1}{k} + \frac{1}{k} h_{n-1}) \leq \frac{1}{k} \cdot \mathcal{D}_{n-1}(h_{n-1}).
$$

□

**Step b.3.** Lastly, for $n \geq 2$, we define $b_n \in l(\bar{B}_n)$ as

$$
b_n = b_n^{(1)} \cdot h''_n + b_n^{(2)} \cdot (1 - h''_n),
$$

where $h''_n \in l(\bar{B}_n)$ is defined as

$$
h''_n(i \cdot w) = h'_{n-1}(w), \quad \forall 1 \leq i \leq k, w \in W_{1,n-1} \setminus \bar{T}_{1,n-1}.
We include the information about boundary values (including left boundary $W_{n,1} \cap \bar{B}_n$, right boundary $W_{n,2} \cap B_n$, upper boundary $W_{1,1} \cdot W_{n-1,3}$ and lower boundary $W_{1,1} \cdot (A_{n-1} \setminus \bar{T}_{1,n-1})$) and energy estimate of the building brick function $b_n \in l(\bar{B}_n)$ in the following Lemma 4.5.

**Lemma 4.5.** Let $n \geq 2$ and $b_n \in l(\bar{B}_n)$ be the function defined through Step b.1 to b.3.

(a). $b_n|_{W_{n,1} \cap \bar{B}_n} = 0$, $b_n|_{W_{n,2} \cap B_n} = 1$.

(b). $b_n(w) = h_n(w)$, $\forall w \in W_{1,1} \cdot W_{n-1,3}$.

(c). $b_n(i \cdot w) = \frac{i-1}{k} + \frac{1}{k} h_{n-1}(w)$, $\forall 1 \leq i \leq k, \forall w \in A_{n-1} \setminus \bar{T}_{1,n-1}$.

(d). $D_{n,B_n}(b_n) \leq C \cdot (D_n(h_n) + D_{n-1}(h_{n-1}) + D_{n-1}(h'_{n-1}))$, where $C > 0$ is independent of $n$.

**Proof.** (a). One can see that $b_n^{(i)}|_{W_{n,1} \cap \bar{B}_n} = 0$ and $b_n^{(i)}|_{W_{n,2} \cap B_n} = 1$ for $i = 1, 2$ from Lemma 4.4 (a), (b). Thus (a) follows immediately.

(b) and (c). By the definition of $h''_n$, one can see

$$h''_n|_{W_{1,1} \cdot W_{n-1,3}} = 1, \quad h''_n|_{W_{1,1} \cdot (A_{n-1} \setminus \bar{T}_{1,n-1})} = 0.$$

So we have

$$b_n|_{W_{1,1} \cdot W_{n-1,3}} = b_n^{(1)}|_{W_{1,1} \cdot W_{n-1,3}}, \quad b_n|_{W_{1,1} \cdot (A_{n-1} \setminus \bar{T}_{1,n-1})} = b_n^{(2)}|_{W_{1,1} \cdot (A_{n-1} \setminus \bar{T}_{1,n-1})}.$$

Then (b) follows immediately from the definition of $b_n^{(1)}$, and (c) follows immediately from the definition of $b_n^{(2)}$.

(d). First, by the $\Gamma_k$-symmetry of $h'_{n-1}$, using a same argument as that in the proof of Lemma 4.3, we have $D_{n,B_n}(h''_n) \leq 2kD_{n-1}(h'_{n-1})$. Then, noticing that $\|b_n^{(1)}\|_{l^\infty(\bar{B}_n)} = 1, \|b_n^{(2)}\|_{l^\infty(\bar{B}_n)} = 1$ and $\|h''_n\|_{l^\infty(\bar{B}_n)} = 1$, we have (d) holds by using Lemma 4.4 (c) and the well known inequality

$$\sqrt{D_{n,B_n}(f \cdot g)} \leq \|f\|_{l^\infty(\bar{B}_n)} \sqrt{D_{n,B_n}(g)} + \|g\|_{l^\infty(\bar{B}_n)} \sqrt{D_{n,B_n}(f)}, \forall f, g \in l(\bar{B}_n).$$

Next, we use the building brick functions to build a function $f_n \in l(W_n)$, whose restriction to the boundary is “linear”. In addition, We will show that $D_n(f_n)$ has an upper bound estimate in terms of $D_m(h_m), 1 \leq m \leq n$.

**Lemma 4.6.** Let $u \in l(K)$ defined by $u(x_1, x_2) = x_1$, $\forall x = (x_1, x_2) \in K$. Then there is $f_n \in l(W_n)$ such that $f_n|_{\partial W_n} = P_n u|_{\partial W_n}$ and

$$D_n(f_n) \leq C \cdot (k^{-n} + \sum_{m=0}^{n-1} k^{-m} D_{n-m}(h_{n-m})), \quad (4.3)$$

for some constant $C > 0$ independent of $n \geq 2$.

**Proof.** For convenience, we first consider $u' \in l(K)$ defined by $u'(x_1, x_2) = \frac{k^n}{k^{-1}(x_1 - \frac{1}{2k})}$ instead of $u$. Then we have $(P_n u')(w) = \frac{e(w)}{w}$, where $e(w) = \sum_{j=1}^{n-1} k^{n-j} (w_j - 1)$ for $w = w_1 \cdots w_n \in W_{n,1}$. In particular, $(P_n u')(1 \cdots 1) = 0$ and $(P_n u')(k \cdots k) = 1$. Now we construct a function $g_n \in l(W_n)$ such that $g_n|_{\partial W_n} = P_n u'|_{\partial W_n}$ by the following two steps.

**Step 1.** We first define $g'_n$ on $\bar{T}_{1,n}$.
We denote the indicator function of a set $A$ by $1_A$. Define
\[ g_n = (P_n u') \cdot 1_{\mathcal{T}_n} + \sum_{m=0}^{n-2} \sum_{w \in W_{m,1}} \frac{1}{k^m} (e(w) + b_{n-m} \circ \Psi_w^{-1}) \cdot 1_{w \cdot B_{n-m}}, \]
where by a little abuse of the notation, $b_{n-m} \circ \Psi_w^{-1}$ is the function supported on $w \cdot B_{n-m}$ such that $b_{n-m} \circ \Psi_w^{-1}(w \cdot \eta) = b_{n-m}(\eta)$ for any $\eta \in B_{n-m}$.

**Step 2.** We use symmetry to extend $g'_n$ to $g_n \in l(W_n)$.

We define
\[ g_n(w) = \begin{cases} 
  g'_n(w), & \text{if } w \in \mathcal{T}_1, \\
  g'_n \circ \Gamma_v(w), & \text{if } w \in \Gamma_v \mathcal{T}_1, \\
  h_n(w), & \text{otherwise}.
\end{cases} \]
Clearly, by Lemma 4.5 (a), $g_n$ satisfies the boundary condition $g_n|_{\partial W_n} = P_n u'|_{\partial W_n}$.

In the following, we estimate $D_n, \mathcal{T}_1 (g_n)$.

Firstly, we consider the energy of $g_n$ on each “layer” of building bricks: for $0 \leq m \leq n-2$, we have
\[ D_{n, \mathcal{T}_m+1, m+2, n} (g_n) = \sum_{w \in W_{m,1}} D_{n, w \cdot B_{n-m}} \left( \frac{1}{k^m} e(w) + \frac{1}{k^m} b_{n-m} \circ \Psi_w^{-1} \right) = \frac{D_{n, B_{n-m}} (b_{n-m})}{k^m}, \]
where the first equality is because $g_n(v) = g_n(v')$ for any $v, v' \in \mathcal{T}_m+1 \setminus \mathcal{T}_m+2$ such that $v \sim n v'$ and $v_1 v_2 \cdots v_m \neq v'_1 v'_2 \cdots v'_m$, as a consequence of Lemma 4.5 (a) and the construction of $g_n$.

Secondly, there is $C_1 > 0$ independent of $n$ such that
\[ D_n, \mathcal{T}_{n-1, n} (g_n) \leq C_1 \cdot k^{-n}, \]
since there are at most $4Nk^{n-1}$ pairs of $n$-cells $v \sim n v'$ in $\mathcal{T}_{n-1, n}$ and by the construction of $g_n$, $|f(v) - f(v')| \leq k^{-n+2}$, where $4$ appears since each $n$-cell is neighbouring to at most $8$ $n$-cells, and $Nk^{n-1}$ is the number of $n$-cells in $\mathcal{T}_{n-1, n}$.

Thirdly, we consider the energy of $g_n$ between two “layers” of building bricks: for $0 \leq m \leq n-1$, let
\[ E_{m, n} = \{ w \in \mathcal{T}_{m+1, n} : w \sim n \mathcal{T}_{m+1, n} \} \cup \{ w \in \mathcal{T}_{m, n} \setminus \mathcal{T}_{m+1, n} : w \sim n \mathcal{T}_{m+1, n} \}, \]
where $\mathcal{T}_{0, n} := W_n$. In particular, we are interested in $E_{m, n}, 1 \leq m \leq n-2$. For $1 \leq m \leq n-2$, one can check that $E_{m, n} = \bigcup_{w \in W_{m,1}} w \cdot E_{0, n-m}$, and in addition, by Lemma 4.5 (b), (c), and the construction of $g_n$, one can see that
\[ g_n(w \cdot v) = e(w) + \frac{1}{k^m} h_{n-m}(v), \quad \forall w \in W_{m,1}, v \in E_{0, n-m}. \]
So we have for $1 \leq m \leq n-2$,
\[ D_n, E_{m, n} (g_n) = \sum_{w \in W_{m,1}} D_{n, w \cdot E_{0, n-m}} \left( \frac{e(w) + h_{n-m} \circ \Psi_w^{-1}}{k^m} \right) \leq \frac{D_{m} (h_{n-m})}{k^m}, \]
where the equality holds due to a same reason as the first case.
Finally, adding up (4.4), (4.5) and (4.6) together, using Lemma 4.3 and Lemma 4.5 (d), and viewing $D_1(h')$ as a constant depending only on $K$, we get the energy estimate

$$D_{n,T_{i,n}}(g_n) \leq C_2 \cdot (k^{-n} + \sum_{m=0}^{n-1} k^{-m}D_{n-m}(h_{n-m})),$$

where $C_2 > 0$ is a constant independent of $n$.

Since $g_n$ is $\Gamma_\nu$-symmetric and $g_n|_{W_{1,1}W_{n-1,3}} = h_n|_{W_{1,1}W_{n-1,3}}$, we get $D_n(g_n) \leq C_3 \cdot (D_{n,T_{i,n}}(g_n) + D_n(h_n))$ for some $C_3 > 0$

At last, we take $f_n = \frac{k^n}{k^n}g_n + \frac{1}{2k^n}$ so that $f_n|_{\partial W_n} = P_n u|_{\partial W_n}$. The energy estimate (4.3) of $f_n$ follows immediately from that of $g_n$. \hspace{1cm} \Box

In the next subsection, we will show the following two estimates.

$$N^{-n+m}\sigma_{n-m} \geq C \cdot \left(\frac{2}{k}\right)^m N^{-n}\sigma_n, \quad \forall n \geq 1, 0 \leq m \leq n - 1, \quad (4.7)$$

$$D_n(h_n) \leq C \cdot N^n\sigma_n^{-1}, \quad \forall n \geq 1, \quad (4.8)$$

for some $C > 0$ independent of $n$. With these estimates, we have a clearer explanation of (4.3).

**Corollary 4.7.** Assume (4.7) and (4.8) hold. Let $f_n$ be the same function in Lemma 4.6. Then

$$D_n(f_n) \leq C \cdot N^n\sigma_n^{-1} \quad (4.9)$$

for some constant $C > 0$ independent of $n \geq 2$.

**Proof.** By (4.7) and (4.8), there is a constant $C_1 > 0$ independent of $n$ so that

$$D_{n-m}(h_{n-m}) \leq C_1 \cdot \left(\frac{k}{2}\right)^m N^n\sigma_n^{-1}, \quad \forall n \geq 1, 0 \leq m \leq n - 1,$$

and

$$C_1 \cdot N^n\sigma_n^{-1} \geq \left(\frac{2}{k}\right)^n, \quad \forall n \geq 1. \quad (4.10)$$

Substituting the above estimates into (4.3), we see that

$$D_n(f_n) \leq C_2 \cdot (k^{-n} + \sum_{m=0}^{n-1} k^{-m}D_{n-m}(h_{n-m})) \leq C_1C_2 \cdot \sum_{m=0}^{n} \frac{1}{2^m}N^n\sigma_n^{-1} \leq 2C_1C_2 \cdot N^n\sigma_n^{-1},$$

where $C_2$ is the constant $C$ in Lemma 4.6. \hspace{1cm} \Box

Lemma 4.6 and Corollary 4.7 are enough for us to verify (B).

**Proposition 4.8.** Assume (4.7) and (4.8) hold. Then (B) holds.

**Proof.** Let $n, m \geq 1$ and $w \in W_m$. We need to estimate $R_{m+n}(w \cdot W_n, N^\circ_{w} \cdot W_n)$.

Let $c_0$ be the constant in (A3), and let $p = (p_1, p_2)$ be the center of $\Psi_w K$. We consider the following four linear functions on $K$,

$$u_1(x_1, x_2) = k^m c_1(x_1 - p_1) + c_2, \quad u_2(x_1, x_2) = k^m c_1(p_1 - x_1) + c_2,$$

$$u_3(x_1, x_2) = k^m c_1(x_2 - p_2) + c_2, \quad u_4(x_1, x_2) = k^m c_1(p_2 - x_2) + c_2,$$
for $x = (x_1, x_2) \in K$, where $c_1 = \sqrt{2}c_0$ and $c_2 = 1 + \frac{1}{2}c_1$. In particular, let $u' = \min_{i=1}^{4} P_{m+n}(u_i)$, one can check that $u'|_{w \cdot W_n} \geq 1$ and $u'|_{\mathcal{N}_w \cdot W_n} \leq 0$.

By Lemma 4.6 and Corollary 4.7, we can construct $g_i \in \mathcal{L}(W_{m+n})$, $1 \leq i \leq 4$ so that $g_i(u' \cdot \tau) = P_n(u_i \circ \Psi_w)(\tau), \forall u' \in W_m \cdot \tau \in \partial W_n$ and $\mathcal{D}_{m+n, w^i \cdot W_n}(g_i) \leq C \cdot c_1^2 N_n \sigma_n^{-1}$, where $C$ is the same constant in Corollary 4.7. Indeed, for the case $i = 1$, for each $u' \in W_m$, by the definition of $u_1$, we see that $u_1(x_1, x_2) = k^n c_1(x_1 - p_1^n) + c_w$ on $\Psi_w K$ for some constant $c_w \in \mathbb{R}$, where we denote the center of $\Psi_w K$ as $p^w = (p_1^n, p_2^n)$. So $g_1(u' \cdot \tau) = c_1 f_n(\tau) + c_w - c_1 \tau$ for each $u' \in W_m$, $\tau \in \partial W_n$, where $f_n$ is the same function in Lemma 4.6. The case $i = 2, 3, 4$ are the same by using symmetry.

In addition, due to the construction of $g_i$, for each pair $u' \sim u''$ in $W_m$, and $\tau', \tau'' \in \partial W_n$ such that $u' \tau' \sim u'' \tau''$, the difference $|g_i(u' \cdot \tau') - g_i(u'' \cdot \tau'')|$ is controlled by $c_1 k^{-n}$. Since there are at most $\frac{3}{2} k^n$ pairs of such $\tau'$ and $\tau''$ for $u' \sim u''$, this gives that

$$\mathcal{D}_{m+n, A}(g_i) \leq C' \cdot N_N \sigma_n^{-1} + C' \cdot k^{-n} \leq C'' \cdot N_n \sigma_n^{-1},$$

where $A = \{v \in W_{m+n} : v \sim N_w \cdot W_n\}$, $C', C'' > 0$ are independent of $n, m, w$, and the second inequality is a consequence of (4.10).

Finally, let $g = \left((\min_{i=1}^{4} g_i) \vee 0\right) \wedge 1$. One can see that $g|_{w \cdot W_n} = 1$, $g|_{\mathcal{N}_w \cdot W_n} = 0$ and $\mathcal{D}_{m+n}(g) = \mathcal{D}_{m+n, A}(g) \leq 4C'' \cdot N_n \sigma_n^{-1}$. The condition (B) follows immediately. □

4.3. Verification of the estimates. We prove formulas (4.7) and (4.8) in this last subsection. The proof is essentially based on $\lambda_n \leq C \cdot \sigma_n$ by (3.3) and Lemma 3.5.

First, we prove (4.8), which is equivalent to the effective resistance estimate

$$R_n(W_{n,2}, W_{n,4}) \geq C \cdot N^{-n} \sigma_n$$

for some $C > 0$ independent of $n$.

Let $n \geq 1$, $m \geq 1$. For a connected subset $A$ of $W_{m,1}$ with $l = \# A \geq 2$, let $w^{(1)}_A$ be the most left one and $w^{(2)}_A$ be the most right one in $A$, viewing $A$ as a chain of cells. Define

$$R_{n,m,A} = \max \{ D_{m+n, A \cdot W_n}^{-1}(f) : f \in \mathcal{L}(A \cdot W_n), f|_{w^{(1)}_A \cdot W_n} = 0, f|_{w^{(2)}_A \cdot W_n} = 1 \}.$$

Clearly, $R_{n,m,A}$ depends only the length of $A$, i.e. $l = \# A$, and is independent of $m$. Denote $R_{n,m,A}$ as $R_{n,l}$. Since we are working on finite graphs, $R_{n,l}$ are positive and finite for any $n \geq 1$ and $l \geq 2$.

Lemma 4.9. For $n \geq 1$, we have $R_n(W_{n,2}, W_{n,4}) \geq \frac{1}{4l} \cdot R_{n,l}$.

Proof. One can see that $R_{n,l}$ is increasing in $l$, i.e. $R_{n,l_1} \leq R_{n,l_2}$ if $l_1 \leq l_2$. To see this, we choose large enough $m$ and $A_1 \subset A_2 \subset W_{m,1}$ such that $\#A_1 = l_1$, $\#A_2 = l_2$, and $111 \cdots 1 \in A_1$. Let $l_1 \in (l_1 \cdot W_n)$ so that $h_1|_{w^{(1)}_A \cdot W_n} = 0, h_1|_{w^{(2)}_A \cdot W_n} = 1$ and $\mathcal{D}_{m+n, A \cdot W_n}(h_1) = R_{n,l_1}^{-1}$.

By extending $h_1$ to $h_2 \in l(A_2 \cdot W_n)$ with

$$h_2|_{A_1 \cdot W_n} = h_1, \quad h_2|_{(A_2 \setminus A_1) \cdot W_n} = 1,$$

we immediately have $h_2|_{w^{(1)}_A \cdot W_n} = 0$ and $h_2|_{w^{(2)}_A \cdot W_n} = 1$, and so

$$R_{n,l_2}^{-1} \leq \mathcal{D}_{m+n, A_2 \cdot W_n}(h_2) = \mathcal{D}_{m+n, A_1 \cdot W_n}(h_1) = R_{n,l_1}^{-1}.$$
The lemma is obvious for \( l = 2 \), so we only consider the case \( l \geq 3 \). We start with the case \( l = 3 \). In this case, \( A = \{ w^{(1)}_A, w', w^{(2)}_A \} \) such that \( w^{(1)}_A \sim w' \sim w^{(2)}_A \). Choose \( f \in l(A \cdot W_n) \) so that \( f|_{w^{(1)}_A \cdot W_n} = 0, f|_{w^{(2)}_A \cdot W_n} = 1 \) and \( D_{m+n,A \cdot W_n}(f) = \tilde{R}^{-1}_{n,3} \). Define \( g \in l(W_n) \) as
\[
g(\tau) = 1_{W_{n,2}}(\tau) + f(w' \cdot \tau) \cdot 1_{W_n \setminus (W_{n,2} \cup W_{n,4})} + 0 \cdot 1_{W_{n,4}}(\tau), \quad \forall \tau \in W_n.
\]
Then, we have \( g|_{W_{n,4}} = 0, g|_{W_{n,2}} = 1 \). In addition,
\[
\begin{align*}
\sum_{v \in W_{n,4}} \sum_{v' \neq v; v' \sim v} (g(v) - g(v'))^2 &= \sum_{v \in W_{n,4}} \sum_{v' \neq v; v' \sim v} (0 - f(w'v'))^2 \\
&\leq 2 \sum_{v \in W_{n,4}} \sum_{v' \neq v; v' \sim v} \left( (0 - f(w'v))^2 + (f(w'v) - f(w'v'))^2 \right) \\
&\leq 2 \sum_{\tau \in W_n \setminus (W_{n,2} \cup W_{n,4})} \sum_{\tau' \in A \cdot W_n \setminus \tau \sim \tau'} (f(\tau) - f(\tau'))^2,
\end{align*}
\]
where the last inequality holds because \( f|_{w^{(1)}_A \cdot W_n} = 0 \) and for each \( \tau \in W' \cdot W_{n,4} \), we have
\[
\# \{ \tau' \in w^{(1)}_A \cdot W_n : \tau' \sim \tau \} \geq \# \{ \tau' \in w' \cdot (W_n \setminus W_{n,4}) : \tau' \sim \tau \}.
\]
One can apply the same argument to \( W_{n,2} \) by \( \Gamma_n \)-symmetry. Thus,
\[
D_n(g) = \sum_{v \in W_n \setminus W_{n,4}} \sum_{v' \neq v; v' \sim v} (g(v) - g(v'))^2 + D_{n,W_n \setminus (W_{n,4} \cup W_{n,2})}(g) \\
\leq 2 \sum_{i=2,4} \sum_{\tau \in W_n \setminus W_{n,i}} \sum_{\tau' \in A \cdot W_n \setminus \tau \sim \tau'} (f(\tau) - f(\tau'))^2 + D_{m+n,A \cdot W_n \setminus (W_{n,4} \cup W_{n,2})}(f) \\
\leq 2 \cdot D_{m+n,A \cdot W_n}(f).
\]

Thus,
\[
R_n(W_{n,2}, W_{n,4}) \geq \frac{1}{2} D^{-1}_{m+n,A \cdot W_n}(f) = \frac{1}{2} \tilde{R}_{n,3}.
\]

Next, we consider the cases \( l = 2^s + 2, s \geq 0 \). By applying symmetry, one can see
\[
\tilde{R}_{n,2^{s+1}+2} \leq 2 \cdot \tilde{R}_{n,2^s+2}.
\]
(4.11)
To see this, we fix \( A \) with \( \# A = 2^{s+1} + 1 \), and choose \( f \in l(A \cdot W_n) \) so that \( f|_{w^{(i)}_A \cdot W_n} = 0 \), \( f|_{w^{(2)}_A \cdot W_n} = 1 \) and \( D_{m+n,A \cdot W_n}(f) = \tilde{R}^{-1}_{n,2^{s+1}+2} \). To apply symmetry, we divide \( A \) in the middle: let \( A = A_1 \cup A_2 \), where \( A_1, A_2 \) are connected and \( \# A_1 = \# A_2 = 2^s + 1 \), \( w^{(1)}_A \in A_1, w^{(2)}_A \in A_2 \). Then, \( f - \frac{1}{2} \) is anti-symmetric with respect to the refection \( \Gamma_A \) which interchanges \( A_1, A_2 \), then one can see
\[
f|_{A_1} \leq \frac{1}{2}, \quad f|_{A_2} \geq \frac{1}{2},
\]
(4.12)
since otherwise one can define \( g = (f \wedge \frac{1}{2}) \cdot 1_{A_1 \cdot W_n} + (f \vee \frac{1}{2}) \cdot 1_{A_2 \cdot W_n} \), and see that \( D_{m+n,A \cdot W_n}(g) \leq D_{m+n,A \cdot W_n}(f) \). (In fact, \( |f(v) - f(v')| \geq |g(v) - g(v')| \), if \( \{v, v'\} \subset A_1 \cdot W_n \) or \( \{v, v'\} \subset A_2 \cdot W_n \); for the cross energy, if \( v \in A_1 \cdot W_n, v' \in A_2 \cdot W_n \), and \( v \sim v' \), without loss of generality we assume \( |f(v) - \frac{1}{2}| \leq |f(v') - \frac{1}{2}| \), then \( \sum_{l=0}^{m+n} |f(v) - f(v')|^2 \geq \sum_{l=0}^{m+n} |g(v) - g(v')|^2 \)).
implies $f_{3.6}$, one can see

Then one can see

Proof. Lemma 4.10.

$\sigma$ desired estimate follows from the monotonicity of $\tilde{\sigma}$.

Proof. Lemma 4.11.

$N$ get a function $f$ with the formula

$\sigma$ $\prod$ $\sum$ $\cap$ $\lor$ $\vee$

Now, we prove (4.8).

$\prod$ $\sum$ $\cap$ $\lor$ $\vee$

... provides 2\(m+1\) + 2. Finally, for general $l \geq 3$, the desired estimate follows from the monotonicity of $R_n$, $l$.

Another useful observation is that $\sigma_{n+m}$ and $\sigma_n$ are comparable if $m$ is small. In particular, $\sigma_{n+m} \leq 8 \cdot \sigma_m \sigma_n$ is another inequality in Theorem 2.1 of Kusuoka and Zhou’s paper [43].

Lemma 4.10. $\frac{N^m}{N^{n+m}} \sigma_n \leq \sigma_{n+m} \leq 8 \cdot \sigma_m \sigma_n, \forall m, n \geq 1$.

Proof. To see the left side of the inequality, we choose $m' \geq 1, w \overset{m'}{\sim} w'$ and $f \in l(\{w, \overline{w'}\} \cdot W_n)$ so that $D_{m'+n}(w, \overline{w'}) \cdot W_n(f) = 1$ and $N^m([f]_{w,w'} - [f]_{w',w''}+1)^2 = \sigma_n$. Next, we define $f' \in l(\{w, \overline{w'}\} \cdot W_{n+m})$ with the formula $f'(v - \tau = f(v), \forall v \in \{w, \overline{w'}\} \cdot W_n, \forall \tau \in W_m$. Then one can see $D_{m'+n+m}(w, \overline{w'}) \cdot W_{n+m}(f') \leq 3k^m$ by a routine argument as before and $N^{n+m}([f']_{w,w'} - [f']_{w',w''}+1)^2 = N^m \sigma_n$, which implies $\sigma_{n+m} \geq \frac{N^m}{N^{n+m}} \sigma_n$.

To see the right side of the inequality, we choose $m' \geq 1, w \overset{m'}{\sim} w'$ and $g \in l(\{w, \overline{w'}\} \cdot W_{n+m})$ so that $D_{m'+n+m}(w, \overline{w'}) \cdot W_{n+m}(g) = 1$ and $N^{n+m}([g]_{w,w'} - [g]_{w',w''}+1)^2 = \sigma_{n+m}$. Then we define $g' \in l(\{w, \overline{w'}\} \cdot W_n)$ to be $g'(v) = [g]_{w,w'}, \forall v \in \{w, \overline{w'}\} \cdot W_n$. Then, just as in Lemma 3.6 one can see $D_{m'+n}(g') \leq 8N^{-m} \sigma_n$ and $N^n([g']_{w,w'} - [g]_{w,w''}+1)^2 = N^{-m} \sigma_{n+m}$, which implies $\sigma_n \geq \frac{N^{-m} \sigma_{n+m}}{8N^{-m} \sigma_m}$. \hfill \Box

Now, we prove (4.8).

Lemma 4.11. There exists $C > 0$ such that for any $n \geq 1$,

$R_n(W_{n,2}, W_{n,4}) = R_n(W_{n,1}, W_{n,3}) \geq C \cdot N^{-n} \sigma_n$.

Proof. Fix a large $n$, and choose $w, w'$ in $W_n$ so that $\sigma_n(w, w') = \sigma_n$. Choose $f' \in l(\{w, \overline{w'}\} \cdot W_n)$ such that $D_{n+w}(w, \overline{w'}) \cdot W_n(f') = 1$ and $[f']_{w,W_n} - [f']_{w',W_n} = \sqrt{N^{-n} \sigma_n}$. Without loss of generality, we assume $f'$ is antisymmetric with respect to the central point of $\Psi_wK \cup \Psi_{w'}K$ (noticing that $\tilde{f}' := \tilde{f}' - \Gamma_{w,w'} \frac{f'}{2}$ will decrease the energy, but keep $[f']_{w,w'} - [f']_{w,w''} = [f']_{w,w'} - [f']_{w',W_n}$, where $\Gamma_{w,w'}$ is the associated rotation). Thus, by letting $f = f' \circ \Psi_w$, we get a function $f \in l(W_n)$ satisfying

$[f]_{W_n} = \frac{1}{2} \sqrt{N^{-n} \sigma_n}$, $D_n(f) \leq \frac{1}{2}$.
In addition, there is $\tau' \in \partial W_n$ such that $f(\tau') \leq \frac{1}{2}$ (it suffices to choose $\tau'$ so that $\Psi_{w\tau'}K$ contains the central point of $\Psi_wK \cup \Psi_{w'}K$). We also choose $\tau \in W_n$ such that $f(\tau) = \max_{v \in W_n} f(v) > \lfloor f \rfloor_{W_n}$.

Next, we choose a sequence $\tau = \tau^{(0)}, \ldots, \tau^{(2L-1)} = \tau' \in W_n$ such that

$$
\begin{aligned}
\tau_1^{(2\ell)} \cdots \tau_{M_1}^{(2\ell)} &= \tau_1^{(2\ell+1)} \cdots \tau_{M_1}^{(2\ell+1)}, & \text{for } 0 \leq \ell < L, \\
\tau^{(2\ell+2)} &\approx \tau^{(2\ell+2)}, & \text{for } 0 \leq \ell < L - 1,
\end{aligned}
$$

where we fix $M_1 > 0$ and $L > 0$ so that $\|f(v) - f(\tau)\| < \frac{1}{2}\lfloor f \rfloor_{W_n}$ for any $v \in W_n$ satisfying $v_m = \tau_m$ for all $1 \leq m \leq M_1$. Noticing that by Lemma 3.5 and the fact $\lambda_n \leq C \cdot \sigma_n$ for some $C > 0$ independent of $n$, $M_1$ can be chosen independently of $n$ (for large $n$), and clearly $L$ is also fixed as it only depends on $M_1$. See Figure 5 for an illustration for such sequence.

**Figure 5.** The sequence $\tau^{(0)} \cdots \tau^{(2L-1)}$ represented by small squares, where we use big squares to represent the associated $M_1$-cells.

As a consequence of the choice of $M_1$, we can find $1 \leq \ell < L$ such that $|f(\tau^{(2\ell)}) - f(\tau^{(2\ell+1)})| \geq \frac{1}{24L}\lfloor f \rfloor_{W_n}$, noticing that $f(\tau^{(2L-1)}) \leq \frac{1}{2}$ and $|f(\tau^{(2\ell+1)}) - f(\tau^{(2\ell+2)})| \leq 1$ are ignorable compared with $\lfloor f \rfloor_{W_n}$ (for large $n$). Now we fix this $\ell$ and consider

$$
g = f \circ \Psi_{\tau_1^{(2\ell)} \cdots \tau_{M_1}^{(2\ell)}} \in l(W_{n-M_1}).
$$

Clearly, $D_{n-M_1}(g) \leq 1$, and we can choose $\eta, \eta'$ from

$$
\{\tau_{M_1+1}^{(2\ell)} \cdots \tau_n^{(2\ell)}, \tau_{M_1+1}^{(2\ell+1)} \cdots \tau_n^{(2\ell+1)}\} \cup \{w \in W_{n-M_1} : q_i \in \Psi_wK \text{ for some } 1 \leq i \leq 4\},
$$

such that $g(\eta) - g(\eta') \geq \frac{1}{6L}\lfloor f \rfloor_{W_n}$ and $\{\eta, \eta'\} \subset W_{n-M_1,j}$ for some $1 \leq j \leq 4$.

Lastly, again by Lemma 3.5, we choose $M_2$ independently of large $n$ and $M_1$ so that

$$
|g(v) - g(\eta)| \leq \frac{1}{24L}\lfloor f \rfloor_{W_n}, \quad |g(v') - g(\eta')| \leq \frac{1}{24L}\lfloor f \rfloor_{W_n},
$$
for any $v, v' \in W_{n-M_1}$ such that $v_1 \cdots v_M = \eta_1 \cdots \eta_M$ and $v'_1 \cdots v'_M = \eta'_1 \cdots \eta'_M$. Thus, by letting $\kappa = \eta_1 \cdots \eta_M$ and $\kappa' = \eta'_1 \cdots \eta'_M$, we have

$$
\min_{v \in \kappa \cdot W_{n-M_1-M_2}} g(v) - \max_{v' \in \kappa' \cdot W_{n-M_1-M_2}} g(v') \geq g(\kappa) - g(\kappa') - 2 \cdot \frac{1}{24L} [f]_{W_n} \geq \frac{1}{24L} \sqrt{N^{-n} \sigma_n}.
$$

Observing that $\kappa, \kappa' \in W_{M_2,j}$, we conclude that $\tilde{R}_{n-M_1-M_2,l} \geq C' \cdot N^{-n} \sigma_n$ for some $C' > 0$, where $l$ is the length of the finite chain in $W_{M_2,j}$ connecting $\kappa, \kappa'$. Since $l$ is bounded above by $kM_2$, by using Lemma 4.9, one can see $\tilde{R}_{n-M_1-M_2}(W_{n-M_1-M_2,2}, W_{n-M_1-M_2,4}) \geq C'' \cdot N^{-n} \sigma_n$ for some $C'' > 0$ depending on $M_1, M_2$.

Finally, we finish the proof by applying Lemma 4.10 to see that

$$
\tilde{R}_{n-M_1-M_2}(W_{n-M_1-M_2,2}, W_{n-M_1-M_2,4}) \geq C \cdot N^{-n+M_1+M_2} \sigma_n - M_1 - M_2
$$

for some $C > 0$ depending only on $M_1, M_2$ (and $K$).

Finally, we prove (4.7) using Lemma 4.11.

Corollary 4.12. There exists $C > 0$ such that $N^{-n} \sigma_n \geq C \cdot \left( \frac{2}{7} \right)^m N^{-n-m} \sigma_{n+m}, \forall n, m \geq 1$.

Proof. By Lemma 4.11, we can find a function $h \in l(W_{m+n})$ so that $h|_{W_{m+n,4}} = -\frac{1}{2}$, $h|_{W_{m+n,4}} = \frac{3}{2}$ and $D_{m+n}(h) \leq C_1 \cdot N^{-n} \sigma_{m+n}^{-1}$ for some $C_1 > 0$. Define $h' \in l(W_m)$ by $h'(w) = [h]_{W_m}$, $\forall w \in W_m$. We can interpret $D_m(h')$ as a lower bound of $N^{-n} \sigma_n$ by using Lemma 3.6

$$
D_m(h') \leq 8N^{-n} \sigma_n D_m(h) \leq 8C_1 \cdot N^{-n} \sigma_n (N^{-n} \sigma_{m+n})^{-1}.
$$

(4.13)

It remains to estimate $D_m(h')$. By a same reason as the proof of Lemma 4.11, there is $M_1 > 0$ independent of $n$ (for large $n$) so that $h|_{W_{m,4}} \leq 0$, $h|_{W_{m,2}} \geq 1$ as long as $m \geq M_1$. Thus for $m \geq M_1$,

$$
h'|_{W_{m,4}} \leq 0, \quad h'|_{W_{m,2}} \geq 1.
$$

We can find $2^m$ disjoint chains of $m$-cells connecting $W_{m,2}$ and $W_{m,4}$: for each $e = e_1 e_2 \cdots e_m \in \{1, 3\}^m := \{e_1 e_2 \cdots e_m : e_1 = 1 \text{ or } 3, \forall 1 \leq l \leq m\}$, we consider

$$
\prod_{l=1}^m W_{1, e_l} := \{w \in W_m : w_l \in W_{1, e_l}, \forall 1 \leq l \leq m\}.
$$

We write $\prod_{l=1}^m W_{1, e_l} = \{w_{e,1}, w_{e,2}, \cdots, w_{e,k^m}\}$, where we order the cells from left to right so that $w_{e,1} \in W_{m,4}, w_{e,k^m} \in W_{m,2}$ and $w_{e,l} \sim w_{e,l'}$ if and only if $|l - l'| \leq 1$. Then

$$
D_m(h') \geq \sum_{e \in \{1, 3\}^m} \sum_{l=1}^{k^m-1} (h'(w_{e,l+1}) - h'(w_{e,l}))^2
$$

$$
\geq \sum_{e \in \{1, 3\}^m} \frac{1}{k^m-1} \left( \sum_{l=1}^{k^m-1} (h'(w_{e,l+1}) - h'(w_{e,l})) \right)^2
$$

$$
= \frac{1}{k^m-1} \sum_{e \in \{1, 3\}^m} (h'(w_{e,1}) - h'(w_{e,k^m}))^2 \geq \frac{2m}{k^m}. \quad (4.14)
$$
For \( m \geq M_1 \), the lemma follows by combining (4.14) and (4.13); for \( m < M_1 \), one simply adjust \( C \) to be small enough by Lemma 4.10.

**Proof of (B).** The condition (B) is a consequence of Proposition 4.8, Lemma 4.11 and Corollary 4.12.

5. Self-similar forms

In this section, we prove the existence of a self-similar Dirichlet form on \( \mathcal{USC} \). The proof is based on the existence of a limit form in Theorem 3.3.

First, we formally introduce the definition of self-similar Dirichlet forms on \( K \). Recall that the capacity associated with \( (\mathcal{E}, \mathcal{F}) \) is defined as

\[
\text{Cap}(U) = \inf \{ \mathcal{E}_1(f) : f|_U \geq 1, \text{a.e., } f \in \mathcal{F} \},
\]

for open \( U \subset K \), where \( \mathcal{E}_1(f) = \mathcal{E}(f) + \|f\|_{L^2(K, \mu)}^2 \). For a general set \( B \), \( \text{Cap}(B) = \inf_{B \subset U} \text{Cap}(U) \), where the infimum is taken over open sets \( U \) containing \( B \). A function \( f \in l(A), A \subset K \) is quasi-continuous if and only if for any \( \varepsilon > 0 \), there is open \( U \subset K \) such that \( \text{Cap}(U) < \varepsilon \) and \( f|_{A \setminus U} \in C(A \setminus U) \).

For \( f, g \in l(A) \), where \( A \) is a Borel subset of \( K \). We say \( f = g \) quasi-everywhere on \( A \) \((f = g \text{ q.e. on } A \text{ for short})\) if \( \text{Cap} \left( \{ x \in A : f(x) \neq g(x) \} \right) = 0 \). In particular, if \( A \) is open in \( K \), and \( f, g \) are both quasi-continuous, then \( f = g \text{ a.e. on } A \) implies \( f = g \text{ q.e. on } A \) (see Lemma 2.1.4. of [24]).

**Definition 5.1.** Let \( K \) be a \( \mathcal{USC} \) and \( \mu \) be the normalized \( d_H \)-dimensional Hausdorff measure on \( K \). A regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) on \( K \) is called self-similar if the following holds:

(a) If \( f \in \mathcal{F} \), then \( f \circ \Psi_i \in \mathcal{F} \) for \( 1 \leq i \leq N \). Conversely, if \( f \) is quasi-continuous and \( f \circ \Psi_i \in \mathcal{F} \) for \( 1 \leq i \leq N \), then \( f \in \mathcal{F} \).

(b) The self-similar identity of the energy holds,

\[
\mathcal{E}(f) = \sum_{i=1}^{N} r^{-1} \mathcal{E}(f \circ \Psi_i), \quad \forall f \in \mathcal{F}, \tag{5.1}
\]

where \( r > 0 \) is called the renormalization factor.

**Remark 1.** Conventionally (see [24]), the notations \( \hat{\mathcal{F}} \) and \( \hat{f} \) are used to indicate the quasi-continuous modification of functions. We are not highlighting this in the paper, as we always have \( \mathcal{F} \subset C(K) \) (by Theorem 3.3(a)) except in a few arguments of Subsection 6.3 and Section 8. We will claim our basic settings there to make things clear.

**Remark 2.** In the first half of Definition 5.1(a), if \( f \in \mathcal{F} \), it suffices to understand \( f \circ \Psi_i \) in the almost sure sense, which already determines the function. But we would like to point out here that \( f \circ \Psi_i \) is quasi-continuous if \( f \) is quasi-continuous, which follows from a short argument by Hino (see Lemma 3.11(i) of [22]). The key observation is that

\[
\text{Cap}(\Psi_i^{-1}E) \leq (N \lor r) \cdot \text{Cap}(E),
\]

for any Borel set \( E \) (or more general nearly Borel sets). In fact, it suffices to show the inequality for any open \( U \). Let \( e_U \) be the function in \( \mathcal{F} \) such that \( \mathcal{E}_1(e_U) = \text{Cap}(U) \) and \( e_U \geq 1 \text{ a.e. on } U \). Then we have \( e_U \circ \Psi_i \geq 1 \text{ a.e. on } \Psi_i^{-1}U \), so

\[
\text{Cap}(\Psi_i^{-1}U) \leq \mathcal{E}_1(e_U \circ \Psi_i) \leq (N \lor r)\mathcal{E}_1(e_U) = (N \lor r)\text{Cap}(U).
\]
Remark 3. The second half of Definition 5.1 (a) is an assumption used to glue the functions in Section 6 and Section 8. In fact, we can replace it with a weaker version: for $f \in C(K)$, if $f \circ \Psi_i \in \mathcal{F}, \forall 1 \leq i \leq N$, then $f \in \mathcal{F}$.

Theorem 5.2. Let $K$ be a USC and $\mu$ be the $d_H$-dimensional normalized Hausdorff measure on $K$. There is a strongly local, regular, irreducible, symmetric, self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$.

In addition, there are constants $C, C' > 0$ such that
\[ C \cdot \tilde{\mathcal{E}}(f) \leq \mathcal{E}(f) \leq C' \cdot \tilde{\mathcal{E}}(f), \quad \forall f \in \mathcal{F}, \]
where $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the same as that in Theorem 3.4.

Remark. In Kusuoka-Zhou’s original proof, Theorem 6.9 in [43], $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is defined to be a limit of the Cesàro mean of bilinear forms $(\tilde{\mathcal{E}}_n, \tilde{\mathcal{F}})$, $n \geq 1$ which are given by
\[ \tilde{\mathcal{E}}_n(f, g) = r^{-n} \sum_{w \in W_n} \sum_{i=1}^4 (P_n f(w) - P_{ni} f(w))(P_n g(w) - P_{ni} g(w)), \quad \forall f, g \in \mathcal{F}, \]
with $P_{ni} f(w) = \int_{L_i} f \circ \Psi_{w} d\nu_i$, and $\nu_i$ the Lebesgue measure on $L_i$. However, the Markov property of the limit form is difficult to prove. Kigami filled up this gap in [38] by developing a fixed point theorem of order preserving additive maps on an ordered topological cone. Using this technique, the existence of a desired Dirichlet form $(\mathcal{E}, \mathcal{F})$ follows, especially with the Markov property, see Theorem 2.3 in [38]. For the self-containedness of the paper, we will still provide a proof following Kusuoka-Zhou’s strategy, but replacing $\tilde{\mathcal{E}}_n(f, g)$ to $\sum_{w \in W_n} r^{-n} \tilde{\mathcal{E}}(f \circ \Psi_w, g \circ \Psi_w)$.

Proof. Recall the form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ in Theorem 3.4, and notice that $\mathcal{F} \subset C(K)$. For $n \geq 0$, define
\[ \tilde{\mathcal{E}}_n(f) = \sum_{w \in W_n} r^{-n} \tilde{\mathcal{E}}(f \circ \Psi_w), \quad \forall f \in \mathcal{F}_n, \]
where $\mathcal{F}_n = \{ f \in C(K) : f \circ \Psi_w \in \mathcal{F}, \forall w \in W_n \}$. It is straightforward to check that $\mathcal{F} \subset \mathcal{F}_n$, since $D_{n+m}(f) \geq \sum_{w \in W_n} D_m (f \circ \Psi_w), \forall m \geq 0$. We claim that
\[ C_1 \cdot r^{-m} D_n(f) \leq \tilde{\mathcal{E}}_n(f) \leq C_2 \cdot \liminf_{m \to \infty} r^{-m} D_m(f), \quad \forall f \in \mathcal{F}_n, \tag{5.2} \]
for some $C_1, C_2 > 0$ independent of $n$. On one hand, by Theorem 3.4, there is $C_2 > 0$ such that $\tilde{\mathcal{E}}(f) \leq C_2 \cdot \liminf_{m \to \infty} r^{-m} D_m(f)$ for any $f \in \mathcal{F}$. It follows from the definition of $D_m$, for any $f \in \mathcal{F}_n$, we have
\[ \tilde{\mathcal{E}}_n(f) = \sum_{w \in W_n} r^{-n} \tilde{\mathcal{E}}(f \circ \Psi_w) \leq C_2 \cdot \liminf_{m \to \infty} \sum_{w \in W_n} r^{-n-m} D_m(f \circ \Psi_w) \leq C_2 \cdot \liminf_{m \to \infty} r^{-n-m} D_{n+m}(f) = C_2 \cdot \liminf_{m \to \infty} r^{-m} D_m(f). \]
On the other hand, still by Theorem 3.4, $\mathcal{F} \subset C(K)$ and $|P_0 f - f(x)|^2 \leq C_3 \cdot \tilde{\mathcal{E}}(f), \forall x \in K, \forall f \in \mathcal{F}$, for some $C_3 > 0$, where $P_0 f = \int_K f(y) \mu(dy)$. Thus, for any $f \in \mathcal{F}_n$ and $w \sim w'$
in $W_n$,
\[ |(P_n f)(w) - (P_n f)(w')|^2 \leq 2C_3 \cdot (\mathcal{E}(f \circ \Psi_w) + \mathcal{E}(f \circ \Psi_{w'})), \]
where $(P_n f)(w) = N^n \int_{\Psi_w K} f(y) \mu(\text{dy})$ as in previous sections. By summing over all $w \sim w'$, we get $C_1 \cdot r^{-n} \mathcal{D}_n(f) \leq \mathcal{E}_n(f)$ for some $C_1 > 0$.

Now, let’s show $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2 \cdots$. For each $n \geq 1$, one can see
\[ \mathcal{F} = \{ f \in \mathcal{F}_n : \sup_{m \geq n} r^{-m} \mathcal{D}_m(f) < \infty \} \]
\[ = \{ f \in \mathcal{F}_n : \sup_{m \geq n} \sum_{w \in W_n} r^{-m} \mathcal{E}_{m-n}(f \circ \Psi_w) < \infty \} \]
\[ = \{ f \in \mathcal{F}_n : \sup_{m \geq 0} \mathcal{E}_m(f \circ \Psi_w) < \infty, \forall w \in W_n \} \]
\[ = \{ f \in \mathcal{F}_n : \sup_{m \geq 0} r^{-m} \mathcal{D}_m(f \circ \Psi_w) < \infty, \forall w \in W_n \} = \mathcal{F}_n, \]
where in the second and fifth equalities, we use the following consequence of (5.2):
\[ C_1 \cdot \sup_{m \geq n'} r^{-m} \mathcal{D}_m(f) \leq \sup_{m \geq n'} \mathcal{E}_m(f) \leq C_2 \cdot \liminf_{m \to \infty} r^{-m} \mathcal{D}_m(f), \forall f \in \mathcal{F}_n, \forall n' \geq 0.
\]

Let $\hat{\mathcal{F}}$ be a $\mathbb{Q}$-vector subspace of $\mathcal{F}$ with countable number of elements, that is dense in $\mathcal{F}$ with respect to the norm $\|f\|_{\tilde{\mathcal{E}}_1} := \sqrt{\mathcal{E}_1(f) + \|f\|_{L^2(K, \mu)}^2}$. To achieve this, one can simply choose a $\mathbb{Q}$-vector dense subspace $H$ of $L^2(K, \mu)$ with countable number of elements, and let $\hat{\mathcal{F}} = U_1(H)$, where $U_1$ is the resolvent operator associated with $\mathcal{E}_1$, i.e. $\mathcal{E}_1(U_1 f, g) = \int_K f g d\mu$, for any $f \in L^2(K, \mu)$ and $g \in \mathcal{F}$.

Then by a diagonal argument, there is a subsequence $\{n_i\}_{i \geq 1}$ such that the limit
\[ \mathcal{E}(f) := \lim_{i \to \infty} \frac{1}{n_i} \sum_{m=1}^{n_i} \tilde{\mathcal{E}}_m(f) \]
exists for any $f \in \hat{\mathcal{F}}$. In addition, by (5.2) and Theorem 3.4 we know
\[ C_4 \cdot \mathcal{E}(f) \leq \mathcal{E}(f) \leq C_5 \cdot \mathcal{E}(f), \forall f \in \mathcal{F}, \]
for some constant $C_4, C_5 > 0$. Thus, $\sqrt{\mathcal{E}}$ is continuous and thus uniformly continuous (by translation invariance of the induced metric) with respect to the norm $\sqrt{\mathcal{E}_1}$, so we can extend $\mathcal{E}$ continuously to $\mathcal{F}$.

By continuity, the extension keeps the properties: $\mathcal{E}(tf) = t^2 \mathcal{E}(f)$, $\mathcal{E}(f + g) + \mathcal{E}(f - g) = 2\mathcal{E}(f) + 2\mathcal{E}(g)$ and $\mathcal{E}((f \lor 0) \land 1) \leq \mathcal{E}(f)$ for any $t \in \mathbb{R}^2$ and $f, g \in \mathcal{F}$. Immediately, the functional $\mathcal{E}$ induces a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $K$.

Because $\mathcal{F}_1 = \mathcal{F}$, if $f$ is quasi-continuous (this simply means $f \in C(K)$, since as an easy consequence of Theorem 3.4 each point has positive capacity uniformly bounded from below) and $f \circ \Psi_i \in \mathcal{F}, \forall 1 \leq i \leq N$, then $f \in \mathcal{F}$. So $(\mathcal{E}, \mathcal{F})$ is self-similar by the construction.

The strongly local property follows from the self-similarity of the form. Finally, since $\mathcal{F} \subset C(K), 1_A \in \mathcal{F}$ implies $\mu(A) \in \{0, 1\}$. So $(\mathcal{E}, \mathcal{F})$ is irreducible by Theorem 1.6.1 of [24], noticing that $1 \in \mathcal{F}$.

**Remark.** Since $1 \in \mathcal{F}$ and $\mathcal{E}(1) = 0$, by Theorem 1.6.3 of [24], $(\mathcal{E}, \mathcal{F})$ is also recurrent. \qed
6. A uniform trace theorem

From now on, we turn to the second topic of this paper. We will consider whether the self-similar form \((\mathcal{E}, \mathcal{F})\) on a \(\text{USC}\) constructed in Theorem 5.2 is unique, and whether the form depends in a continuous way on the underlying \(\text{USC}\).

As a preparation, we will develop a uniform trace theorem for forms on \(\text{USC}\). Here uniform means the constants \(C\) appearing in this section do not depend on \(K\) (only depend on \(k\)). Some results in this section will be redundant for the proof of uniqueness, but all the materials will be useful in Section 9 for the weak convergence.

The Besov type characterization of the trace of \(\mathcal{F}\) on the boundary has been studied in previous papers [14, 33, 34]. In particular, Kumagai and Hino [33] develop a trace theorem on a large variety of fractals.

**Basic setting.** Throughout Section 6–9, we fix \(k\) and \(N\). We introduce the following conditions on a form \((\mathcal{E}, \mathcal{F})\) on \(K\) for short.

(C1). \((\mathcal{E}, \mathcal{F})\) is strongly local, regular, irreducible, symmetric, and self-similar with a renormalization factor \(0 < r \leq \frac{N}{k^2}\). In addition,

\[ R(L_1, L_3) = 1. \]  

(6.1)

Here, for any disjoint closed \(A, B\) in \(K\) with positive capacity, \(R(A, B)\) denotes the effective resistance between \(A, B\), defined as

\[ R(A, B) = \left( \inf \{ \mathcal{E}(\tilde{f}) : \tilde{f}|_A = 0, \tilde{f}|_B = 1, f \in \mathcal{F} \} \right)^{-1}, \]

where we write \(\tilde{f}\) to emphasize that \(\tilde{f}\) is a quasi-continuous version of \(f\) (whose existence is guaranteed by Theorem 2.1.3 [24]).

(C2). Every point in \(K\) has positive capacity.

**Remark 1.** Assume \((\mathcal{E}, \mathcal{F})\) is a strongly local, regular, and irreducible Dirichlet form on a compact set \(K\). Let \(A\) be a Borel subset of \(K\) with positive capacity, let \(u \in l(A)\) be bounded, and assume

\[ M = \{ f \in \mathcal{F} : \tilde{f}|_A = u, \text{q.e.} \} \neq \emptyset, \]

then there is a unique \(h \in M\) such that \(\mathcal{E}(h) = \inf_{f \in M} \mathcal{E}(f)\).

First, there is a uniformly bounded sequence (by Markov property) \(f_n \in M\) so that

\[ \lim_{n \to \infty} \mathcal{E}(f_n) = \inf_{f \in M} \mathcal{E}(f). \]

So, by taking the limit (with respect to \(\mathcal{E}_1\)-norm) of the Cesàro mean of \(f_n\), we find such an \(h\). In particular, to see \(h \in M\), it suffices to apply Theorem 2.1.3 [24] to see a subsequence of quasi-continuous modifications of Cesàro mean of the subsequence of \(\tilde{f}_n\) converges q.e to \(\tilde{h}\).

The uniqueness of \(h\) will follow by applying Remark 2 below and noticing that \(h\) can be characterized as \(h \in M, \mathcal{E}(h, v) = 0, \forall v \in \{ u \in \mathcal{F} : \tilde{u}|_A = 0, \text{q.e.} \}\). In particular, if \(A\) is closed, \(\{ u \in \mathcal{F} \cap C(K) : u|_A = 0 \}\) is dense in \(\{ u \in \mathcal{F} : \tilde{u}|_A = 0, \text{q.e.} \}\) by Lemma 2.3.4 of [24].

We say \(h\) is harmonic in \(K \setminus A\). For a more general definition and a probabilistic explanation, see [24].

**Remark 2.** If \((\mathcal{E}, \mathcal{F})\) is irreducible, then \(\mathcal{E}(f) > 0\) for any \(f \in \mathcal{F} \setminus \text{constants}\). We can see this by contradiction. Assume \(f \in \mathcal{F} \setminus \text{constants}\) and \(\mathcal{E}(f) = 0\), by replacing \(f\) with \((|f| - s)\vee 0\) for some suitable \(s \geq 0\), we can assume \(f \geq 0\) and both \(A = \{ f > 0 \}\) and \(A^c = \{ f = 0 \}\)
have positive measure. Since $\mathcal{E}(f) = 0$, we have $P_{t}f|_{A^{c}} = f|_{A^{c}} = 0$ a.e., $\forall t > 0$, where $P_{t}$ is the heat operator associated with $(\mathcal{E}, \mathcal{F})$. This implies that $A$ is $P_{t}$-invariant, which is a contradiction to the irreducibility of $(\mathcal{E}, \mathcal{F})$. In particular, if $(\mathcal{E}, \mathcal{F})$ is also regular, then $\mathcal{E}(f) = 0$ implies that $\tilde{f} = \text{constant}$ q.e.

Remark 3. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, and irreducible Dirichlet form on a compact set $K$. Then by the regular property of $(\mathcal{E}, \mathcal{F})$ and Remark 1 and 2, $0 < R(A, B) < \infty$ for any disjoint closed $A, B \subset K$ with positive capacity.

Remark 4. The form constructed in Theorem 5.2 satisfies (C1) and (C2) if we normalize it with a constant so that $R(L_{1}, L_{2}) = 1$.

Remark 5. Any function appearing in the rest of this section will be quasi-continuous (for simplicity, we will not write $\tilde{f}$ and introduce standard notations like $\tilde{F}$). Moreover, “$f = g$ on $A$” means “$f = g$ q.e. on $A$”, i.e. $f(x) = g(x), \forall x \in A \setminus N$ for some $N$ with 0 capacity. In particular, when (C2) is assumed, “$f = g$ on $A$” simply means “$f(x) = g(x), \forall x \in A$”.

The trace theorem is stated as follows, where $\Lambda_{2,2}^{\sigma(r)}(\partial_{n}K)$ with $\sigma(r) := -\frac{\log r}{2\log k} + \frac{1}{2}$ will be introduced in Section 6.1.

**Theorem 6.1.** Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $K$ satisfying (C1) and (C2). For $n \geq 0$, we define $\partial_{n}K = \bigcup_{w \in W_{n}} \Psi_{w} \partial_{n}K$. Then we have $\mathcal{F}|_{\partial_{n}K} = \Lambda_{2,2}^{\sigma(r)}(\partial_{n}K)$.

In addition, there are $C_{1}, C_{2} > 0$ depending only on $k$ such that

$$C_{1} \cdot \|[h|_{\partial_{n}K}]^{2}_{\Lambda_{2,2}^{\sigma(r)}} \leq \mathcal{E}(h) \leq C_{2} \cdot \|[h|_{\partial_{n}K}]^{2}_{\Lambda_{2,2}^{\sigma(r)}}, \forall h \in \mathcal{H}_{n},$$

where

$$\mathcal{H}_{n} = \{ h \in \mathcal{F} : \mathcal{E}(h) = \inf \{ \mathcal{E}(f) : f|_{\partial_{n}K} = h|_{\partial_{n}K}, f \in \mathcal{F} \} \}$$

is the space of functions that are harmonic in $K \setminus \partial_{n}K$.

We will break the proof of Theorem 6.1 into two parts: an extension theorem in Subsection 6.3, where (C1) is enough, and a restriction theorem in Subsection 6.5, where (C2) is used.

6.1. **Besov spaces.** The Besov spaces on the line segment $[0, 1]$ naturally appear in the trace theorem of Dirichlet spaces. See [14, 33, 34] for related works. In this paper, we will admit the following discrete version of Besov spaces on $[0, 1]$.

**Definition 6.2.** Let $\sigma > \frac{1}{2}$. For each $u \in C[0, 1]$, define

$$\|[u] \|_{\Lambda_{2,2}^{\sigma}[0, 1]} = \sqrt{\sum_{m=0}^{\infty} k^{2\sigma-1} \sum_{l=0}^{k^{m}-1} \left( u(l/k^{m}) - u(l+1/k^{m}) \right)^{2}}$$

and

$$\Lambda_{2,2}^{\sigma}[0, 1] = \{ u \in C[0, 1] : \|[u] \|_{\Lambda_{2,2}^{\sigma}[0, 1]} < \infty \}.$$

**Remark.** $\Lambda_{2,2}^{\sigma}[0, 1] = \text{Constants}$ if $\sigma \geq 1$ by a simple estimate using the Cauchy–Schwarz inequality. For $\frac{1}{2} < \sigma < 1$, $\Lambda_{2,2}^{\sigma}[0, 1]$ is a discrete characterization of the classical Besov space $\mathcal{B}_{2,2}^{\sigma}[0, 1]$ [35]. See also Lemmas 9.12 and 9.13 for a few more discussions on $\Lambda_{2,2}^{\sigma}[0, 1]$. 
Throughout this section, we always choose \( \sigma = \sigma(r) := \frac{-\log r}{2 \log k} + \frac{1}{2} \), so that

\[
[u]_{\Lambda_{2,2}^{\sigma}(0,1)}^2 = \sum_{m=0}^{\infty} r^{-m} \sum_{l=0}^{k^m-1} (u(x) - u(x - l/k^m))^2.
\]

We remark that \( k \) is a fixed integer and \( \sigma(r) \geq \frac{3}{2} - \frac{\log N}{2 \log k} \) by (C1). We also write

\[
\|u\|_{\Lambda_{2,2}^{\sigma}(0,1)} = \sqrt{\|u\|_{\Lambda_{2,2}^{\sigma}(0,1)}^2 + \|u\|_{L^2(0,1)}^2}
\]

for the norm on \( \Lambda_{2,2}^{\sigma}(0,1) \), where \( L^2(0,1) \) is with respect to the Lebesgue measure on \([0,1]\).

For a line segment \( x, y \), with \( x \neq y \) in \( \mathbb{R}^2 \), we identify \( x, y \) with \([0,1]\) by the linear map \( \gamma_{x, y} : [0,1] \rightarrow [x, y] \) defined as \( \gamma_{x, y}(t) = (1 - t)x + ty, \forall t \in [0,1] \). For \( u \in C([x, y]) \), we simply write

\[
[u]_{\Lambda_{2,2}^{\sigma}(\gamma_{x, y})} = |x-y|^{1-\sigma}(u \circ \gamma_{x, y})_{\Lambda_{2,2}^{\sigma}(0,1)},
\]

and \( \Lambda_{2,2}^{\sigma}(\gamma_{x, y}) = \{u \in C([x, y]) : \|u\|_{\Lambda_{2,2}^{\sigma}(\gamma_{x, y})} < \infty\} \), where \( |x, y| \) is the length of \( x, y \). Notice that \( \|u\|_{\Lambda_{2,2}^{\sigma}(\gamma_{x, y})} = \|u\|_{\Lambda_{2,2}^{\sigma}(x, y)} \).

Finally, we introduce the space \( \Lambda_{2,2}^{\sigma}(\partial K) \) appeared in Theorem 6.1.

**Definition 6.3.** For \( u \in C(\partial K) \), we write

\[
[u]_{\Lambda_{2,2}^{\sigma}(\partial K)} = \sum_{w \in W_n} \sum_{i=1}^{4} [u \circ \Psi_w]_{\Lambda_{2,2}^{\sigma}(\Psi_w L_i)}^2,
\]

and define \( \Lambda_{2,2}^{\sigma}(\partial K) = \{u \in C(\partial K) : \|u\|_{\Lambda_{2,2}^{\sigma}(\partial K)} < \infty\} \).

6.2. **Building bricks and boundary graph.** Let \( K \) be a USC. For \( m \geq 1 \), recall that in Definition 4.1, we have defined \( T_{m,K} = \bigcup_{w \in W_{m,1}} \Psi_w K \), and \( B_K = \text{cl}(T_{1,K} \setminus T_{2,K}) \). In addition, \( T_{1,K} \) can be decomposed into

\[
T_{1,K} = ( \bigcup_{w \in W_{1,1}} \Psi_w B_K ) \cup L_1,
\]

where each \( \Psi_w B_K \) is called a level-\(|w|\) building brick.

We introduce two graphs based on \( B_K \) and \( T_{1,K} \).

**Definition 6.4.** Let \( K \) be a USC.

(a). Define \( V_{B_K} = \{ \Psi_1 q_4, \Psi_3 k q_3 \} \cup \{ \Psi_1 iq_4, \Psi_1 k q_3 \}_{i=1}^k \),

\[
E_{B_K} = \{ \{ \Psi_1 q_4, \Psi_3 k q_3 \}, \{ \Psi_1 q_4, \Psi_1 k q_4 \}, \{ \Psi_3 k q_3, \Psi_1 k q_3 \} \} \cup \{ \{ \Psi_1 q_4, \Psi_1 k q_3 \}_{i=1}^k \},
\]

and denote \( G_{B_K} = (V_{B_K}, E_{B_K}) \) the associated finite graph.

(b). Define \( V_K = \bigcup_{w \in W_{1,1}} \Psi_w V_{B_K} \) and \( E_K = \bigcup_{w \in W_{1,1}} \Psi_w E_{B_K} \), where \( \Psi_w E_{B_K} := \{ \{ \Psi_w x, \Psi_w y \} : \{x, y\} \in E_{B_K} \} \) for each \( w \in W_{1,1} \). Denote \( G_K = (V_K, E_K) \) the associated infinite graph.

See Figure 7 for an illustration.
The points in $V_{B_K}$ locate on the boundary of $B_K$, while the set $V_K$ consists of boundary vertices of all building bricks, with the lower boundary $L_1$ as its limit. We call $G_K$ the lower boundary graph of $K$.

As before, we use the $\mathbb{R}^2$-coordinate to represent points in $V_K$. One can check that

$$V_{B_K} = \{(0, \frac{1}{k}), (1, \frac{1}{k})\} \cup \left\{\left(\frac{l}{k}, \frac{1}{k^2}\right) : 0 \leq l \leq k\right\},$$

and

$$V_K = \bigcup_{m=0}^{\infty} \left\{\left(\frac{l}{k^m}, \frac{1}{k^{m+1}}\right) : 0 \leq l \leq k^m\right\}.$$

In the following, we define an energy on $G_K$.

**Definition 6.5.** (a). For $f \in l(V_{B_K})$, define

$$D_{G_{B_K}}(f) = \sum_{\{x,y\} \in E_{B_K}} (f(x) - f(y))^2.$$

(b). For $f \in l(V_K)$, define

$$D_{G_K}(f) = \sum_{w \in \mathbb{W}_{-1}} r^{-|w|} D_{G_{B_K}}(f \circ \Psi_w),$$

where $r$ is the renormalization factor in (C1).

6.3. **An extension theorem.** In this subsection, we always assume $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $K$ satisfying (C1). We begin the story with an extension theorem. The method is similar to that in Section 4.2.

First, we introduce a finite dimensional space of functions that are supported on $B_K$.

**Lemma 6.6.** Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $K$ satisfying (C1).

(a). There is $h \in \mathcal{F} \cap C(K)$ such that $0 \leq h \leq 1$, $h|_{L_1} = 0$, $h|_{L_2} = 1$ and $\mathcal{E}(h) < 2$.

(b). There is $h' \in \mathcal{F} \cap C(K)$ such that $0 \leq h' \leq 1$, $h'|_{K \setminus T_1,K} = 1$, $h'|_{T_2,K} = 0$ and $\mathcal{E}(h') < 2k^2r^{-2}$. 

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**Figure 6.** The infinite graph $G_K$. 

The points in $V_{B_K}$ locate on the boundary of $B_K$, while the set $V_K$ consists of boundary vertices of all building bricks, with the lower boundary $L_1$ as its limit. We call $G_K$ the lower boundary graph of $K$.
Proof. (a). It suffices to prove
\[ \inf \{ E(g) : g \in \mathcal{F} \cap C(K), g|_{L_2} = 0, g|_{L_4} = 1 \} = 1, \]
where \( 1 = R(L_2, L_4)^{-1} \) by assumption (C1). We choose a sequence \( f_n \in \mathcal{F} \cap C(K) \) such that \( 0 \leq f_n \leq 1, f_n|_{L_2} = 0, f_n|_{L_4} = 1 \) and \( \lim_{n \to \infty} E(f_n) = \inf \{ E(g) : g \in \mathcal{F} \cap C(K), g|_{L_2} = 0, g|_{L_4} = 1 \} \). By choosing Cesàro mean of a subsequence of \( f_n \), we have a limit \( f \in \mathcal{F} \) in the \( E_1 \)-norm sense. It is direct to check that \( E(f + v) \geq E(f) \) for any \( v \in \mathcal{F} \cap C(K) \) satisfying \( v|_{L_2 \cup L_4} = 0 \), which implies \( E(f + v) \geq E(f) \) for all \( v \in \mathcal{F}, v|_{L_2 \cup L_4} = 0 \) by Lemma 2.3.4 in \( [24] \). Thus,
\[ E(f) = \inf \{ E(g) : g \in \mathcal{F}, g|_{L_2} = 0, g|_{L_4} = 1 \} = 1. \]

On the other hand, by the construction of \( f \), \( E(f) = \inf \{ E(g) : g \in \mathcal{F} \cap C(K), g|_{L_2} = 0, g|_{L_4} = 1 \} \) (noticing that Cesàro mean of \( f_n \) is still in \( C(K) \)). This finishes the proof.

(b). We can use the function \( h \) in part (a) to construct \( h' \). For \( x \in K, w \in W_2, \) define
\[
h'(\Psi_w x) = \begin{cases} 
1, & \text{if } w \in W_2 \setminus \bar{T}_{1,2}, \\
h \circ \Gamma_3 (x), & \text{if } w \in W_{1,1} \cdot W_{1,3}, \\
0, & \text{if } w \in \bar{T}_{1,2} \setminus (W_{1,1} \cdot W_{1,3}).
\end{cases}
\]
Since \( h \) is \( \Gamma_v \)-symmetric, \( h' \) is continuous on \( K \). It is direct to check that \( 0 \leq h' \leq 1 \), \( h'|_{K \setminus T_{1,K}} = 1 \), \( h'|_{T_{2,K}} = 0 \), and \( E(h') = k^2 r^{-2} \cdot E(h) \). In particular, \( h'|_{B_K} \) is a continuous version of \( h''_n \) introduced in Step b.3 of Section 4.2.

**Definition 6.7.** Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form on \( K \) satisfying (C1). Let \( h, h' \) be the functions defined in Lemma 6.6. For each \( f \in l(V_{B_K}) \), we define \( \mathcal{B}_f^{(1)}, \mathcal{B}_f^{(2)} \in C(B_K) \) as
\[
\mathcal{B}_f^{(1)}(x) = f(\Psi_1 q_4 + (f(\Psi_k q_3) - f(\Psi_1 q_4)) \cdot h(x), \ \forall x \in B_K; \\
\mathcal{B}_f^{(2)}(\Psi_i x) = f(\Psi_i q_4 + (f(\Psi_i q_3) - f(\Psi_i q_4)) \cdot h(x), \ \forall 1 \leq i \leq k, \forall x \in cl(K \setminus T_{1,K}),
\]
and define \( \mathcal{B}_f \in C(B_K) \) as
\[
\mathcal{B}_f(x) = \mathcal{B}_f^{(1)}(x) \cdot h'(x) + \mathcal{B}_f^{(2)}(x) \cdot (1 - h'(x)), \ \forall x \in B_K.
\]
Call \( \mathcal{B}_f \) a building brick function on \( B_K \) induced by \( f \in l(V_{B_K}) \).

Next, for convenience, on a simple set \( A = \bigcup_{i=1}^{M_A} \Psi_{w(i)} K, \) where \( M_A < \infty, w(i) \in W_s \) and \( \mu(\Psi_{w(i)} K \cap \Psi_{w(i')} K) = 0 \) for any \( 1 \leq i \neq i' \leq M_A \), we define
\[
\mathcal{F}_A = \{ f \in l(A) : f \circ \Psi_{w(i)} \in \mathcal{F} \text{ and } f \circ \Psi_{w(i)} \text{ is quasi-continuous}, \forall 1 \leq i \leq M_A \},
\]
and
\[
E_A(f, g) = \sum_{i=1}^{M_A} r^{-|w(i)|} E(f \circ \Psi_{w(i)}, g \circ \Psi_{w(i)}), \ \forall f, g \in \mathcal{F}_A.
\]
Write \( E_A(f) := E_A(f, f) \) for short. It is easy to check that \((E_A, \mathcal{F}_A)\) is a Dirichlet form on \( L^2(A, \mu) \). The Markov property is obvious. \( \mathcal{F}_A \) is dense in \( L^2(A, \mu) \) because \( \mathcal{F} \supseteq \mathcal{F}|_A \) by the self-similarity of \( \mathcal{F} \). We only need to prove that the form is closed. Let \( f_n \in \mathcal{F}_A \) be a \( E_{A,1} \)-Cauchy sequence where \( E_{A,1} \) := \( E_A \) + \( \| \cdot \|_{L^2(A, \mu)} \). Then by applying Theorem 2.1.4 (i) of \([24]\), one can find a subsequence \( f_{n_j} \) such that \( f_{n_j} \circ \Psi_{w(i)} \) converges in \( E_1 \) and q.e. for
1 \leq i \leq M_A$. One then simply define $f(x) = \lim_{l \to \infty} f_{n_l}(x)$ for each $x$ such that the limit exists, and define $f(x) = 0$ if the limit does not exist. So, for $1 \leq i \leq M_A$, $f_{n_l} \circ \Psi^{(i)}_w$ converges in $\mathcal{E}_1$ and q.e. to $f \circ \Psi^{(i)}_w$, whence $f \circ \Psi^{(i)}_w \in \mathcal{F}$ and $f \circ \Psi^{(i)}_w$ is quasi-continuous by Theorem 2.1.4 (ii) of [24]. Thus $f \in \mathcal{F}_A$ is the desired limit of $f_{n_l}$.

We have the following lemma.

**Lemma 6.8.** Let $f, f' \in l(V_{B_K})$, and $B_f, B_{f'}$ be the functions defined in Definition 6.7.

(a) For $x \in \mathbb{V}_{1q4}, \mathbb{V}_{kq3}$,

$$B_f(x) = f(\Psi_{1q4}) \cdot (1 - h(x)) + f(\Psi_{kq3}) \cdot h(x).$$

In addition, if $f \equiv c$ on $V_{B_K}$, then $B_f \equiv c$ on $B_K$.

(b) $B_f \in \mathcal{F}_{B_K} \cap C(B_K)$ and there exists $C > 0$ depending only on $k$ and $r$ so that

$$\mathcal{E}_{B_K}(B_f) \leq C \cdot \mathcal{D}_{G_{B_K}}(f).$$

(c) If $f(\Psi_{kq3}) = f'(\Psi_{1q4})$ and $f(\Psi_{kkq3}) = f'(\Psi_{11q4})$, then

$$B_f(1, x_2) = B_{f'}(0, x_2), \quad \forall x_2 \in \left[ \frac{1}{K^2}, \frac{1}{K} \right].$$

(d) If $f(\Psi_{1q4}) = f'(\Psi_{11q4})$ and $f(\Psi_{kq3}) = f'(\Psi_{ikq3})$ for some $1 \leq i \leq k$, then

$$B_{f'} \circ \Psi_i(x) = B_f(x), \quad \forall x \in \mathbb{V}_{1q4}, \mathbb{V}_{kq3}.$$

**Proof.** (a) is immediate from the construction of $B_f$.

(b) First, by Lemma 6.6 and Definition 6.7, $h'|_{B_K} \in C(B_K)$, $h'|_{B_K} \in \mathcal{F}|_{B_K} \subset \mathcal{F}_{B_K}$ and $\mathcal{E}_{B_K}(h'|_{B_K}) \leq \mathcal{E}(h') \leq 2k^2r^{-2}$.

Next, by Lemma 6.6 and Definition 6.7, $B_f^{(1)} \in C(B_K)$, $B_f^{(1)} \in \mathcal{F}|_{B_K} \subset \mathcal{F}_{B_K}$ and $\mathcal{E}_{B_K}(B_f^{(1)}) \leq (f(\Psi_{1q4}) - f(\Psi_{kq3}))^2 \cdot \mathcal{E}(h) < 2 \cdot (f(\Psi_{1q4}) - f(\Psi_{kq3}))^2$.

Still by Lemma 6.6 and Definition 6.7, $B_f^{(2)} \in C(B_K)$, $B_f^{(2)} \in \mathcal{F}_{B_K}$ since $B_f^{(2)} \circ \Psi_w \in \mathcal{F}, \forall w \in \mathcal{T}_2 \setminus W_{2,1},$ and $\mathcal{E}_{B_K}(B_f^{(2)}) \leq 2r^{-1} \cdot \sum_{i=1}^{k} (f(\Psi_{i1q4}) - f(\Psi_{ikq3}))^2$.

So by Theorem 1.4.2 of [24], $B_f \in \mathcal{F}_{B_K} \cap C(B_K)$. To get the energy estimate of $B_f$, without loss of generality, we assume $f(\Psi_{1q4}) = 0$ so that $\|B_f^{(1)}\|^2_{L^\infty(B_K, \mu)} \leq C_1 \cdot \mathcal{D}_{G_{B_K}}(f)$ and $\|B_f^{(2)}\|^2_{L^\infty(B_K, \mu)} \leq C_1 \cdot \mathcal{D}_{G_{B_K}}(f)$ for some $C_1 > 0$ depending only on $k$ and $r$. Then, the desired estimate of $\mathcal{E}_{B_K}(B_f)$ follows from the inequality that $\mathcal{E}_{B_K}(g \cdot g') \leq 2\|g\|^2_{L^\infty(B_K, \mu)} \mathcal{E}_{B_K}(g') + 2\|g'\|^2_{L^\infty(B_K, \mu)} \mathcal{E}_{B_K}(g)$, $\forall g, g' \in \mathcal{F}_{B_K} \cap L^\infty(B_K, \mu)$.

(c) For $x_2 \in \left[ \frac{1}{K^2}, \frac{1}{K} \right]$, one can check that

$$B_f(1, x_2) = f(\Psi_{kq3}) \cdot h'(1, x_2) + f(\Psi_{kkq3}) \cdot (1 - h'(1, x_2)),$$

while

$$B_{f'}(0, x_2) = f'(\Psi_{1q4}) \cdot h'(0, x_2) + f'(\Psi_{11q4}) \cdot (1 - h'(0, x_2)).$$

Noticing that $h'(1, x_2) = h'(0, x_2)$ by the $\Gamma_h$-symmetry of $h'$, we immediately have $B_f(1, x_2) = B_{f'}(0, x_2)$.

(d) can be verified in a same way as (c).
Remark. Lemma 6.8 (c), (d) imply that the functions $B_f$ can be scaled and glued together easily by only looking at the values of $f$ on $V_{BK}$, and Lemma 6.8 (b) provides a good energy estimate of $B_f$ on $B_K$ in terms of the graph energy of $f$ on $G_{BK}$.

**Theorem 6.9.** Let $f \in l(V_K)$ with $D_{G_K}(f) < \infty$, and define $g \in C(T_{1,K} \setminus L_1)$ as

$$g(x) = B_f \circ \Psi_w \circ \Psi_w^{-1}(x), \text{ if } x \in \Psi_w B_K, w \in W_{*1}.$$ 

Then, $g$ extends continuously to $g \in F_{T_{1,K}} \cap C(T_{1,K})$. In addition, there is $C > 0$ depending only on $k$ and $r$, such that

$$\mathcal{E}_{T_{1,K}}(g) \leq C \cdot D_{G_K}(f).$$

**Proof.** Since $r \leq \frac{N}{2r} < 1$, $f$ is uniformly continuous on $(V_K, d)$, so $f$ extends continuously to $\tilde{f} \in C(cl(V_K))$, where clearly $cl(V_K) = V_K \cup L_1$. We extend $g$ to $g \in C(T_{1,K})$ by letting $g|_{L_1} = \tilde{f}|_{L_1}$, noticing that by definition $\min_{p \in V_{BK}} f \circ \Psi_w(p) \leq g(x) \leq \max_{p \in V_{BK}} f \circ \Psi_w(p)$ for each $w \in W_{*1}$ and $x \in \Psi_w B_K$.

Let $h$ be the function defined in Lemma 6.6 (a). For $m \geq 2$, define $g_m$ on $T_{1,K}$ by

$$g_m(x) = \begin{cases} g(x), & \text{if } x \in T_{1,K} \setminus T_{m,K}, \\ f(\Psi_{u1} q_4) \cdot (1 - h(\Psi_w^{-1} x)) + f(\Psi_{wk} q_3) \cdot h(\Psi_w^{-1} x), & \text{if } x \in \Psi_w T_{1,K}, \text{ for some } w \in W_{m-1,1}. \end{cases}$$

By Lemma 6.8, it is easy to see that $g_m \in F_{T_{1,K}} \cap C(T_{1,K})$ and

$$\mathcal{E}_{T_{1,K}}(g_m) = \mathcal{E}_{T_{1,K}}(g_m) + \mathcal{E}_{T_{m,K}}(g_m)$$

$$\leq C_1 \cdot \sum_{n=0}^{m-2} \sum_{w \in W_n} \mathcal{E}_{w B_K}(B_f \circ \Psi_w \circ \Psi_w^{-1}) + \sum_{w \in W_{m-1,1}} \mathcal{E}_{w T_{1,K}}(g_m)$$

$$\leq C_1 \cdot \sum_{n=0}^{m-2} r^{-n} D_{G_B}(f) + r^{-m+1} \sum_{w \in W_{m-1,1}} (f(\Psi_{u1} q_4) - f(\Psi_{w} q_3))^2 \cdot \mathcal{E}(h)$$

$$\leq C \cdot D_{G_K}(f),$$

for some $C_1, C > 0$ depending only on $k$ and $r$, where the first inequality follows from Lemma 6.8 (b), and the second inequality follows from the definition of $D_{G_K}(f)$ and $\mathcal{E}(h) < 2$. Clearly, $\{ \|g_m\|_{L^2(T_{1,K},d)} \}$ is uniformly bounded, so we can find a subsequence of $\{g_m\}$ converging weakly to $g$ in $F_{T_{1,K}}$. Thus, $g \in F_{T_{1,K}}$, and the desired energy estimate holds. \[\square\]

**Corollary 6.10.** Let $u \in \Lambda_{2,2}^{(r)}(L_1)$. There exists $g \in F_{T_{1,K}}$ such that

$$\begin{cases} g|_{L_1} = u, \\ \mathcal{E}_{T_{1,K}}(g) \leq C \cdot [u]_{\Lambda_{2,2}^{(r)}(L_1)}^2 \end{cases}$$

for some constant $C > 0$ depending only on $k$ and $r$. In addition, if $u(q_1) = u(q_2) = c$, we can require $g|_{L_2 \cap T_{1,K}} = g|_{L_4 \cap T_{1,K}} = g|_{L_{3} \cap T_{1,K}} = \Psi_x L_3 = c, \forall 1 \leq i \leq k$.

**Proof.** Define $f \in l(V_K)$ as $f(x_1, x_2) = u(x_1)$ for any $x = (x_1, x_2) \in V_K$. From the definition, $D_{G_K}(f) \leq (1 + r)[[u]]_{\Lambda_{2,2}^{(r)}(L_1)}^2$. Then we can construct a function $g \in F_{T_{1,K}}$ following from Theorem 6.9 which satisfies the desired requirement. Finally, if $u(q_1) = u(q_2) = c$, by
the construction of building brick functions and Lemma 6.8 (a), we can see \( g|_{L_2 \cap T_{1,K}} = g|_{L_3} = c, \forall 1 \leq i \leq k \).

We will deal with \( L_i, i = 1, 2, 3, 4 \) together at the end of this section after we have established the associated restriction theorem.

6.4. Resistance between \( q_1, q_2, q_3, q_4 \). Next, we will consider the converse direction to the last subsection, the restriction of \( F \) to \( L_i, i = 1, 2, 3, 4 \). Before doing that, we need to estimate the resistance between boundary points.

**Lemma 6.11.** Let \((\mathcal{E}, F)\) be a Dirichlet form on \( K \) satisfying (C1) and (C2). Then, \( R(x,y) \leq 2k^3 \cdot R(q_1, q_2), \forall x, y \in \partial_0 K \).

**Proof.** It is well known that the effective resistance \( R \) is a metric on \( K \). So, for \( x = (\sum_{m=1}^{M} \alpha_m k^{-m}, 0) \), with \( 0 \leq \alpha_m < k \), \( \alpha_m \in \mathbb{Z} \) and \( M < \infty \), by the triangle equality, we have

\[
R(x, q_1) \leq \sum_{m=1}^{M} \alpha_m r^m R(q_1, q_2) \leq (k-1) R(q_1, q_2) \sum_{m=1}^{\infty} r^m \leq \frac{r(k-1) R(q_1, q_2)}{1-r} \leq (k^3-1) R(q_1, q_2),
\]

where in the last inequality we use \( r \leq \frac{N}{k^2} \leq \frac{k^2-1}{k^2} \). Now, for \( x = (x_1, x_2), y = (y_1, y_2) \in \partial_0 K \) such that \( x_1, x_2, y_1, y_2 \) have finite \( k \)-adic expansion (we denote \( \partial_0 K \) the set of all such points), the desired estimate of \( R(x, y) \) follows immediately by using symmetry, by using the triangle inequality, and by inserting at most 2 boundary vertices from \( \{q_i\}_{i=1}^4 \).

Next, we will extend the above estimate by continuity. For general \( x, y \in \partial_0 K \), we have \( R(x,y) = \sup \left\{ \mathcal{E}(f)^{-1} : f(x) = 1, f(y) = 0, f \in \mathcal{F} \cap C(K) \right\} \) by a same argument as in the proof of Lemma 6.6. Now, for every \( \varepsilon > 0 \), we can find \( f \in \mathcal{F} \cap C(K) \) with \( f(x) = 1, f(y) = 0, \) so that \( \mathcal{E}(f) \leq R^{-1}(x, y) + \varepsilon \); and \( x', y' \in \partial_0 K \) with \( d(x', x) < \varepsilon, d(y', y) < \varepsilon, f(x') - f(y') > 1 - \varepsilon \). So \( R^{-1}(x', y') \leq \frac{\mathcal{E}(f)}{(1-\varepsilon)^2} \leq \frac{R^{-1}(x, y) + \varepsilon}{(1-\varepsilon)^2} \), thus the estimate \( R(x, y) \leq 2k^3 \cdot R(q_1, q_2) \) still holds by letting \( \varepsilon \to 0 \).

**Proposition 6.12.** Let \((\mathcal{E}, F)\) be a Dirichlet form on \( K \) satisfying (C1) and (C2). Then \( R(q_1, q_2) \leq C \) for some \( C > 0 \) depending only on \( k \).

**Proof.** Let \( h \) be the unique function on \( K \) such that

\[
h(q_1) = -1, h(q_2) = 1, \mathcal{E}(h) = \frac{4}{R(q_1, q_2)},
\]

i.e. \( h \) is harmonic in \( K \setminus \{q_1, q_2\} \). It suffices to show that \( \mathcal{E}(h) \geq C \cdot \mathcal{E}(h') \), where \( C > 0 \) depends only on \( k \) and \( h' \) is the unique function such that

\[
h'|_{L_4} = 0, h'|_{L_2} = 1, \mathcal{E}(h') = \frac{1}{R(L_2, L_4)} = 1.
\]

Choose the smallest positive integer \( l \geq \frac{\log 32k^3}{\log k^2 - \log (k^2-1)} \geq -\frac{\log 32k^3}{\log k} \). Then by Lemma 6.11, for any \( x \in \Psi_1 \partial_0 K \), one can see that \( R(q_1, x) \leq r^l R(q_1, \Psi_1^{-1} x) \leq \frac{1}{10} R(q_1, q_2) \), and so

\[
|h(q_1) - h(x)|^2 \leq R(q_1, x) \mathcal{E}(h) \leq \frac{1}{4}.
\]

As a consequence \( h|_{\Psi_1 \partial_0 K} \leq -\frac{1}{2} \), and by symmetry we also have \( h|_{\Psi_4 \partial_0 K} \geq \frac{1}{2} \).
Now, let $h'' \in \mathcal{F}_{T_i,k}$ be defined as $h''(\Psi_w x) = \frac{e(w)}{k^l} + \frac{1}{k^l} h' \circ \Psi_w(x) - \frac{1}{2}$ for each $w \in T_{i,j}, x \in K$, where $e(w) = \sum_{l=1}^{l'} (w l - 1) k^{l''}$. It is easy to see that $h''|_{\Psi^1_{i} L_4} = -\frac{1}{2}$ and $h''|_{\Psi^1_{i} L_2} = \frac{1}{2}$. We will prove that

$$\mathcal{E}(h) \geq \mathcal{E}_{T_i,k}(h) \geq \mathcal{E}_{T_i,k}(h'') = k^{-l} r^{-l} \cdot \mathcal{E}(h'),$$

(6.2)

where the first inequality and the last equality are trivial.

To see the second inequality in (6.2), we claim that

$$\mathcal{E}_{T_i,k}(h'') = \inf \{ \mathcal{E}_{T_i,k}(g) : g|_{\Psi^1_{i} L_4} = -\frac{1}{2}, g|_{\Psi^1_{i} L_2} = \frac{1}{2}, g \in \mathcal{F}_{T_i,k} \},$$

i.e. $h''$ is harmonic in $T_{i,K} \setminus (\Psi^1_{i} L_4 \cup \Psi^1_{i} L_2)$.

First, for each $w, w' \in T_{i,j}$ with $e(w') - e(w) = 1$, one can check $h''|_{\Psi_w K \cup \Psi_{w'} K}$ is harmonic in $(\Psi_w L_4 \cup \Psi_{w'} L_4) \setminus (\Psi_w L_4 \cup \Psi_{w'} L_2)$. In fact, if $g$ is the unique function in $\mathcal{F}_{\Psi_w K \cup \Psi_{w'} K}$ such that $g|_{\Psi_w L_4} = \frac{e(w)}{k^l} - \frac{1}{2}, g|_{\Psi_{w'} L_2} = \frac{e(w) + 2}{k^l} - \frac{1}{2}$ and $g$ is harmonic in $(\Psi_w K \cup \Psi_{w'} K) \setminus (\Psi_w L_4 \cup \Psi_{w'} L_2)$, then we can see $g + \frac{1}{2} - \frac{e(w) + 1}{k^l}$ is anti-symmetric with respect to the horizontal reflection about $\Psi_w L_4$, whence $g|_{\Psi_w L_2} = \frac{e(w) + 1}{k^l} - \frac{1}{2}$. It follows $h''|_{\Psi_w K \cup \Psi_{w'} K} = g$, since both $h''$ and $g$ are harmonic in $\Psi_w K \setminus (\Psi_w L_2 \cup \Psi_{w'} L_2)$ and $\Psi_{w'} K \setminus (\Psi_w L_2 \cup \Psi_{w'} L_4)$ and taking the same boundary values.

Second, for each $v \in T_{i,K} \cap C(T_{i,K})$ such that $v|_{\Psi^1_{i} L_4 \cup \Psi^1_{i} L_2} = 0$, it is easy to find a partition $v = \sum_{w \in T_{i,j}} 0 \leq e(w) \leq k^l - 2 v_w \cdot 1_{\Psi_w K \cup \Psi_{w'} K}$ where for each $0 \leq e(w) \leq k^l - 2$ with $e(w') - e(w) = 1$, $v_w \in \mathcal{F}_{\Psi_w K \cup \Psi_{w'} K} \cap C(\Psi_w K \cup \Psi_{w'} K)$ and $v_w|_{\Psi_w L_4 \cup \Psi_{w'} L_2} = 0$. Thus, $\mathcal{E}_{T_i,k}(h'', v) = \sum_w \mathcal{E}_{T_i,k}(h'', v_w \cdot 1_{\Psi_w K \cup \Psi_{w'} K}) = \sum_w \mathcal{E}_{T_i,k}(h'', v_w) = 0$, where the last equality holds since $h''|_{\Psi_w K \cup \Psi_{w'} K}$ is harmonic in $(\Psi_w K \cup \Psi_{w'} K) \setminus (\Psi_w L_2 \cup \Psi_{w'} L_2)$ as showed in the last paragraph. This implies that $h''$ is harmonic in $T_{i,K} \setminus (\Psi^1_{i} L_4 \cup \Psi^1_{i} L_2)$.

Hence the claim holds, and thus (6.2) holds since $h|_{\Psi^1_{i} \partial_K} \leq -\frac{1}{2}$ and $h|_{\Psi^1_{i} \partial_K} \geq \frac{1}{2}$. So $\mathcal{E}(h) \geq k^{-l} r^{-l} \cdot \mathcal{E}(h') \geq (\frac{k}{k^2 - 1})^l \cdot \mathcal{E}(h')$ since $r \leq \frac{N}{k^2} \leq \frac{k^2}{k^2 - 1}$. The proof is completed.  \hfill \Box

6.5. A restriction theorem. Now, for a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $K$ satisfying (C1) and (C2), we proceed to study the trace of $\mathcal{F}$ on $L_i, i = 1, 2, 3, 4$. First, we remark that the proof of the upper bound estimate of resistance in Theorem 7.2 only uses Lemma 6.11 and Proposition 6.12, so we always have $\mathcal{F} \subset C(K)$. In addition, as an immediate application of Proposition 6.12, we have the following estimate, as a reverse direction of Lemma 6.8 (b).

**Lemma 6.13.** There is $C > 0$ depending only on $k$ such that

(a) $D_{GBK}(f) \leq C \cdot \mathcal{E}_{BK}(f)$ for any $f \in \mathcal{F}_{BK}$,

(b) $D_{GK}(f) \leq C \cdot \mathcal{E}(f)$ for any $f \in \mathcal{F}$.

We will focus on $L_1$ for convenience. For a function $u$ on $L_1$, we introduce the notation

$$D_n(u) = \sqrt{\sum_{l=0}^{k^n-1} (u(l/k^n, 0) - u(l + 1/k^n, 0))^2}.$$

Then, $\left[ \left[ u \right] \right]_{L^{2,2}_{2,2}(L_1)} = \left\| r^{-n/2} D_n(u) \right\|_2 = \sqrt{\sum_{n=0}^{\infty} r^{-n} |D_n(u)|^2}.$
Also, for \( f \in C(\text{cl}(V_K)) \), we write

\[
\tilde{D}_n(f) = \sqrt{\sum_{w \in W_{n,1}} \mathcal{D}_{G_{BK}}(f \circ F_w)}.
\]

Then, \( \mathcal{D}_K(f) = \sum_{n=0}^{\infty} r^{-n} |\tilde{D}_n(f)|^2 \).

**Lemma 6.14.** Let \( f \in C(\text{cl}(V_K)) \), then we have

\[
D_n(f|_{L_1}) \leq 3 \sum_{m=n}^{\infty} \tilde{D}_m(f), \quad \forall n \geq 0.
\]

**Proof.** Clearly, for \( 0 \leq l \leq k^n \), we have

\[
f(\frac{l}{k^n}, 0) = f(\frac{l}{k^n}, \frac{1}{k^{n+1}}) + \sum_{m=n}^{\infty} (f(\frac{l}{k^n}, \frac{1}{k^{m+2}}) - f(\frac{l}{k^n}, \frac{1}{k^{m+1}})).
\]

Thus by Minkowski inequality,

\[
D_n(f|_{L_1}) = \sqrt{\sum_{l=0}^{k^n-1} (f(\frac{l}{k^n}, 0) - f(\frac{l+1}{k^n}, 0))^2} \\
= \sqrt{\sum_{l=0}^{k^n-1} \left(f(\frac{l}{k^n}, \frac{1}{k^{n+1}}) - f(\frac{l+1}{k^n}, \frac{1}{k^{n+1}})\right)^2 + \sum_{m=n}^{\infty} \left((-1)^l \sum_{l'=0}^{k^n-1} (f(\frac{l+l'}{k^n}, \frac{1}{k^{m+2}}) - f(\frac{l+l'}{k^n}, \frac{1}{k^{m+1}}))^2\right)} \\
\leq \sqrt{\sum_{l=0}^{k^n-1} \left(f(\frac{l}{k^n}, \frac{1}{k^{n+1}}) - f(\frac{l+1}{k^n}, \frac{1}{k^{n+1}})\right)^2 + \sum_{m=n}^{\infty} \sum_{l'=0}^{k^n-1} \left(f(\frac{l+l'}{k^n}, \frac{1}{k^{m+2}}) - f(\frac{l+l'}{k^n}, \frac{1}{k^{m+1}})^2\right)} \\
\leq 3 \sum_{m=n}^{\infty} \tilde{D}_m(f).
\]

\[\square\]

Combining Lemma 6.13 and 6.14, we can easily prove the following restriction theorem.

**Theorem 6.15.** Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form on \(K\) satisfying \(C1\) and \(C2\). There exists \(C > 0\) depending only on \(k\) such that

\[
[[f|_{L_1}]]_{\mathcal{A}(\sigma(L_1))}^2 \leq C \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F}.
\]
Proof. The theorem follows from the following inequality,
\[ ([f|_{L_1}]_{\Lambda_2^{(r)}}(L_1)) = \left\| r^{-n/2}D_n(f|_{L_1}) \right\|_{L^2} \]
\[ \leq 3 \cdot \left\| \sum_{m=n}^{\infty} r^{-n/2}D_m(f|_{L_1}) \right\|_{L^2} \]
\[ = 3 \cdot \left\| \sum_{m=0}^{\infty} r^{m/2}r^{-(m+n)/2}D_{m+n}(f) \right\|_{L^2} \]
\[ \leq 3 \cdot \left\| \sum_{m=0}^{\infty} r^{m/2}r^{-(m+n)/2}D_{m+n}(f) \right\|_{L^2} \]
\[ \leq 3 \cdot \left\| \sum_{m=0}^{\infty} r^{m/2}r^{-(m+n)/2}D_{m+n}(f) \right\|_{L^2} \]
where we use Lemma 6.14 in the second line and use Lemma 6.13 (b) in the last inequality. Here \( C' \) depends only on \( k \) noticing that \( r \leq \frac{N}{k^2} \).

Remark. As an immediate consequence of Theorem 6.15, it holds that \( r > \frac{1}{2} \) since otherwise \( \Lambda_2^{(r)}(L_1) = \text{Constants} \). So all the constants in Section 6.2, 6.3 depends only on \( k \) as also \( r \leq \frac{N}{k^2} \leq \frac{k^2}{k^2-1} \).

Finally, we finish this section with the proof of Theorem 6.1

Proof of Theorem 6.1. We consider the \( n = 0 \) case only, since the \( n \geq 1 \) case follows immediately from it.

First, by Theorem 6.15, one can see \( \mathcal{H}_0|_{\partial_bK} \subset \Lambda_2^{(r)}(\partial_bK) \), and there is \( C_1 > 0 \) depending only on \( k \) so that for any \( h \in \mathcal{H}_0 \),
\[ ([h|_{\partial_bK}]_{\Lambda_2^{(r)}}(\partial_bK)) \leq C_1 \cdot \mathcal{E}(h). \]
(6.3)

For the other direction, it suffices to show that for any \( u \in \Lambda_2^{(r)}(\partial_bK) \), there is \( f \in \mathcal{F} \) such that
\[ \begin{cases} f|_{\partial_bK} = u, \\ \mathcal{E}(f) \leq C_2 \cdot [u]_{\Lambda_2^{(r)}}(\partial_bK) \end{cases} \]
for some \( C_2 > 0 \) depending only on \( k \). To achieve this, we apply the construction in Corollary 6.10. We take two steps to construct a desired \( f \).

Step 1. We find \( f' \in \mathcal{F} \) such that \( f'|_{L_1} = u|_{L_1} \), \( f'|_{L_3} = u|_{L_3} \), and
\[ \mathcal{E}(f') \leq C_3 \cdot [u]_{\Lambda_2^{(r)}}(\partial_bK) \]
(6.4)
for some \( C_3 > 0 \) depending only on \( k \).

To complete Step 1, we first use Corollary 6.10 to construct \( f' \) on \( T_{1,K} \) and \( \Gamma_vT_{1,K} \) (recall \( \Gamma_v \) in (2.1)) satisfying \( f'|_{L_1} = u|_{L_1} \), \( f'|_{L_3} = u|_{L_3} \). Then for the middle part, we take
\[ f'(x) = u(q_1)(1 - h(x))h''(x) + u(q_2)h(x)h''(x) \\
+ u(q_3)h(x)(1 - h''(x)) + u(q_4)(1 - h(x))(1 - h''(x)) \quad \forall x \in K \setminus (T_{1,K} \cup \Gamma_vT_{1,K}), \]
where $h$ is the same function defined in Lemma 6.6 which satisfies $h \in \mathcal{F} \cap \mathcal{C}(K)$, $h|_{L_2} = 0$ and $h|_{L_2} = 1$; for $h''$, as usual one can use scaled copies of $h \circ \Gamma_{t_1}$ to build $h'' \in \mathcal{F} \cap \mathcal{C}(K)$ such that $h''|_{\Gamma_{t_1}} = 1, h''|_{\Gamma_{t_1}} = 0$. Clearly, $f' \in \mathcal{F}$ and satisfies the desired energy estimate.

Now, let $u' = u - f'|_{\partial K}$. We have

$$\left[\left[u'\right]\right]_{A_{2,2}^1(\partial K)} \leq \left[\left[u\right]\right]_{A_{2,2}^1(\partial K)} + \left[\left[f'|_{\partial K}\right]\right]_{A_{2,2}^1(\partial K)} \leq C_4 \cdot \left[\left[u\right]\right]_{A_{2,2}^1(\partial K)}$$

(6.5)

for some $C_4 > 0$ depending only on $k$, where in the last inequality we use a combination of (6.3) and (6.4). In addition, we see that

$$u'|_{L_1 \cup L_3} = 0.$$

Step 2. We construct $f'' \in \mathcal{F}$ such that $f''|_{\partial K} = u'$ and $\mathcal{E}(f'') \leq C_3 \cdot \left[\left[u\right]\right]_{A_{2,2}^1(\partial K)}^2$.

This can be fulfilled with a similar argument as Step 1. First, we construct $f''$ on $\Gamma_{d_1} \cap \Gamma_{d_2} \cap K$ using Corollary 6.10 so that $f''|_{L_2 \cup L_4} = u'|_{L_2 \cup L_4}$ (recall $\Gamma_{d_1}, \Gamma_{d_2}$ in (2.1)). Then we extend $f''$ to $K$ with continuously since $f''|_{\bigcup_{i \in \mathcal{W}_1} \Psi_i L_2} = f''|_{\bigcup_{i \in \mathcal{W}_1} \Psi_i L_4} = 0$. The energy estimate

$$\mathcal{E}(f'') \leq C_3 \cdot \left[\left[u\right]\right]_{A_{2,2}^1(\partial K)}^2$$

(6.6)

is same as in Step 1, and $f''|_{L_1 \cup L_3} = 0$ is guaranteed by the construction.

Finally, we take $f = f' + f''$. We then have $f|_{\partial K} = u$, and by combining (6.4)-(6.6),

$$\mathcal{E}(f) \leq C_2 \cdot \left[\left[u\right]\right]_{A_{2,2}^1(\partial K)}^2$$

for some $C_2 > 0$ depending only on $k$.

7. A uniform resistance estimate

In this section, we provide a uniform estimate on the resistance metric $R(\cdot, \cdot)$ for a Dirichlet form on $\mathcal{G}$ satisfying (C1), (C2). Same as the last section, here uniform means that the estimate only depends on $k$, does not depend on the choice of $K$.

To achieve a uniform estimate, we will adopt the geodesic metric on $\mathcal{G}$, instead of the Euclidean metric. For a $\mathcal{G}$, the geodesic metric $d_G$ on $K$ is defined as

$$d_G(x, y) = \inf \left\{ \text{length} \left( \gamma \right) : \gamma : [0, 1] \to K \text{ is a rectifiable curve}, \gamma(0) = x, \gamma(1) = y \right\},$$

for any $x \neq y \in K$. It is not so different compared with the Euclidean metric $d$ on $K$.

**Lemma 7.1.** There exists $C > 0$ depending only on $K$ such that

$$d(x, y) \leq d_G(x, y) \leq C \cdot d(x, y), \quad \forall x, y \in K.$$

**Proof.** We first observe that

$$d_G(x, \partial K) \leq C', \forall x \in K, \text{ for some } C' > 0 \text{ depending only on } k.$$  

(7.1)

In fact, a very rough chain argument gives that $C' \leq 4Nk^{-1} + 4Nk^{-2} + \cdots$, where the first term $4Nk^{-1}$ is greater than the total length of $\partial K$.

Now, given $x, y \in K$, we choose $n = 0 \lor \min \left\{ m \in \mathbb{Z} : d(x, y) < c_0k^{-m} \right\}$, where $c_0$ is the constant in (A3) in Section 2. Then, by (A3), we can find $w, w'' \in \mathcal{W}_n$ so that $x \in \Psi_w K, y \in \Psi_{w''} K$ and $\Psi_w K \cap \Psi_{w''} K \neq \emptyset$. By (7.1), there is $z_x \in \Psi_w (\partial K)$ so that $d_G(x, z_x) \leq C'k^{-n}$; there is $z_y \in \Psi_{w''} (\partial K)$ so that $d_G(y, z_y) \leq C'k^{-n}$. Then, by (A3), we can find $w, w'' \in \mathcal{W}_n$ so that $x \in \Psi_w K, y \in \Psi_{w''} K$ and $\Psi_w K \cap \Psi_{w''} K \neq \emptyset$. By (7.1), there is $z_x \in \Psi_w (\partial K)$ so that $d_G(x, z_x) \leq C'k^{-n}$; there is $z_y \in \Psi_{w''} (\partial K)$ so that $d_G(y, z_y) \leq C'k^{-n}$. Then, by (A3), we can find $w, w'' \in \mathcal{W}_n$ so that $x \in \Psi_w K, y \in \Psi_{w''} K$ and $\Psi_w K \cap \Psi_{w''} K \neq \emptyset$. By (7.1), there is $z_x \in \Psi_w (\partial K)$ so that $d_G(x, z_x) \leq C'k^{-n}$; there is $z_y \in \Psi_{w''} (\partial K)$ so that $d_G(y, z_y) \leq C'k^{-n}$.
Thus, \( d_G(x, y) \leq d_G(x, z_x) + d_G(z_x, z_y) + d_G(z_y, y) \leq 2C'k^{-n} + 12k^{-n} \). The lemma follows immediately by the choice of \( n \).

The following theorem is the main result in this section, which is a first application of the trace theorem, Theorem 6.1

**Theorem 7.2.** Let \( K \) be a US Ą and \((E,F)\) be a Dirichlet form satisfying (C1), (C2). Let \( \theta = -\frac{\log r}{\log k} \) Then there are \( C_1, C_2 > 0 \) depending only on \( k \) such that

\[
C_1 \cdot d_G(x, y) \theta \leq R(x, y) \leq C_2 \cdot d_G(x, y) \theta, \quad \forall x, y \in K,
\]

\[
C_1 \cdot \rho^\theta \leq R(x, K \setminus B_\rho^{(G)}(x)) \leq C_2 \cdot \rho^\theta, \quad \forall x \in K, 0 < \rho \leq 1,
\]

where \( B_\rho^{(G)}(x) = \{ y \in K : d_G(x, y) < \rho \} \).

**Proof.** It suffices to show \( R(x, y) \leq C_2 \cdot d_G(x, y) \theta \) and \( C_1 \cdot \rho^\theta \leq R(x, K \setminus B_\rho^{(G)}(x)) \) for some \( C_1, C_2 > 0 \) depending only on \( k \).

**Proof of \( \text{"}R(x, y) \leq C_2 \cdot d_G(x, y) \theta\text{"} \).** First, by Lemma 6.11 and Proposition 6.12, \( R(x', y') \leq C_3, \forall x', y' \in \partial_0 K \), for some \( C_3 > 0 \) depending only on \( k \). Then, by a routine chain argument and an argument of extension by continuity as the proof of Lemma 6.11, \( R(x', y') \leq C_4, \forall x', y' \in K \) for some \( C_4 > 0 \) depending only on \( k \). As a consequence, \( R(x, y) \leq C_4 \cdot r^n \) if there is \( w \in W_n \) such that \( \{ x, y \} \subset \Psi_w K \).

Now, we consider \( x, y \) in the statement. We choose \( n \) such that \( k^{-n-1} < d_G(x, y) \leq k^{-n} \), and \( w, w' \in W_n \) such that \( x \in \Psi_w K, y \in \Psi_{w'} K \). Noticing that \( d(\cdot, \cdot) \leq d_G(\cdot, \cdot) \), there are at most \( C_5 = [\pi(1 + \sqrt{2})^2] \) many \( w'' \in W_n \) such that \( d_G(x, \Psi_{w''} K) \leq k^{-n} \), since \( \Psi_{w''} K \subseteq B(x, k^{-n}(1 + \sqrt{2})) \) (ball centered at \( x \) with respect to \( d \)) for any such \( w'' \). Since the geodesic path between \( x, y \) will only intersect cells \( \Psi_{w''} K \) such that \( d_G(x, \Psi_{w''} K) \leq d_G(x, y) \leq k^{-n} \), we can find a chain \( w = w(1), w(2), \ldots, w(l) = w' \) in \( W_n \) such that

\[
l \leq C_5, \quad \Psi_{w(l')} K \cap \Psi_{w(l-1')} K \neq \emptyset, \quad \forall 1 < l' \leq l.
\]

Thus, we get \( R(x, y) \leq C_4 C_5 \cdot r^n \leq C_2 \cdot d_G(x, y) \theta \) for some \( C_2 > 0 \) depending only on \( k \).

**Proof of \( \text{"}C_1 \cdot \rho^\theta \leq R(x, K \setminus B_\rho^{(G)}(x))\text{"} \).** The proof is based on Theorem 6.1 and the observation (7.1). We choose \( n \) so that \( k^{-n} \leq C_6^{-1} \cdot \frac{\pi}{4} \leq k^{-n+1} \), where \( C_6 = C' + 2 \) and \( C' \) is the constant in (7.1). Define \( u \in C(\partial_0 K) \) to be

\[
u(y) = (1 - \rho^{-1} d_G(x, y)) \vee 0, \quad \forall y \in \partial_0 K.
\]

Clearly, \( u \) is Lipschitz on each edge \( \Psi_{w(i)} L_i, w \in W_n, i \in \{1, 2, 3, 4\} \), i.e. \( |u(y) - u(y')| \leq \rho^{-1} d(y, y'), \forall y, y' \in \Psi_{w(i)} L_i \).

Let \( h \in \mathcal{H}_n \) such that \( h|_{\partial_0 K} = u \) as in Theorem 6.1. Then we have

1. \( h(x) \geq \frac{3}{4} \) and \( h|_{K \setminus B_\rho^{(G)}(x)} \leq \frac{1}{4} \).
2. \( h|_{\Psi_w K} = 0 \) for any \( w \in W_n \) such that \( \Psi_w K \cap B_\rho^{(G)}(x) = \emptyset \). In particular, \( h|_{\Psi_w K} \) is nonzero on at most \( C_7 = \lceil 4C_6k + \sqrt{2} \rceil + 2 \) many cells \( \Psi_w K \) with \( w \in W_n \).

In fact, let \( x \in F_u K \) for some \( u \in W_n \). Then for any \( y \in \Psi_u (\partial_0 K) \), by (7.1), \( d_G(x, y) \leq C_6 k^{-n} \leq \frac{\pi}{4} \). So \( u(y) \geq \frac{3}{4} \), and it follows \( h(x) \geq \frac{3}{4} \). On the other hand, for any \( y \in K \setminus B_\rho^{(G)}(x) \), it follows that \( y \in F_u' K \) for some \( u' \in W_n \) and \( d_G(x, F_u' (\partial_0 K)) \geq d_G(x, y) \)
Theorem 4.5.1 in [24]. Thus, if $A$ is a Dirichlet form on $L^2(K,\mu)$, Lemma 8.2.

Proof. In this proof, all the functions are taken to be quasi-continuous, and we say symmetric Dirichlet form $(K,\mu)$ on $L^2(K,\mu)$ that is self-similar in the sense of Definition 5.1 and has a spectral gap:

\[ \|f - \int_{K} f d\mu\|_{L^2(K,\mu)}^2 \leq C \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F}. \] (8.1)

Let’s first show that the conditions (C1), (C2) are satisfied for any form under consideration in the above theorem.

Lemma 8.2. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, irreducible, symmetric, self-similar Dirichlet form on $L^2(K,\mu)$. There is $C > 0$ such that $(C\mathcal{E}, \mathcal{F})$ satisfies (C1).

Proof. In this proof, all the functions are taken to be quasi-continuous, and we say $f = g$ on $A$ if $f(x) = g(x)$ for quasi every $x \in A$.

Let’s write $(\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ for the Hunt process associated with $(\mathcal{E}, \mathcal{F})$ and $L^2(K,\mu)$, and write $\sigma_A = \inf\{t \geq 0 : X_t \in A\}$ for a subset $A \subset K$. Since $(\mathcal{E}, \mathcal{F})$ is irreducible and local, $\mathbb{P}_x(\sigma_{K\setminus\Psi_1K} < \infty) = \mathbb{P}_x(\sigma_{\Psi_1(\partial_0K)} < \infty) > 0$ for quasi every $x \in \Psi_1K$ by Theorem 4.7.1 and Theorem 4.5.1 in [24]. Thus, $\text{Cap}(\Psi_1(\partial_0K)) > 0$ by Theorem 4.2.1 in [24].

Now we show that $\text{Cap}(\partial_0K) > 0$, which implies $\text{Cap}(L_2) = \text{Cap}(L_4) \geq \frac{\text{Cap}(\partial_0K)}{4} > 0$, so we can find $C > 0$ such that $C^{-1} \cdot R(L_2, L_4) = 1$. With the previous paragraph, it suffices to show that

\[ \text{Cap}(\partial_0K) \geq (N^{-1} r \land 1) \cdot \text{Cap}(\partial_1K). \]

Since $\partial_0K$ is compact, by Lemma 2.2.7 of [24], there is $f \in \mathcal{F} \cap C(K)$ such that $f|_{\partial_0K} = 1$ and $\mathcal{E}_1(f) \leq \text{Cap}(\partial_0K) + \epsilon$ for any small $\epsilon > 0$. Define $g \in C(K)$ by $g \circ \Psi_i(x) = f(x), \forall 1 \leq i \leq N$. Then, by self-similarity (Definition 5.1), we see that $g \in \mathcal{F}$ and

\[ \text{Cap}(\partial_1K) \leq \mathcal{E}_1(g) = Nr^{-1} \mathcal{E}(f) + \|f\|_{L^2(K,\mu)}^2 \leq (N r^{-1} \lor 1) (\text{Cap}(\partial_0K) + \epsilon). \]

The desired estimate follows immediately noticing that $\epsilon$ is arbitrary.

It remains to prove $0 < r \leq \frac{N}{r^2}$. We can assume that $r > \frac{1}{r^2}$ since otherwise there is nothing to prove. With this, the linear function $u : [0, 1] \to \mathbb{R}$ defined by $u(x) = x$ is in $\Lambda_{2,2}^\alpha(0,1]$ (see

\[ d_G(y, F_u(\partial_0K)) - 2k^{-n} \geq \rho - C_0 k^{-n} \geq \frac{3\rho}{4}. \] This gives that $h|_{K \setminus B^G(\rho)}(x) \leq \frac{1}{4}$. Thus (1) follows. (2) is trivial as each $\Psi_uK$ with $\Psi_uK \cap B^G(\rho) \neq \emptyset$ is contained in $B(x, \rho + \sqrt{2k^{-n}})$.

Thus, by (2) and Theorem 6.1

\[ \mathcal{E}(h) \leq C_8 \cdot \sum_{w \in W_n} r^{-n} \left[ \left[ (h \circ \Psi_w)\right]|_{\partial_0K} \right]^2 \Lambda_{2,2}^\alpha(\partial_0K) \leq C_9 \cdot r^{-n}, \]

for some $C_8, C_9 > 0$ depending only on $k$. This together with (1) yields the desired estimate. \hfill \Box

8. Uniqueness

In this section, we prove the uniqueness of the self-similar Dirichlet form on $\mathcal{USC}$ under some assumptions.

Theorem 8.1. Let $K$ be a $\mathcal{USC}$ and $\mu$ be the normalized $d_H$-dimensional Hausdorff measure on $K$. There is a unique (up to a constant multiplier) strongly local, regular, irreducible, symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\mu)$ that is self-similar in the sense of Definition 5.1 and has a spectral gap:

\[ \|f - \int_{K} f d\mu\|_{L^2(K,\mu)}^2 \leq C \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F}. \] (8.1)
Definition 6.2, and by a same argument in Section 6.3 one can see that there is $h \in \mathcal{F} \cap C(K)$ such that
\[ h(x_1, x_2) = x_1, \quad \forall x = (x_1, x_2) \in \partial_0 K. \]
By self-similarity, and by glueing scaled and translated copies of $h$ together, we can obtain a $h_n \in \mathcal{F}$ such that
\[ h_n(x_1, x_2) = x_1, \quad \forall x = (x_1, x_2) \in \partial_n K, \]
and
\[ \mathcal{E}(h_n) = \left( \frac{r^{-1} N}{k^2} \right)^n \cdot \mathcal{E}(h). \]
On the other hand, as $h_n|_{\partial_0 K} = h|_{\partial_0 K}$, $\mathcal{E}(h_n)$, $n \geq 1$ is uniformly bounded away from 0 by Remark 1 and 2 at the beginning of Section 6. This implies $r \leq \frac{N}{k^2}$.

Lemma 8.3. Let $(\mathcal{E}, \mathcal{F})$ be a form satisfying all the assumptions of Theorem 8.1 Then (C2) holds.

Proof. By using 8.1, for any $f \in \mathcal{F}$, one immediately see that $\|Q_1 f - Q_0 f\|_{L^2(K, \mu)} \leq C_1 \cdot \mathcal{E}(f)$ for some $C_1 > 0$, where $Q_n$, $n \geq 0$ is defined in the proof Theorem 3.4, and in particular $Q_0 f = \int_K f(x) \mu(dx)$. Then, by using self-similarity, we have $\|Q_{n+1} f - Q_n f\|_{L^2(K, \mu)} \leq C_1 \cdot r^{-n} \mathcal{E}(f)$.

Hence, by the fact that $r \leq \frac{N}{k^2}$ in (C1), we see that $\|f - Q_0 f\|_{L^2(K, \mu)} \leq C_2 \cdot \mathcal{E}(f)$ for some $C_2 > 0$.

As a consequence, for any $f_n \to f$ in $\mathcal{E}_1$-norm, if we add the restriction that $\int_K f_n(x) \mu(dx) = 0$, $\forall n \geq 1$ and $\int_K f(x) \mu(dx) = 0$ (by subtracting suitable constants), then $f_n \to f$ in $L^\infty(K, \mu)$. Whence, $\mathcal{F} \subset C(K)$ since $\mathcal{F} \cap C(K)$ in dense in $\mathcal{F}$.

Finally, for any $x \in K$, $f \in \mathcal{F}$, with $f(x) = 1$, we have $\mathcal{E}_1(f) \geq \mathcal{E}(f) + (\int_K f(y) \mu(dy))^2 \geq \mathcal{E}(f) + (1 - \sqrt{C_2 \cdot \mathcal{E}(f)})^2 \geq (1 + C_2)^{-1}$, hence $\text{Cap}\{\{x\}\} \geq (1 + C_2)^{-1}$. Thus (C2) holds.

Thus, in the rest of this section, we can always assume (C1), (C2). The proof of Theorem 8.1 is an application of the heat kernel estimate. To be more precise, let $p_t(x, y)$ be the heat kernel associated with the heat operator $P_t$ of a Dirichlet form $(\mathcal{E}, \mathcal{F})$, i.e. $\{P_t\}_{t \geq 0}$ is the unique strongly continuous semigroup on $L^2(K, \mu)$ such that
\[ \mathcal{E}(f) = \lim_{t \to 0} \frac{1}{t} < (1 - P_t)f, f >_{L^2(K, \mu)}, \]
where $< f, g >_{L^2(K, \mu)} = \int_K f(x) g(x) \mu(dx)$. The following heat kernel estimate,
\[ c_1 \cdot t^{-d_H/d_W} \exp \left( - c_2 \cdot \left( \frac{d(x, y)_{dW}}{t} \right)^{\frac{1}{d_W - 1}} \right) \leq p_t(x, y) \]
\[ \leq c_3 \cdot t^{-d_H/d_W} \exp \left( - c_4 \cdot \left( \frac{d(x, y)_{dW}}{t} \right)^{\frac{1}{d_W - 1}} \right), \quad \forall 0 < t \leq 1, x, y \in K \]
holds for some constants $c_1, c_2, c_3, c_4 > 0$, where $d_H = \frac{\log N}{\log K}$ is the Hausdorff dimension of $K$, $d_W = \frac{\log r}{\log K} + d_H$ is named walk dimension in various contents [2] [26]. For short, we will write $\theta = -\frac{\log r}{\log K}$.

There have been many deep works on the relation of resistance estimate and heat kernel estimate in various setting of metric measure spaces [2] [9] [22] [28] [40]. In particular, by
applying Theorem 15.10 and 15.11 in [40] by Kigami, (8.2) is an immediate consequence of the resistance estimate, Theorem 7.2.

Proposition 8.4. Let $K$ be a USC and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form satisfying (C1), (C2). Let $\theta = -\frac{\log r}{\log k}$, $d_H = \frac{\log N}{\log k}$ and $d_W = \theta + d_H$. We have (8.2) holds for some $c_1, c_2, c_3, c_4 > 0$ depending only on $K$.

The rest of this subsection is just the same as that in [8]. First, we refer to [26] for a characterization of $\mathcal{F}$ as a Besov space $B_{2,\infty}^\sigma(K)$.

For $\sigma > 0$, define the Besov space $B_{2,\infty}^\sigma(K)$ on $K$ as

$$B_{2,\infty}^\sigma(K) = \{f \in L^2(K, \mu) : \|f\|_{B_{2,\infty}^\sigma(K)} < \infty\},$$

where

$$\|f\|_{B_{2,\infty}^\sigma(K)} = \left\{ \sup_{0 < \mu < 1} \rho^{-2\sigma - d_H} \int_K \int_{B_\mu(x)} |f(x) - f(y)|^2 \mu(dy)\mu(dx) \right\}^{\frac{1}{2}}.$$

Notice that the Hausdorff dimension $d_H$ does not depend on the form $(\mathcal{E}, \mathcal{F})$.

Proposition 8.5 ([26]). Let $K$ be a USC and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $K$ satisfying (8.2).

(a) $\mathcal{F} = B_{2,\infty}^{d_W/2}(K)$, and there are $C_1, C_2 > 0$ depending only on $c_1, c_2, c_3, c_4$ such that

$$C_1 \cdot \|f\|_{B_{2,\infty}^{d_W/2}(K)} \leq \mathcal{E}(f) \leq C_2 \cdot \|f\|_{B_{2,\infty}^{d_W/2}(K)}, \quad \forall f \in \mathcal{F}.$$

(b) $d_W/2 = \inf \{\sigma > 0 : B_{2,\infty}^\sigma(K) = \text{Constants}\}$.

As an immediate consequence of Proposition 8.4 and 8.5, we see any Dirichlet forms satisfying (C1), (C2) are not very different.

Corollary 8.6. Let $K$ be a USC, and $(\mathcal{E}, \mathcal{F}), (\mathcal{E}', \mathcal{F}')$ be two Dirichlet forms on $L^2(K, \mu)$ satisfying (C1), (C2). Then, we have $\mathcal{F}' = \mathcal{F}$, and there is a constant $C > 0$ depending only on $K$ so that

$$C^{-1} \cdot \mathcal{E}'(f) \leq \mathcal{E}(f) \leq C \cdot \mathcal{E}'(f), \quad \forall f \in \mathcal{F} = \mathcal{F}'.$$

We finalize the proof of Theorem 8.1 with an argument of [8]. The following Proposition 8.7 is essentially the same as Theorem 2.1 in [8].

Proposition 8.7 ([8]). Let $K$ be a USC, and $(\mathcal{E}, \mathcal{F}), (\mathcal{E}', \mathcal{F})$ be two Dirichlet forms on $L^2(K, \mu)$ satisfying (C1), (C2), with a renormalization factor $0 < r \leq \frac{N}{2^7}$. If

$$\mathcal{E}(f) \leq \mathcal{E}'(f), \quad \forall f \in \mathcal{F},$$

then for $\eta > 0$, there exists $C > 0$ such that $(C((1 + \eta)\mathcal{E}' - \mathcal{E}), \mathcal{F})$ is also a Dirichlet form on $L^2(K, \mu)$ satisfying (C1), (C2) with the renormalization factor $r$.

Proof of Theorem 8.1. By Lemma 8.2, 8.3, it suffices to consider Dirichlet forms satisfying (C1), (C2).
The existence follows from Theorem 5.2. We prove the uniqueness by contradiction. Assume $(E, F)$ and $(E', F')$ be two different Dirichlet forms satisfying (C1), (C2). By Corollary 8.6 we know $F = F'$. In addition, letting
\[
\sup(E'|E) = \sup \left\{ E'(f) / E(f) : f \in F \setminus \text{Constants} \right\}, \quad \inf(E'|E) = \inf \left\{ E'(f) / E(f) : f \in F \setminus \text{Constants} \right\},
\]
we know that
\[
\frac{\sup(E'|E)}{\inf(E'|E)} \leq C^2, \tag{8.3}
\]
where $C$ is the same constant in Corollary 8.6. Now for $\eta > 0$, let $E'' = (1+\eta)E' - \inf(E''|E)$. By Proposition 8.7 and by choosing a suitable constant $C' > 0$, one can see that $(C'E'', F)$ is also a Dirichlet form satisfying (C1), (C2). In addition,
\[
\frac{\sup(C'E''|E)}{\inf(C'E''|E)} = \frac{\sup(E''|E)}{\inf(E''|E)} = \frac{(1+\eta)\sup(E'|E) - \inf(E'|E)}{\eta \inf(E'|E)} > \frac{1}{\eta} \left( \frac{\sup(E'|E)}{\inf(E'|E)} - 1 \right).
\]
When $\eta$ is small, this contradicts Corollary 8.6, since a same bound estimate as (8.3) with $E'$ replaced by $C'E''$ should hold by Corollary 8.6. \qed

9. Weak convergence: sliding the USC

By removing the constrain that squares living on grids, we are able to view the USC family (with same $k, N$) as a collection of moving fractals, instead of isolated ones in the classical setting. In this section, we no longer focus on one USC, but let the USC slide around. This situation is essentially new compared with previous ones.

Recall the Hausdorff metric on compact subsets of $\mathbb{R}^s$ is defined as
\[
\delta(A, A') = \inf \{ \rho : A \subset U_\rho(A'), A' \subset U_\rho(A) \}, \quad \forall \text{ compact } A, A' \subset \mathbb{R}^s,
\]
where $U_{\rho}(A) = \bigcup_{x \in A} B_{\rho}(x)$ and $B_{\rho}(x) = \{ y \in \mathbb{R}^s : d(x, y) < \rho \}$. We say $A_n$ converges to $A$ with respect to the Hausdorff metric if $\delta(A_n, A) \to 0$ as $n \to \infty$, and we simply write $A_n \to A$ for short. Clearly $A_n \to A$ implies $A_n \times A_n \to A \times A$ as well.

We will use the following notations in this section. See Appendix B for related definitions and useful results.

**Definition 9.1.** For $n \geq 1$, let compact $A_n \subset \mathbb{R}^s$ and $f_n \in C(A_n)$.

(a). We say $f_n, n \geq 1$ are equicontinuous if and only if
\[
\lim_{\rho \to 0} \left( \sup \{ |f_n(x) - f_n(y)| : n \geq 1, x, y \in A_n, d(x, y) < \rho \} \right) = 0.
\]

(b). Let $A_n \to A$ and $f \in C(A)$ for a compact $A \subset \mathbb{R}^d$. We write $f_n \rightharpoonup f$ if $f_n(x_n) \to f(x)$ for any $x_n \to x$ with $x_n \in A_n, n \geq 1$ and $x \in A$.

In this section, we will frequently assume that $K_n, n \geq 1$ (and also $K$) be USC’s with the same $k, N$, and assume $K_n \to K$. For convenience, and to shorten the statements of the theorems, we will always use the notations as follows:

1). let $\{ \Psi_i \}_{i=1}^N$ be the i.f.s. generating $K$, and let $\{ \Psi_{n,i} \}_{i=1}^N$ be the i.f.s. generating $K_n$. In addition, for each $w = w_1 w_2 \cdots w_m \in W_s$, similarly to $\Psi_w$, we write $\Psi_{n,w} := \Psi_{n,w_1} \circ \Psi_{n,w_2} \circ \cdots \circ \Psi_{n,w_m}$ for short;
2). let $\mu$ be the normalized Hausdorff measure on $K$, and let $\mu_n$ be the normalized Hausdorff measure on $K_n$;
3). let $d_G$ be the geodesic metric on $K$, and let $d_{G,n}$ be the geodesic metric on $K_n$;
4). let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on $K$ satisfying (C1), (C2), and let $(\mathcal{E}_n, \mathcal{F}_n)$ be the Dirichlet form on $K_n$ satisfying (C1), (C2). Recall that Theorem 8.1 ensures the existence and uniqueness of such forms;
5). let $R$ be the resistance metric associated with $(\mathcal{E}, \mathcal{F})$, and let $R_n$ be the resistance metric associated with $(\mathcal{E}_n, \mathcal{F}_n)$;
6). let $M = (\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ be the Hunt process associated with $(\mathcal{E}, \mathcal{F})$, and let $M_n = (\Omega, \mathcal{M}_n, X_t^{(n)}, \mathbb{P}_x^{(n)})$ be the Hunt process associated with $(\mathcal{E}_n, \mathcal{F}_n)$.

The following is the main result in this section.

**Theorem 9.2.** Let $K_n, n \geq 1$ and $K$ be USC’s with the same $k, N$, and assume $K_n \rightarrow K$. Then the following two statements are equivalent:

(i). $R_n \Rightarrow R$. Moreover, for any $x_n \rightarrow x$, with $x_n \in K_n, n \geq 1$ and $x \in K$, we have

$$\mathbb{P}_{x_n}((X_t^{(n)})_{t \geq 0} \in \cdot) \Rightarrow \mathbb{P}_x((X_t)_{t \geq 0} \in \cdot),$$

where the weak convergence “$\Rightarrow$” is in the sense of probability measures on $D(\mathbb{R}_+, \square)$ (the space of càdlàg processes on $D(\mathbb{R}_+, \square)$ equipped with the usual Skorohod J1-topology).

(ii). $d_{G,n}, n \geq 1$ are equicontinuous, viewed as functions on $K_n \times K_n \subset \square^2 \subset \mathbb{R}^4$, equipped with the Euclidean metric on $\mathbb{R}^4$.

**Remark 1.** By continuous embedding, the weak convergence holds on $D(\mathbb{R}_+, \mathbb{R}^2)$ as well.

**Remark 2.** Since probability distributions forms a complete separable metric space with Prohorov metric (on $D(\mathbb{R}_+, \square)$ in Skorohod J1-topology), one can replace the sequence $K_n, n \geq 1$ with a continuous family $K_s, s \in [0, 1]$, such that $K_s \rightarrow K_{s_0}$ with $s \rightarrow s_0 \in [0, 1]$, in the theorem.

In [54], one of the author constructed self-similar resistance forms on fractals which have finitely ramified cell structures in the sense of [54] but are not post critically finite [39] or finitely ramified self-similar sets [54], by using the technique of convergence of resistance metrics and the associated $\Gamma$-convergence on the space of continuous functions (see Appendix B and part of Appendix C for necessary results, that will also be used in the USC setting). A result analogous to Theorem 9.2 is also proved.

For the USC, additional difficulty will arise on the boundaries of cells, so that we can not obtain the self-similarity of the limit form directly. This will be resolved with the trace theorem, Theorem 6.1 and an estimate of energy measure.

### 9.1. Basic geometric properties

Before proving Theorem 9.2, we need to have a close look at the geometry of USC, concerning the normalized Hausdorff measures and geodesic metrics.

**Lemma 9.3.** Let $K_n, n \geq 1$ and $K$ be USC’s with the same $k, N$.

(a). Assuming that for each $1 \leq i \leq N$, $\Psi_{n,i}$ converges to $\Psi_i$ (in the finite dimensional linear space of affine mappings), then $K_n \rightarrow K$.

(b). Conversely, if $K_n \rightarrow K$, then by reordering $\Psi_{n,i}, 1 \leq i \leq N$ for each $n \geq 1$, we have $\Psi_{n,i}$ converges to $\Psi_i$ for each $1 \leq i \leq N$. 

Proof. (a) One can easily check that \( \Psi_{n,w} \to \Psi_w \) and hence \( \delta(\Psi_{n,w}\square, \Psi_w\square) \to 0 \) as \( n \to \infty \) for each \( w \in W_* \). As a consequence,
\[
\delta\left( \bigcup_{w \in W_m} \Psi_{n,w}\square, \bigcup_{w \in W_m} \Psi_w\square \right) \to 0, \quad \text{as} \quad n \to \infty.
\]
(a) then follows immediately, noticing that
\[
\delta(K, \bigcup_{w \in W_m} \Psi_w\square) \leq k^{-m}, \quad \text{and} \quad \delta(K_n, \bigcup_{w \in W_m} \Psi_{n,w}\square) \leq k^{-m}, \forall n \geq 1.
\]

(b) By choosing a subsequence \( n_{l,j} \), \( l \geq 1 \), we have \( \Psi_{n_{l,j}} \) converges for each \( 1 \leq j \leq N \). Take the limit to be \( \Psi_j, 1 \leq j \leq N \). Then one can check that \( \{\Psi_j\}_{j=1}^N \) is still an i.f.s. satisfying Definition 2.1, noticing that \( \Psi_{n_{l,j}}\square \to \Psi_j\square \). In addition, we have \( \bigcup_{j=1}^N \Psi_j\square = \bigcup_{j=1}^N \Psi_j\square \), hence \( \{\Psi_j\}_{j=1}^N = \{\Psi_1\}_{j=1}^N \) since there is a unique way to divide \( \bigcup_{j=1}^N \Psi_j\square \) into \( N \) disjoint squares with side length \( k^{-1} \) (cut \( \square \) from outer to middle to see this).

Noticing that the above argument works for any subsequence, for each \( 1 \leq i \leq N \), we can find a sequence \( \{j(n,i)\}_{j \geq 1} \) so that \( \Psi_{n_{j(n,i)}} \to \Psi_i \) (otherwise, for some subsequence \( n_{l} \), we can not find a further subsequence so that \( \Psi_{n_{l,i}}\square \to \Psi_i \) for some \( j \) as shown in the previous discussion). In addition, for each \( i \neq i' \), the sequence \( \{n_{j(n,i)}\}_{n \geq 1} \) and \( \{n_{j(n,i')}\}_{n \geq 1} \) will be eventually disjoint since they converge to different limits, so we can assume that all the sequences are disjoint. Thus, by replacing \( \Psi_{n,i} \) with \( \Psi_{n_{j(n,i)}} \), we get a proper reordering so that \( \Psi_{n,i} \to \Psi_i \) for each \( 1 \leq i \leq N \).

\[ \square \]

Remark. Due to Lemma 9.3 in the following context, without explicitly mention, we will always assume \( \Psi_{n,i} \) converges to \( \Psi_i \) for \( 1 \leq i \leq N \), whenever \( K_n \to K \).

Proposition 9.4. Let \( K_n, n \geq 1 \) and \( K \) be USC’s with the same \( k, N \). Assume \( K_n \to K \). Then \( \mu_n \Rightarrow \mu \), where “\( \Rightarrow \)” means weak convergence on \( (\square, d) \).

Proof. By Lemma 9.3 (b), by properly reordering \( \{\Psi_{n,i}\}_{1 \leq i \leq N} \), we can assume \( \Psi_{n,i} \to \Psi_i \) as \( n \to \infty \) (in the finite dimensional linear space of affine mappings). Then, for each \( f \in C(\square) \), we have
\[
N^{-m} \sum_{w \in W_m} f(\Psi_w q_1) = \lim_{n \to \infty} N^{-m} \sum_{w \in W_m} f(\Psi_{n,w} q_1).
\]
In addition, for any \( m \geq 0 \), we have
\[
\left\{ \left| \int_K f(x) \mu(dx) - N^{-m} \sum_{w \in W_m} f(\Psi_w q_1) \right| \leq Osc_f(k^{-m}), \right.
\]
\[
\left. \left| \int_{K_n} f(x) \mu_n(dx) - N^{-m} \sum_{w \in W_m} f(\Psi_{n,w} q_1) \right| \leq Osc_f(k^{-m}), \quad \forall n \geq 1, \right. \]
where \( Osc_f(\eta) = \sup \{ \left| f(x) - f(y) \right| : x, y \in \square, d(x,y) \leq \eta \} \), which is continuous at 0 since \( f \) is uniformly continuous. The proposition follows easily from the above estimates. \[ \square \]

Finally, we consider the geodesic metrics. In general, the geodesic metric \( d_{G,n} \) on \( K_n \) (or \( d_G \) on \( K \)) is complicated, so it’s important to have an equivalent but easier characterization of the equicontinuity of \( d_{G,n} \). In the following, we show it’s enough to consider the geodesic metric on \( \bigcup_{i=1}^N \Psi_{n,i}\square \).
Proposition 9.5. Let $K, n \geq 1$ be USC’s with the same $k, N$ and assume $K_n \to K$. For $n \geq 1$, let $\tilde{K}_n = \bigcup_{i=1}^{N} \Psi_{n,i,K}$, and let $\tilde{d}_{G,n}$ be the geodesic metric on $\tilde{K}_n$. Then, the following 
(i),(ii), (iii) are equivalent.

(i). $d_{G,n}, n \geq 1$ are equicontinuous.

(ii). There do not exist $x,y \in \partial \square$ and $i \neq j$ so that

$$\lim_{n \to \infty} \Psi_{n,i,x} = \lim_{n \to \infty} \Psi_{n,j,y}, \text{ and } \liminf_{n \to \infty} d_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y}) > 0.$$ 

(iii). There do not exist $x,y \in \partial \square$ and $i \neq j$ so that

$$\lim_{n \to \infty} \Psi_{n,i,x} = \lim_{n \to \infty} \Psi_{n,j,y}, \text{ and } \liminf_{n \to \infty} \tilde{d}_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y}) > 0.$$ 

Proof. (i)$\Rightarrow$(ii). Since $d_{G,n}$ are equicontinuous, for any $x_n, y_n \in K_n, n \geq 1$ such that $d(x_n, y_n) \to 0$, we have $d_{G,n}(x_n, y_n) \to 0$.

(ii)$\Rightarrow$(iii). This is trivial since $d_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y}) \geq \tilde{d}_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y})$, $\forall 1 \leq i, j \leq N$ and $x, y \in \square$.

(iii)$\Rightarrow$(ii). Let $x, y \in \partial \square$ and $i \neq j$, and let $\gamma_n$ be a geodesic path connecting $\Psi_{n,i,x}, \Psi_{n,j,y}$ in $\tilde{K}_n$. Then, we can find a path $\gamma'_n$ in $\bigcup_{i=1}^{N} \Psi_{n,i,K} \subset K_n$ connecting $\Psi_{n,i,x}, \Psi_{n,j,y}$ in $K_n$, by replacing line segments going through the interior of $\Psi_{n,i,K}$ with line segments lying along $\Psi_{n,i'}(\partial \square)$. Then $\text{length}(\gamma'_n) \leq \sqrt{2} \text{length}(\gamma_n)$, so $\sqrt{2} d_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y}) \geq d_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y})$.

This observation gives (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i). We assume by contradiction that $d_{G,n}, n \geq 1$ are not equicontinuous. We will show there are $x, y \in \partial \square$ and $i \neq j$, so that $\lim_{n \to \infty} \Psi_{n,i,x} = \lim_{n \to \infty} \Psi_{n,j,y}$, and $\liminf_{n \to \infty} d_{G,n}(\Psi_{n,i,x}, \Psi_{n,j,y}) > 0$, which contradicts (ii).

Since $d_{G,n}, n \geq 1$ are not equicontinuous, we can find $x_l, y_l \in K, n_1 < n_2 < \cdots$, so that $d(x_l, y_l) \to 0$ and $\lim_{l \to \infty} d_{G,n_l}(x_l, y_l) > \eta$ for some $\eta > 0$. By passing to a subsequence, by compactness, we can in addition assume that $x_l \to z$ for some $z \in K$, so $y_l \to z$ as well. Next, noticing that by (7), $x_l, y_l$ belong to different $m_0$-cells in $K_n$, for some $m_0$ depending only on $\eta$. So by passing to a further subsequence, we can in addition assume that $x_l \in \Psi_{ni,w_0}K_n$ and $y_l \in \Psi_{n_i,v}K_n$ for some $w \neq w' \in W_{m_0}$ and for any $l \geq 1$. Then, we choose $m < m_0, v \in W_m$ so that $w_1w_2 \cdots w_m = w_1'w_2' \cdots w_m' = v$ and $w_{m+1} \neq w'_{m+1}$. Let $i = w_{m+1}, j = w'_{m+1}$ and $x = \Psi^{-1}_i \Psi^{-1}_v z, y = \Psi^{-1}_j \Psi^{-1}_v z$.

Clearly, $z$ is in the intersection of two $(m+1)$-cells, so $x, y \in \partial \square$. In addition, $\Psi_{n_i,x} \to \Psi^{-1}_v z, \Psi_{n_j,y} \to \Psi^{-1}_v z$ so $\lim_{n \to \infty} \Psi_{n_i,x} = \lim_{n \to \infty} \Psi_{n_j,y}$. Moreover, let $\tilde{x}_l = \Psi_{n_l,i}^{-1} \Psi_{n_l,i,x_l}$ and $\tilde{y}_l = \Psi_{n_l,j}^{-1} \Psi_{n_l,j,y_l}$. Then $\tilde{x}_l \to x$, and it’s easy to check that $d_{G,n_l}(\tilde{x}_l, x) \to 0$ by taking the advantage that $x \in \partial \square$. Similarly, $\tilde{y}_l \to y$ and $d_{G,n_l}(\tilde{y}_l, y) \to 0$. Noticing that $d_{G,n_l}(\Psi_{n_l,i}\tilde{x}_l, \Psi_{n_l,j}\tilde{y}_l) \geq k^m d_{G,n_l}(x_l, y_l) \geq k^m \eta$, we have limit $l \to \infty$ $d_{G,n_l}(\Psi_{n_i,x_l}, \Psi_{n_j,y_l}) \geq k^m \eta$. Thus we have proved (ii)$\Rightarrow$(i). \qed

Example 9.6. Let $k = 7$ and $N = 4(k-1) + 8 = 32$. For $1 \leq i \leq 24$, $\Psi_i$ is defined as usual. Define $\Psi_{25}(x) = \frac{1}{2} x + (z + \frac{1}{2}, \frac{1}{2})$, with $0 \leq z \leq \frac{1}{11}$, to satisfy the symmetry conditions, the contraction maps $\Psi_i$ for $25 \leq i \leq 32$ are uniquely determined. We write $K(z)$ for the unique compact $K$ such that $K = \bigcup_{i=1}^{32} \Psi_i K$, see Figure 7. We can see there are two cases.

Case 1. $z \neq 0, z \neq \frac{1}{11}$. In this case, if $z_n \to z$, then the geodesic metric $d_{G,n}$ on $K(z_n), n \geq 1$ are equicontinuous.
Case 2. $z = 0$ or $z = \frac{1}{14}$. In this case, if $z_n \to z$, then the geodesic metric $d_{G,n}$ on $K(z_n)$, $n \geq 1$ are not equicontinuous (unless $z_n = z$ for all $n \geq n_0$ for some $n_0$).

Figure 7. The USC $K(z)$.

9.2. The limit form. As before, we let $K_n, n \geq 1$ and $K$ be USC’s with the same $k, N$. We view $K_n, K$ as subsets of the compact metric space $(\square, d)$ from now on, and we will apply the results in Appendix B and C in this subsection. In particular, all the proofs will heavily use Appendix B, so readers are recommended to read Appendix B briefly before reading this part.

One important tool we will use is a modified version of $\Gamma$-convergence: we say $(\mathcal{E}_n, \mathcal{F}_n)$ $\Gamma$-converges to $(\mathcal{E}, \mathcal{F})$ on $C(\square)$ if the associated quadratic forms $\mathcal{E}_n$ $\Gamma$-converges to $\mathcal{E}$ (with extended real values), that is, if and only if (a),(b) hold:

(a). If $f_n \rightharpoonup f$, where $f_n \in C(K_n), n \geq 1$ and $f \in C(K)$, then
$$\mathcal{E}(f) \leq \liminf_{n \to \infty} \mathcal{E}_n(f_n).$$

(b). For each $f \in C(K)$, there exists a sequence $f_n \in C(K_n), n \geq 1$ such that $f_n \rightharpoonup f$ and
$$\mathcal{E}(f) = \lim_{n \to \infty} \mathcal{E}_n(f_n).$$

This is equivalent to Definition B.5 noticing that by Lemma B.2 (c): (c1)$\Leftrightarrow$(c3), $f_n \rightharpoonup f$ if and only if there are $g_n \in C(\square), n \geq 1$ and $g \in C(\square)$ so that $g_n \Rightarrow g$, $f_n = g_n|_{K_n}, n \geq 1$ and $f = g|_K$. Here “$\Rightarrow$” means uniform convergence.

We begin with an observation by Theorem 7.2, Proposition 9.4, Theorem B.4, Theorem B.6 and Corollary C.7.

Lemma 9.7. Assume the geodesic metrics $d_{G,n}$ on $K_n$ are equicontinuous, then there is a subsequence $n_l$ and a resistance metric $R_\infty$ on $K$ so that $R_{n_l} \rightharpoonup R_\infty$. In particular, $(\mathcal{E}_{n_l}, \mathcal{F}_{n_l})$ $\Gamma$-converges on $C(\square)$ to $(\mathcal{E}_\infty, \mathcal{F}_\infty)$, where $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ is the form associated with $R_\infty$.

In addition, $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ is strongly local and regular. Also, there is $\frac{1}{k} < r_\infty \leq \frac{N}{k^2}$ so that $r_{n_l} \to r_\infty$, where $r_n$ is the renormalization factor of $(\mathcal{E}_n, \mathcal{F}_n)$. 

Proof. For any \( n \geq 1 \) and \( x, y \in K_n \), by using Theorem 7.2 and noticing that \( d(x, y) < d_{G,n}(x, y) \), we have
\[
C_1 \cdot d^{\theta^*}(x, y) \leq R_n(x, y) \leq C_2 \cdot d^{\theta^*}(x, y)
\]
with \( \theta^* = \sup_{n \geq 1} \frac{-\log r_n}{\log k} \geq 2 - \frac{\log N}{\log k} \) and \( \theta^* = \inf_{n \geq 1} \frac{-\log r_n}{\log k} < 1 \) for some constants \( C_1, C_2 > 0 \). Then, by the assumption \( d_{G,n}, n \geq 1 \) are equicontinuous, all the assumptions (uniform lower and upper bound estimate of \( R_n, n \geq 1 \)) of Theorem B.4 are satisfied, so one can find a subsequence that \( R_{n_i} \to R_{\infty} \), where \( R_{\infty} \) is a resistance metric on \( K \).

Then, using Theorem B.6, we see \((E_{n_i}, F_{n_i})\) \(\Gamma\)-converges on \( C(\square) \) to \((E_{\infty}, F_{\infty})\), the form \( K \) associated with \( R_{\infty} \). By Proposition 9.4 and Corollary C.7, we know that \((E_{\infty}, F_{\infty})\) is local (hence strongly local since it’s a resistance form on compact space).

Finally, by passing to a further subsequence if necessary, one has \( r_{n_i} \to r_{\infty} \) for some \( r_{\infty} \). Noticing that we always have \( \frac{1}{k} < r_{n_i} \leq \frac{N}{k} \) (see the remark below Theorem 6.15), \( r_{\infty} \) has the same bounds.

Remark. To prove Theorem 9.2, we need to verify \( R_{\infty} = R \). For convenience, in most lemmas in the rest of this section, let’s assume \( R_n \to R_{\infty}, r_n \to r_{\infty} \), and write \((E_{\infty}, F_{\infty})\) for the associated resistance form on \( K \). At present it is not know if \( R_{\infty} = R \), and we aim to prove this.

We will apply the uniqueness theorem, Theorem 8.1 but the main difficulty is to verify that \((E_{\infty}, F_{\infty})\) is a self-similar form. Nevertheless, it is not hard to see a partial result.

Lemma 9.8. Let \( K_n, n \geq 1 \) and \( K \) be \( \text{USC} \)'s with the same \( k, N \). In addition, assume \( R_n \to R_{\infty} \) for some resistance metric \( R_{\infty} \) on \( K \). Let \( f \in C(K) \) and assume \( f|_{\partial_1 K} = 0 \), then \( E_{\infty}(f) = \sum_{i=1}^{N} r_{\infty}^{-1} E_{\infty}(f \circ \Psi_i) \). In particular, \( f \in F_{\infty} \) if and only if \( f \circ \Psi_i \in F_{\infty}, \forall 1 \leq i \leq N \).

Proof. By Theorem B.6, \( E_n \Gamma\)-converges in \( C(\square) \) to \( E_{\infty} \). So, there is a sequence \( f_n \in C(K_n), n \geq 1 \) such that \( f_n \to f \) and \( E_{\infty}(f) = \lim_{n \to \infty} E_n(f_n) \). For each \( 1 \leq i \leq N \), \( f_n \circ \Psi_{n,i} \to f \circ \Psi_i \), since \( x_n \to x \) implies \( \Psi_{n,i} x_n \to \Psi_i x \) by Lemma 9.3 (b), so
\[
E_{\infty}(f) = \lim_{n \to \infty} E_{\infty}(f_n) = \lim_{n \to \infty} \sum_{i=1}^{N} r_{\infty}^{-1} E_n(f_n \circ \Psi_{n,i}) \geq \sum_{i=1}^{N} r_{\infty}^{-1} E_{\infty}(f \circ \Psi_i).
\]

To see the other direction, we use the condition \( f|_{\partial_1 K} = 0 \). For each \( 1 \leq i \leq N \), we pick a sequence \( g_{n,i} \in C(K_n) \), such that \( g_{n,i} \to f \circ \Psi_i \) and \( E_{\infty}(f \circ \Psi_i) = \lim_{n \to \infty} E_n(g_{n,i}) \).

In addition, we can assume \( g_{n,i}|_{\partial_1 K} = 0 \), since otherwise, for each \( 1 \leq i \leq N \), by Lemma B.2 (c): (c1) \( \Rightarrow \) (c3) and by the Markov property, we can find a sequence of positive numbers \( \varepsilon_{n,i}, n \geq 1 \) that converges to 0, such that \( g_{n,i} - (g_{n,i} \vee (-\varepsilon_{n,i})) \wedge \varepsilon_{n,i} \) satisfies the desired requirement. Then, for each \( n \), we glue \( g_{n,i} \) together: define \( g_n \in C(K_n) \) by \( g_n \circ \Psi_{n,i} = g_{n,i} \), \( 1 \leq i \leq N \). Then,
\[
\sum_{i=1}^{N} r_{\infty}^{-1} E_{\infty}(f \circ \Psi_i) = \lim_{n \to \infty} \sum_{i=1}^{N} r_{\infty}^{-1} E_n(g_{n,i}) = \lim_{n \to \infty} E_n(g_n) \geq E_{\infty}(f).
\]

□
Since the form \((\mathcal{E}_\infty, \mathcal{F}_\infty)\) is strongly local and regular, we can apply the tool of energy measure. For each function \(f \in \mathcal{F}_\infty\), noticing that \(f \in C(K)\) in our setting, we can define the energy measure \(\nu_f\) associated with \(f\) to be the unique Radon measure on \(K\) such that

\[
\int_K g(x)\nu_f(dx) = 2\mathcal{E}_\infty(fg, f) - \mathcal{E}_\infty(f^2, g), \quad \forall g \in \mathcal{F}_\infty.
\]

In particular, it is well known that \(\nu_f(K) = 2\mathcal{E}_\infty(f)\) (see Lemma 3.2.3 of \([24]\)). A useful fact is that the measure of \(\nu_f\) on a Borel subset \(A \subset K\) only depends on the value of \(f\) on \(A\).

**Lemma 9.9.** Let \(f, g \in \mathcal{F}_\infty\) and assume \(f = g\) on a Borel subset \(A \subset K\), then \(\nu_f(A) = \nu_g(A)\).

**Proof.** We have the triangle inequality for the energy measures \(\sqrt{\nu_f(A)} \leq \sqrt{\nu_g(A)} + \sqrt{\nu_{f-g}(A)}\) (see \([24]\), page 123 for a proof). Also, \(\nu_h\{\{x \in K : h(x) = 0\}\} = 0\) for any \(h \in \mathcal{F}_\infty\) (see for example Lemma 2.7 of \([8]\)). These two observations give that \(\nu_f(A) \leq \nu_g(A)\). In a same way, \(\nu_g(A) \leq \nu_f(A)\). The lemma follows. \(\square\)

We can make Lemma 9.8 a little stronger with the language of the energy measure.

**Lemma 9.10.** Let \(f \in \mathcal{F}_\infty\), then for any Borel set \(A \subset K \setminus \partial_1 K\), we have

\[
\nu_f(A) = \sum_{i=1}^{N} r_i^{-1} \nu_{f \circ \Psi_i}(\Psi_i^{-1}(A)).
\]

**Proof.** First, we assume \(f|_{\partial_1 K} = 0\). Then, for any \(g \in \mathcal{F}_\infty\) such that \(g|_{\partial_1 K} = 0\), we can see from Lemma 9.8 that

\[
\int_K g(x)\nu_f(dx) = 2\mathcal{E}_\infty(fg, f) - \mathcal{E}_\infty(f^2, g)
\]

\[= \sum_{i=1}^{N} r_i^{-1} \left(2\mathcal{E}_\infty((f \circ \Psi_i \cdot (g \circ \Psi_i), f \circ \Psi_i) - \mathcal{E}_\infty(f^2 \circ \Psi_i, g \circ \Psi_i)\right)\]  \hspace{1cm} (9.1)

\[= \sum_{i=1}^{N} r_i^{-1} \int_K g \circ \Psi_i(x) \nu_{f \circ \Psi_i}(dx).
\]

We can check that \(\{g \in \mathcal{F}_\infty : g|_{\partial_1 K} = 0\}\) is dense in \(\{g \in C(K) : g|_{\partial_1 K} = 0\}\) if \(g \in C(K)\) and \(g|_{\partial_1 K} = 0\), then for any \(\varepsilon > 0\), there is \(g' \in \mathcal{F}_\infty\) such that \(\|g' - g\|_{C(K)} \leq \varepsilon\) since \((\mathcal{E}_\infty, \mathcal{F}_\infty)\) is regular, hence by letting \(g'' = g' - (g' \vee (-\varepsilon)) \wedge \varepsilon\), we have \(g'' \in \mathcal{F}_\infty, g''|_{\partial_1 K} = 0\), and \(\|g'' - g\|_{C(K)} \leq 2\varepsilon\). So \(9.1\) holds for any \(g \in C(K)\) satisfying \(g|_{\partial_1 K} = 0\). Hence, the lemma holds for any \(f \in \mathcal{F}_\infty\) with \(f|_{\partial_1 K} = 0\).

For general \(f \in \mathcal{F}_\infty\), we consider compact \(A \subset K \setminus \partial_1 K\). Since \((\mathcal{E}_\infty, \mathcal{F}_\infty)\) is regular, we can find \(\psi \in \mathcal{F}_\infty\) so that \(\psi|_A = 1\) and \(\psi|_{\partial_1 K} = 0\). So by Lemma 9.9

\[
\nu_f(A) = \nu_f\psi(A) = \sum_{i=1}^{N} r_i^{-1} \nu_{(f \psi) \circ \Psi_i}(\Psi_i^{-1}(A)) = \sum_{i=1}^{N} r_i^{-1} \nu_{f \circ \Psi_i}(\Psi_i^{-1}(A)).
\]

Since \(\nu_f\) is a Radon measure, the lemma follows immediately. \(\square\)
9.3. Energy on the boundary. We now show that \( \nu_f(\partial_1 K) = 0 \) for any \( f \in \mathcal{F}_\infty \), which together with Lemma 9.10, will yield the self-similar property of \( \mathcal{E}_\infty \). At the same time, we will also see that \( \mathcal{F}_\infty \) satisfies the self-similar property by proving a trace theorem of \( \mathcal{F}_\infty \). We will show the following trace theorem by using \( \Gamma \)-convergence on \( C(\mathbb{D}) \).

Proposition 9.11. For \( m \geq 0 \), let
$$
\mathcal{H}'_m = \{ h \in \mathcal{F}_\infty : \mathcal{E}_\infty(h) = \inf \{ \mathcal{E}_\infty(f) : f|_{\partial_mK} = h|_{\partial_mK}, f \in \mathcal{F}_\infty \} \},
$$
which is the space of functions that are harmonic in \( K \setminus \partial_mK \) with respect to \( (\mathcal{E}_\infty, \mathcal{F}_\infty) \).

Then, we have \( \mathcal{H}'_m|_{\partial_mK} = \Lambda_{2,2}^{\sigma(r_m)}(\partial_mK) \), where \( \sigma(r_m) = \frac{\log r_m}{2\log k} + \frac{1}{2} \) (recall Definition 6.2). In addition, there are \( C_1, C_2 > 0 \) depending only on \( k \) such that
$$
C_1 \cdot [h|_{\partial_mK}]^2_{\Lambda_{2,2}^{\sigma(r_m)}(\partial_mK)} \leq \mathcal{E}_\infty(h) \leq C_2 \cdot [h|_{\partial_mK}]^2_{\Lambda_{2,2}^{\sigma(r_m)}(\partial_mK)} , \quad \forall h \in \mathcal{H}'_m.
$$

We first collect some useful facts about the Besov type spaces. For convenience, we look at \( \Lambda_{2,2}^\sigma([0,1]) \). The same results for \( \Lambda_{2,2}^\sigma(\partial_mK) \) will follow directly.

Lemma 9.12. Let \( H^1[0,1] := \{ u \text{ is absolutely continuous on } [0,1] : \int_0^1 \left( \frac{d}{ds} u(s) \right)^2 ds < \infty \} \), and we let \([u]_{H^1[0,1]} = \sqrt{\int_0^1 \left( \frac{d}{ds} u(s) \right)^2 ds}\).

(a). For \( u \in H^1[0,1] \), \( \sigma \in (\frac{1}{2}, 1) \), we have \([u]_{\Lambda_{2,2}^\sigma[0,1]} \leq \sqrt{\frac{1}{1-k^{2\sigma-2}}} \cdot [u]_{H^1[0,1]} \).

(b). For \( u_n \in H^1[0,1], n \geq 1 \), and \( u \in H^1[0,1] \), assume \( u_n \rightharpoonup u \) in \( H^1[0,1] \), i.e. \( u_n \Rightarrow u \) and \([u_n-u]_{H^1[0,1]} \to 0 \) as \( n \to \infty \). Then, \( \lim_{n \to \infty} [u_n]_{\Lambda_{2,2}^\sigma[0,1]} = [u]_{\Lambda_{2,2}^\sigma[0,1]} \) for \( \sigma \to \sigma \in (\frac{1}{2}, 1) \).

Proof. (a). For \( 0 \leq s_1 < s_2 \leq 1 \), by Hölder inequality, we have \( |u(s_2) - u(s_1)| \leq (s_2 - s_1)^{1/2} \int_{s_1}^{s_2} \left( \frac{d}{ds} u(s) \right)^2 ds \)^{1/2}, so
$$
[u]_{\Lambda_{2,2}^\sigma[0,1]}^2 = \sum_{m=0}^{\infty} k^{(2\sigma-1)m} \sum_{l=0}^{k^m-1} \left( u\left( \frac{l}{k^m} \right) - u\left( \frac{l+1}{k^m} \right) \right)^2 \
\leq \sum_{m=0}^{\infty} k^{(2\sigma-2)m} \cdot \int_0^1 \left( \frac{d}{ds} u(s) \right)^2 ds = (1 - k^{2\sigma-2})^{-1} \cdot [u]_{H^1[0,1]}^2.
$$

(b). Fix \( M \geq 1 \) and let \( \sigma_M = \inf_{n \geq M} \sigma_n \) and \( \overline{\sigma}_M = \sup_{n \geq M} \sigma_n \). By (a), we have \([u_n-u]_{\Lambda_{2,2}^{\sigma_M}[0,1]} \to 0 \) as \( n \to \infty \). Noticing that \([v]_{\Lambda_{2,2}^\sigma[0,1]} \) is increasing in \( \sigma \), this gives that
$$
\liminf_{n \to \infty} [u_n]_{\Lambda_{2,2}^{\sigma_{n}}[0,1]} \geq \liminf_{n \to \infty} [u_n]_{\Lambda_{2,2}^{\sigma_{M}}[0,1]} \geq [u]_{\Lambda_{2,2}^{\sigma_{M}}[0,1]}.
$$

Similarly, we have
$$
\limsup_{n \to \infty} [u_n]_{\Lambda_{2,2}^{\sigma_{n}}[0,1]} \leq \limsup_{n \to \infty} [u_n]_{\Lambda_{2,2}^{\sigma_{M}}[0,1]} \leq [u]_{\Lambda_{2,2}^{\sigma_{M}}[0,1]}.
$$
The lemma follows by letting \( M \to \infty \). \( \square \)

Lemma 9.13. Let \( \sigma \in (\frac{1}{2}, 1) \) and \( u \in \Lambda_{2,2}^\sigma[0,1] \). Let \( V \) be a finite subset of \([0,1]\) satisfying the following properties for some \( m \geq 0 \):

1. \( \{ \frac{l}{k^m} \}_{l=0}^{k^m} \subset V. \)
2. \( \#V \cap (\frac{l-1}{k^m}, \frac{l}{k^m}] \leq 1 \) for each \( 1 \leq l \leq k^m \).
Let $v$ be a piecewise linear function on $[0, 1]$ such that $v|_V = u|_V$ and $v$ is linear in $[0, 1]\setminus V$. Then there is a constant $C > 0$ depending only on $\sigma$, such that

$$[[v]]_{A_{2,2}^\sigma[0,1]} \leq C \cdot [[u]]_{A_{2,2}^\sigma[0,1]}.$$  \hspace{1cm} (9.2)

**Proof.** The case $V = \{0, 1\}$ follows easily from Lemma 9.12. Let’s now prove (9.2) for another easy case, where $V = \{0, s\}$ for some $0 < s < 1$ (the $m = 0$ case). First, we need a claim.

**Claim.** $s^{1-2\sigma}(u(s) - u(0))^2 + (1-s)^{1-2\sigma}(u(1) - u(s))^2 \leq C_1 \cdot [[u]]_{A_{2,2}^\sigma[0,1]}^2$ for some $C_1 > 0$.

Call an interval $[\frac{l-1}{k^n}, \frac{l}{k^n})$ with $1 \leq l \leq k^n$, $m \geq 0$ a level-$m$ $k$-adic interval. We split $[0, s)$ into countably many disjoint $k$-adic intervals $\{I_i = [a_i, b_i]\}_{i \geq 1}$ in a canonical way: first choose all level-$1$ disjoint intervals $I_1, I_2, \cdots, I_s$ contained in $[0, s)$, then choose all level-$2$ disjoint intervals contained in $[0, s) \setminus \bigcup_{i=1}^s I_i$, and continue. Then,

$$|u(s) - u(0)|^2 \leq \left(\sum_{i \geq 1} |u(b_i) - u(a_i)|\right)^2 \leq \left(\sum_{i \geq 1} s^{2\sigma-1}(b_i - a_i)/s^{2\sigma-1}\right) \cdot \left(\sum_{i \geq 1} (b_i - a_i)^{1-2\sigma}|u(b_i) - u(a_i)|^2\right).$$

Immediately, noticing that $\sum_{i \geq 1} ((b_i - a_i)/s)^{2\sigma-1} \leq C_2$ for some $C_2 > 0$ depending only on $\sigma$, we get $s^{1-2\sigma}(u(s) - u(0))^2 \leq C_2 \cdot [[u]]_{A_{2,2}^\sigma[0,1]}^2$. Similarly, $(1-s)^{1-2\sigma}(u(1) - u(s))^2$ is controlled by $[[u]]_{A_{2,2}^\sigma[0,1]}^2$ in a same way. The claim holds.

Now we proceed to estimate $[[v]]_{A_{2,2}^\sigma[0,1]} = \sum_{n=0}^{\infty} \sum_{l=1}^{k^n} k^{(2\sigma-1)n} (v(l/k^n) - v(l-1/k^n))^2$. There is nothing to say about the first term as $(v(1) - v(0))^2 = (u(1) - u(0))^2$. So we let $n \geq 1$. We will encounter three cases.

**Case 1.** $s < k^{-n}$. In this case, we have

$$\sum_{l=1}^{k^n} (v(l/k^n) - v(l-1/k^n))^2 = 2(v(s) - v(0))^2 + 2(v(k^{-n}) - v(s))^2 + \sum_{l=2}^{k^n} (v(l/k^n) - v(l-1/k^n))^2 \leq 2(v(s) - v(0))^2 + (k^n + 1)(k^{-n}v(1) - v(s))^2 \leq 2(u(s) - u(0))^2 + 2k^{-n}(1-s)^{-2}(u(1) - u(s))^2.$$  

Noticing that $(1-s)^{-1} \leq 2$, we conclude:

$$k^{(2\sigma-1)n} \sum_{l=1}^{k^n} (v(l/k^n) - v(l-1/k^n))^2 \leq 2(k^n)^{2\sigma-1} s^{1-2\sigma}(u(s) - u(0))^2 + 4(k^n(1-s))^{2\sigma-2}(1-s)^{1-2\sigma}(u(1) - u(s))^2.$$
Case 2. $\frac{i}{k^n} \leq s \leq \frac{i+1}{k^n}$ for some $1 \leq i \leq k^n - 2$. In this case,

$$\sum_{l=1}^{k^n} (v(l/k^n) - v(l-1/k^n))^2 \leq 2(v(s) - v(i/k^n))^2 + 2(v(i+1/k^n) - v(s))^2 + \sum_{l \neq i+1} (v(l/k^n) - v(l-1/k^n))^2$$

$$\leq (i + 2)(k^n v(s) - v(0))^2 + (k^n - i + 1)(k^n v(1) - v(s))^2.$$ 

Noticing that $i + 2 \leq [k^n s] + 2 \leq 3(k^n s)$ and similarly $k^n - i + 1 \leq 3k^n(1-s)$, we get

$$k^{(2\sigma - 1)n} \sum_{l=1}^{k^n} (v(l/k^n) - v(l-1/k^n))^2$$

$$\leq 3(k^n s)^{2\sigma - 2} s^{1-2\sigma} (u(s) - u(0))^2 + 3(k^n(1-s))^{2\sigma - 2} (1-s)^{1-2\sigma} (u(1) - u(s))^2.$$ 

Case 3. $1 - s < k^{-n}$. This case is same as Case 1.

By collecting Case 1-3, one can see

$$k^{(2\sigma - 1)n} \sum_{l=1}^{k^n} (v(l/k^n) - v(l-1/k^n))^2$$

$$\leq 4\phi(k^n s)s^{1-2\sigma} (u(s) - u(0))^2 + 4\phi(k^n(1-s))(1-s)^{1-2\sigma} (u(1) - u(s))^2,$$

where $\phi$ is a function defined by $\phi(t) = t^{2\sigma - 1} \wedge t^{2\sigma - 2}$. By summing the above estimate over $n \geq 0$, and by the Claim, we get $[[v]]_{A_{2,2}[0,1]} \leq C \cdot [[u]]_{A_{2,2}[0,1]}$ for some constant $C > 0$.

Finally, for general $V$ satisfying the requirement of the lemma, we notice that

$$[[u]]^2_{A_{2,2}[0,1]} = \sum_{n=0}^{m-1} \sum_{l=1}^{k^n} k^{(2\sigma - 1)n} (u(l/k^n) - u(l-1/k^n))^2$$

$$+ \sum_{l=1}^{k^n} k^{(2\sigma - 1)m} [[u \circ \psi_l]]^2_{A_{2,2}[0,1]},$$

where $\psi_l$ is the linear map from $[0,1]$ to $[(l-1)/k^n, l/k^n]$. So when we consider the piecewise linear function $v$, by using the preceding discussion, we immediately see that $[[v]]^2_{A_{2,2}[0,1]} = \sum_{n=0}^{m-1} \sum_{l=1}^{k^n} k^{(2\sigma - 1)n} (u(l/k^n) - u(l-1/k^n))^2 + \sum_{l=1}^{k^n} k^{(2\sigma - 1)m} [[u \circ \psi_l]]^2_{A_{2,2}[0,1]} \leq C \cdot [[u]]^2_{A_{2,2}[0,1]}$ as desired.

**Proof of Proposition 9.11.** First, we show the upper bound estimate. Let $V_0 = \{q_1, q_2, q_3, q_4\}$ and $V_{m'} = \bigcup_{w \in W_{m'}} F_w V_0 \cap \partial_m K$ for $m' \geq 0$ (they are same for all USC with same $k, N$). For a function $u \in \Lambda^\sigma_{2,2}(\partial_m K)$ and $m' \geq 0$, $n \geq 1$, we define $u_{n,m'} \in C(\partial_m K_n)$ by

$$u_{n,m'}(x) = \min \{ u(\Psi_w y) : x = \Psi_{n,w} y, y \in V_{m'}, w \in W_m \}, \quad \forall x \in \bigcup_{w \in W_m} \Psi_{n,w} V_{m'},$$

and extending $u_{n,m'}$ linearly elsewhere in $\partial_m K_n$ (noticing that $\partial_m K_n$ is a finite union of intervals); also define $u_{m'} \in C(\partial_m K)$ by $u_{m'}|_{\cup w \in W_m \Psi_w V_{m'}} = u|_{\cup w \in W_m \Psi_w V_{m'}}$ and extending linearly elsewhere in $\partial_m K$. 


By Lemma 9.13, we have \([u_{m'}]_{L^2(\partial K)^m} \leq C_3 \cdot \|[u]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)}, \forall m' \geq 0\), for some \(C_3 > 0\). In addition, for each \(m' \geq 0\), by Lemma 9.12, we have \([u_{m'\infty}]_{L^2(\partial K)^m} \to \|[u_{m'\infty}]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)}\) as \(n \to \infty\). Thus, by choosing \(m'\) growing slowly enough to \(n \to \infty\), \(m' \leq n\), use a same argument as in the proof of Theorem 9.11, we have \(\liminf_{n \to \infty} \|[u_{m'\infty}]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)} \leq C_3 \cdot \|[u]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)}\).

Let \(g_n\) be the harmonic extension of \(u_{m'\infty}\) on \(K_n\), \(n \geq 1\). Clearly \(g_n\) are equicontinuous (since \(R_n\) are equicontinuous by Lemma 3.2 (b), and \(E_n(g_n)\) are uniformly bounded by using Theorem 6.1), so by Lemma 3.2 (a), we can find a limit \(g \in C(K)\) and a subsequence, denoted by \(n_l\), \(l \geq 1\), such that \(g_{n_l} \to g\). By Theorem 6.1, we see that

\[E_\infty(h) \leq E_\infty(g) \leq \liminf_{l \to \infty} E_{n_l}(g_{n_l}) \leq C_4 \liminf_{l \to \infty} \|[u_{n_l,m_l'\infty}]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)} \leq C_3 C_4 \|[u]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)},\]

where \(C_4 > 0\) is the constant \(C_4\) in Theorem 6.1 and \(h \in H'\) such that \(h|_{\partial K} = u\).

Next, we show the lower bound estimate. For \(h \in H'\), by \(R_n \to R_\infty\) thus \(E_n\) \(\Gamma\) converging to \(E_\infty\) (see the remark below Lemma 9.7), we can find a sequence of functions \(g_n \in F\) such that \(g_n \to h\) and \(E_\infty(h) = \lim_{n \to \infty} E_{n}(g_{n})\). Then by Theorem 6.1,

\[E_\infty(h) = \lim_{n \to \infty} E_{n}(g_{n}) \geq C_1 \cdot \liminf_{n \to \infty} \|[g_{n}|_{\partial K}]\|_{\Lambda_2^{\sigma(\infty)}(\partial K)} \geq C_1 \cdot \|[h]|_{\partial K} \|_{\Lambda_2^{\sigma(\infty)}(\partial K)}\]

where \(C_1 > 0\) is the same constant in Theorem 6.1 and in the last inequality we use Fatou’s lemma.

As before, we use \((E,F)\) to denote the unique form satisfying \((C1)\) and \((C2)\) on \(K\). The following is an immediate corollary of Proposition 9.11.

**Corollary 9.14.** \(r = r_\infty, F = F_\infty\) and \(E \propto E_\infty\).

**Proof.** First, assuming \(r = r_\infty\), we show \(F \subset F_\infty\) and \(E_\infty \leq C_1 \cdot E\) for some constant \(C_1 > 0\). For each \(m \geq 0\) and \(f \in F\), we consider the unique \(h_m \in H_m\) (recall \(H_m\) in Proposition 9.11 defined by \((E_\infty,F_\infty)\)) such that \(h_m|_{\partial K} = f|_{\partial K}\). Then, by Theorem 6.1 and Proposition 9.11, we see immediately that \(\sup_{m \geq 0} E_\infty(h_m) \leq C_1 \cdot E(f)\) for some \(C_1 > 0\) independent of \(f\). So \(h_m\) admits a subsequence converging uniformly, and the limit can only be \(f\). Thus \(f\) is the uniform limit of \(h_m\) (since the argument works for any subsequence). So \(f \in F_\infty\) and \(E_\infty(f) \leq \liminf_{m \to \infty} E_\infty(h_m) \leq C_1 \cdot E(f)\), since the quadratic form \(E_\infty\) is lower-semicontinuous.

The proof of \(F_\infty \subset F\) and \(E \leq C_2 \cdot E_\infty\) for some \(C_2 > 0\), under the assumption \(r = r_\infty\), is the same.

Finally, let’s show \(r = r_\infty\) by contradiction. Without loss of generality, we assume \(r < r_\infty\), then the previous argument still works, except that now we have \(E_\infty(h_m) \leq C_1 \cdot (\frac{r}{r_\infty})^m E(f)\), which implies \(E_\infty(f) = 0\). A contradiction.

**Corollary 9.15.** For any \(f \in F_\infty\), we have \(\nu_f(\partial K) = 0\).

**Proof.** By Corollary 9.14 \(f \in F\) as well. Let \(\tilde{\nu}_f\) denote the energy measure of \(f\) associated with \((E,F)\). Then it is direct to show \(\tilde{\nu}_f(\partial K) = 0\). First, by self-similarity, and by using an easier version of the arguments as in Lemma 9.10, one can show \(\tilde{\nu}_f(A) = 0\) when \(A \subset K\) is a compact set. By density of compact sets, we have \(\tilde{\nu}_f(A) = 0\) for any \(A \subset F_\infty\). Thus \(\tilde{\nu}_f = 0\) is a measure on \(\partial F_\infty\).
Theorem 6.9 and Corollary 6.10, one can see that there is \( f' \in \mathcal{F} \) so that \( f'|_{L_1} = f|_{L_1} \) and

\[
\bar{\nu}'(\bigcup_{w \in W_{n+1.1}} \Psi_w K) \leq C_1 \cdot \sum_{m=n}^{\infty} \sum_{l=1}^{k^m} (f(l/k^m,0) - f(l-1/k^m,0))^2, \quad \forall n \geq 0,
\]

for some \( C_1 > 0 \) independent of \( n \), which goes to 0 as \( n \to \infty \). So \( \bar{\nu}'(L_1) = \bar{\nu}'(L_1) = 0 \) by Lemma 9.9 (replace \( \mathcal{F}_\infty \) there with \( \mathcal{F} \)). Finally, again by self-similarity, \( \bar{\nu}_f(\partial_1 K) = \sum_{i=1}^{N} r^{-1} \bar{\nu}_f \Psi_i(\partial K) = 0 \).

To complete the proof, it suffices to apply Corollary 9.14 and use the domination principle of energy measures (see page 389 of [48] for a proof), which says that \( \nu_f(A) \leq C \cdot \bar{\nu}_f(A), \forall A \subset K \) provided \( \mathcal{E}_\infty \leq C \cdot \mathcal{E} \) for some \( C > 0 \).

**Proof of Theorem 9.2.** (i) \( \Rightarrow \) (ii). If \( R_n \to R \), then \( R_n, n \geq 1 \) are equicontinuous by Lemma B.2(b), hence \( d_{G,n}, n \geq 1 \) are equicontinuous by Theorem 7.2.

(ii) \( \Rightarrow \) (i). Let \( (\mathcal{E}_\infty, \mathcal{F}_\infty) \) be the limit form as in Lemma 9.7. Then \( \mathcal{F}_\infty \subset C(K) \), and by Lemma 9.8 Proposition 9.11 we see that \( f \in \mathcal{F}_\infty \) if and only if \( f \circ \Psi_i \in \mathcal{F}_\infty, 1 \leq i \leq N \). It follows from Lemma 9.10 and Corollary 9.15 that \( (\mathcal{E}_\infty, \mathcal{F}_\infty) \) is a self-similar form. One can also check \( \mathcal{R}_{\infty}(L_1, L_2) = 1 \) by a frequently used \( \Gamma \)-convergence argument. So we see that \( (\mathcal{E}_\infty, \mathcal{F}_\infty) \) satisfies (C1), (C2). By the uniqueness theorem, Theorem 8.3, it follows that \( \mathcal{R}_\infty = R \), where \( R \) is the resistance metric associated with the unique form \( (\mathcal{E}, \mathcal{F}) \) on \( K \) satisfying (C1), (C2).

As a consequence of Lemma 9.7, there is a subsequence \( R_{n_l}, l \geq 1 \) such that \( R_{n_l} \to R \). We have the same result holds for any subsequence of \( R_n \), so we claim that \( R_n \to R \). (Otherwise, we can find \( \epsilon > 0 \), a subsequence \( n'_l \) and \( x_l \to x, y_l \to y \) with \( x_l, y_l \in K_{n'_l} \) and \( x, y \in K \) such that \( |R_{n'_l}(x_l, y_l) - R(x, y)| > \epsilon \).)

Finally, by Proposition 9.4 and Theorem C.1 \( \mathbb{P}^{(n)}_x((X^{(n)}_t)_{t \geq 0} \in \cdot) \Rightarrow \mathbb{P}_x((X_t)_{t \geq 0} \in \cdot) \) whenever \( x_n \to x \) with \( x_n \in K_n, n \geq 1, x \in K \).

10. Further Questions

There are a few questions that are closely related to this paper, and careful readers may have noticed them. We list them here, and hope further researches will be done.

**Q1 (Higher dimensional case).** We may defined the \( \text{USC} \) in a higher dimensional setting, by letting the cells of generalized Sierpinski carpets living off the rational grids. In this paper, we take the advantage of the two dimensional setting. In particular, the process always hits each single point in \( \text{USC} \). However, for the higher dimensional case, this is not true if the spectral dimension is greater than or equal to 2 (in other words \( d_W \leq d_H \)). So the non-diagonal assumption will be necessary. Also, new method should be introduced to establish an elliptic Harnack inequality.

**Q2 (The geometry of \( \text{USC} \)).** The \( \text{USC} \) can slide around. As the remark below Theorem 9.2 says, the continuity theorem still holds if we consider instead a continuous family \( \Gamma : [0,1] \to \{ \text{the family of} \ \text{USC with same} \ k, N \} \) such that \( \Gamma(s) \to \Gamma(s_0) \) with \( s \to s_0 \). Say \( \Gamma \) an
admissible sliding path if in addition,

\[ d_{G,n}(x_n, y_n) \to 0, \text{ for any } s_n \to s_0 \in [0, 1], x_n, y_n \in \Gamma(s_n), \text{ and } d(x_n, y_n) \to 0. \]

Then if \( \Gamma \) is an admissible sliding path, the resistance metric \( R(s, x, y) \) is continuous on \( \mathcal{K} := \{(s, x, y) : x, y \in \Gamma(s), s \in [0, 1]\} \), where \( R(s, \cdot, \cdot) \) is the resistance metric on \( \Gamma(s) \) (associated with the unique form satisfying (C1), (C2)), and the Hunt process on \( \Gamma(s) \) also varies continuously in sense of weak convergence. It is of interest to study the characterization of such admissible sliding paths connecting two USC.

**Q3 (Breaking the boundary).** The proof in Section 4 relies heavily on the boundary structure of the fractal. It would be challenging to break the condition posed on the boundary by allowing the contraction ratios of the i.f.s. along boundary to be distinct.

**Appendix A. Proof of Proposition 3.2**

In this appendix, we prove Proposition 3.2. The proof is essentially the same as that in [43] (Theorem 2.1), with slight adjustment due to the change of condition (A3). We reproduce it here for the convenience of readers. In the following context, all the positive constants \( C \) appeared depend only on the USC.

We begin with an observation that all the constants \( \lambda_n, \sigma_n \) are positive and finite, while \( R_n \) are bounded from below by a multiple of \((k^2N^{-1})^n\).

**Lemma A.1** ([43], Proposition 2.7). All the constants \( \lambda_n, \sigma_n \) and \( R_n \) for \( n \geq 1 \) are positive and finite. In addition, there exists \( C > 0 \) so that (3.2) holds:

\[ R_n \geq C \cdot (k^2N^{-1})^n, \quad \forall n \geq 1. \]

**Proof.** It is straightforward to see that \( \lambda_n > 0, \sigma_n > 0 \) for all \( n \geq 1 \) since they are defined as suprema and all the graphs are finite. To see \( \lambda_n \) is always finite, we take any function \( f \in l(W_n) \) with \( D_n(f) = 1 \), then we can see that

\[
\sum_{w \in W_n} (f(w) - [f]_{W_n})^2 = \sum_{w \in W_n} \left( \frac{1}{N^n} \sum_{w' \in W_n} (f(w) - f(w')) \right)^2 \leq \frac{1}{N^n} \sum_{w, w' \in W_n} (f(w) - f(w'))^2 \leq N^{2n} \sum_{w \sim w'} (f(w) - f(w'))^2 = N^{2n},
\]

which gives that \( \lambda_n \) is finite for all \( n \geq 1 \). The finiteness of \( \sigma_n \) follows in a similar way. In fact, for any \( m \geq 1, w \sim_m w' \) in \( W_m \), and function \( f \in l(W_{m+n}) \) with \( D_{m+n}(w, w') = 1 \), we have

\[
([f]_{W_m} - [f]_{W_n} \cdot w) = \frac{1}{N^{2n}} \sum_{u, u' \in W_n} (f(w \cdot u) - f(w' \cdot u'))^2 \leq 2N^n \sum_{w \sim w'} (f(v) - f(v'))^2 = 2N^n,
\]

and thus \( \sigma_n \) is finite for all \( n \geq 1 \). To see the finiteness of \( R_n \) for \( n \geq 1 \), we only need to check \( R_n(A, B) < \infty \) for any \( A, B \subset W_n \) with \( A \cap B = \emptyset \). But this is also straightforward through a standard argument.
Now, we prove (3.2) by closely following the construction of [43]. We remark that (3.2) can also be derived as an immediate consequence of Lemma 4.6.15 of [41]. For $m \geq 1$, $w \in W_m$, we let

$$f'(x) = \max \{0, 1 - \frac{d(x, \Psi_w K)}{c_0 k^{-m}}\} \in C(K),$$

where $c_0$ is the constant in (A3). Then, $f'|_\Psi w K = 1$ and $f'|_\cup_{w' \in \mathcal{N}_w} \Psi_{w'} K = 0$, noticing that $d(\Psi_w K, \Psi_{w'} K) \geq c_0 k^{-m}, \forall w' \in \mathcal{N}_w$. We define $f = P_{|w| + n} f'$, i.e.

$$f(v) = N^{m+n} \int_{\Psi_v K} f'(x) \mu(dx), \quad \forall v \in W_{m+n}.$$

Then, if $v \sim_n v'$, we have

$$|f(v) - f(v')| \leq c_0^{-1} k^m \cdot \sup_{x \in \Psi_v K, y \in \Psi_{v'} K} d(x, y) \leq 2 \sqrt{2} c_0^{-1} k^{-n}.$$

Thus

$$D_{m+n}(f') = \sum_{v \sim_n v', v} (f(v) - f(v'))^2$$

$$\leq \sum_{v \in \mathcal{N}_w \cdot W_m} \sum_{v \sim_n v'} (f(v) - f(v'))^2 \leq 64 N^m \cdot 8 \cdot 8 c_0^{-2} k^{-2n}$$

where in the first inequality we use the fact that $|f(v) - f(v')| > 0$ only if $f(v) > 0$ or $f(v') > 0$, in the second inequality 64 comes as an upper bound of $\#\mathcal{N}_w$ and the middle 8 is an upper bound for the number of neighbours of $v$. Thus,

$$R_n(w \cdot W_n, \mathcal{N}_w \cdot W_n) \geq 2^{-12} c_0^2 (k^2 N^{-1})^n.$$

This gives the desired estimate.

For $l \geq 1$ and $w \in W_n, n \geq 1$, let’s define

$$\mathcal{N}_{i,w} = \{w' \in W_n : \text{there exist } w(i), 0 \leq i \leq l, w(0) = w, w(l) = w', \Psi_{w(i)} K \cap \Psi_{w(i+1)} K \neq \emptyset, \forall 0 \leq i < l\}.$$

In particular, note that $\mathcal{N}_{2,w} = \mathcal{N}_w$ in Definition 3.1(b).

Next, we prove the inequality $\lambda_n N^m R_m \leq C \cdot \lambda_{n+m}$ in Proposition 3.2, which might be affected by the change of (A3).

**Lemma A.2.** Let $m, n \geq 1$, and $\{g_w\}_{w \in W_n}$ be a collection of nonnegative functions in $l(W_{n+m})$ such that $\sum_{w \in W_n} g_w = 1$ and $g_w(v) = 0, \forall w \in W_n, v \in \mathcal{N}_w \cdot W_m$.

Then for each $f \in l(W_n)$, we have

$$D_{n+m}(\tilde{f}) \leq C \cdot \left(\max_{w \in W_n} D_{n+m}(g_w)\right) D_n(f),$$

for some constant $C > 0$, where $\tilde{f} = \sum_{w \in W_n} f(w) g_w \in l(W_{n+m})$.  

Proof. For any \( v, v' \in W_{n+m} \), we write \( S(v, v') = \{ w \in W_n : g_w(v) + g_w(v') > 0 \} \) and \( a(v, v') = [f]_{S(v, v')} \). Note that \( \#S(v, v') \) is uniformly bounded depending only on the USC. Then,

\[
D_{n+m}(\tilde{f}) = \sum_{v, v'} (\tilde{f}(v) - \tilde{f}(v'))^2
\]

\[
= \sum_{v, v'} \left( \sum_{w \in S(v, v')} (f(w) - a(v, v'))(g_w(v) - g_w(v')) \right)^2
\]

\[
\leq C_1 \cdot \sum_{v, v'} \sum_{w \in S(v, v')} \left( f(w) - a(v, v') \right)^2 \left( g_w(v) - g_w(v') \right)^2
\]

\[
= C_1 \cdot \sum_{w \in W_n} \sum_{v, v' : w \in S(v, v')} \left( f(w) - a(v, v') \right)^2 \left( g_w(v) - g_w(v') \right)^2,
\]

where the second equality is due to \( \sum_{w \in W_n} g_w = 1 \), \( C_1 = \max_{v, v'} \#S(v, v') \). To continue the estimate, we notice that

\[
\left( (f(w) - a(v, v'))^2 \right) \leq \max_{w' \in N_5} (f(w) - f(w'))^2 \leq 5 \cdot D_{n, N_5, w}(f),
\]

where 5 appears since each \( w' \) in \( S(v, v') \) locates in \( N_5, w \). Thus, we have

\[
D_{n+m}(\tilde{f}) \leq 5C_1 \cdot \sum_{w \in W_n} D_{n, N_5, w}(f) \sum_{v, v' : w \in S(v, v')} (g_w(v) - g_w(v'))^2
\]

\[
\leq 5C_1 \cdot \sum_{w \in W_n} D_{n, N_5, w}(f) \left( \max_{w \in W_n} D_{n+m}(g_w) \right).
\]

The lemma follows immediately, noticing that \( \sum_{w \in W_n} D_{n, N_5, w}(f) \leq C_2 \cdot D_n(f) \) for some \( C_2 > 0 \).

Lemma A.3. Let \( m, n \geq 1 \). There exists a collection of nonnegative functions \( \{ g_w \}_{w \in W_n} \) in \( l(W_{n+m}) \) such that \( \sum_{w \in W_n} g_w = 1 \), \( g_w(v) = 0, \forall w \in W_n, v \in N^c_w \cdot W_m \), and

\[
D_{n+m}(g_w) \leq C \cdot R_m^{-1}, \quad \forall w \in W_n,
\]

for some constant \( C > 0 \).

Proof. For each \( w \in W_n \), let \( \tilde{g}_w \) be the unique function supported on \( N^c_w \cdot W_m \) such that

\[
D_{n+m}(\tilde{g}_w) = R_{n+m}^{-1}(w \cdot W_m, N^c_w \cdot W_m) \leq R_m^{-1}, \quad \tilde{g}_w |_{W_m} = 1.
\]

Let \( \tilde{g} = \sum_{w \in W_n} \tilde{g}_w \) and \( g_w = \frac{\tilde{g}_w}{\tilde{g}} \), noticing that \( \tilde{g} \geq 1 \). We immediately have \( \sum_{w \in W_n} g_w = 1 \) and \( g_w(v) = 0, \forall w \in W_n, v \in N^c_w \cdot W_m \). It remains to estimate the energy of each \( g_w \).

First, we notice that \( \tilde{g} \geq 1 \), so \( \left| \frac{1}{\tilde{g}(v)} - \frac{1}{\tilde{g}(v')} \right| \leq \left| \tilde{g}(v) - \tilde{g}(v') \right| \). Thus, we have

\[
D_{n+m, N_3, W_m}(1/\tilde{g}) \leq D_{n+m, N_3, W_m}(\tilde{g}) = D_{n+m, N_3, W_m} \left( \sum_{w' : N^c_{w', \cap N_3, w} \neq \emptyset} \tilde{g}_{w'} \right)
\]

\[
\leq C_1 \cdot \max_{w' : N^c_{w', \cap N_3, w} \neq \emptyset} D_{n+m}(\tilde{g}_{w'}) \leq C_1 \cdot R_m^{-1},
\]

where \( C_1 < 1 \). Thus, we have

\[
D_{n+m, N_3, W_m}(\tilde{g}) \leq C_1 \cdot R_m^{-1},
\]

for some constant \( C_1 > 0 \).
for some $C_1 > 0$, where the equality is because $\tilde{g}(v) = \sum_{w \in \mathcal{N}_u \cap \mathcal{N}_m} g_w(v)$, $\forall v \in \mathcal{N}_3 \cup W_m$. As $g_w$ is supported on $\mathcal{N}_u \cdot W_m = \mathcal{N}_3 \cup W_m$, $|g_w(v) - g_w(v')| > 0$ only if $\{v, v'\} \subset \mathcal{N}_3 \cup W_m$. Thus we have

$$\sqrt{D_{n+m}(g_w)} = \sqrt{D_{n+m, \mathcal{N}_{3,m} \cup W_m} (\tilde{g} / \tilde{\gamma})} \leq \sqrt{D_{n+m, \mathcal{N}_{3,m} \cup W_m} (\tilde{g})} + \sqrt{D_{n+m, \mathcal{N}_{3,m} \cup W_m} (1 / \tilde{\gamma})} \leq (C_1^{1/2} + 1) \cdot R_m^{-1/2},$$

where the first inequality is due to $\|\tilde{g}_w\|_{L^\infty(\mathcal{N}_{3,m} \cup W_m)} \leq 1$ and $\|1 / \tilde{\gamma}\|_{L^\infty(\mathcal{N}_{3,m} \cup W_m)} \leq 1$. \hfill \Box

**Proposition A.4.** There is a constant $C > 0$ such that

$$\lambda_n N^m R_m \leq C \cdot \lambda_{n+m+2}, \quad \forall n, m \geq 1.$$

**Proof.** Let $f \in l(W_m)$ so that $D_n(f) = 1$ and $\sum_{w \in W_n} (f(w) - [f]_{W_n})^2 = \lambda_n$. Let $\tilde{f} \in l(W_{n+2})$ be defined as $\tilde{f}(w) = f(w), \forall w \in W_n, \tau \in W_2$. Clearly,

$$D_{n+2}(\tilde{f}) \leq 3k^2 D_n(f) = 3k^2,$$

where $3$ comes from the fact that for any distinct pair $w, w' \in W_n$, each $(n+2)$-subcell in $\Psi_w K$ is neighbouring to at most $3$ $(n+2)$-subcells in $\Psi_w K$. Next, we let $\{g_w\}_{w \in W_{n+2}}$ in $l(W_{n+m+2})$ be defined as in Lemma A.3, and define $\tilde{f} = \sum_{w \in W_{n+2}} \tilde{f}(w) g_w$. By Lemma A.2 and [A.3], we know there is a $C_1 > 0$ such that

$$D_{n+m+2}(\tilde{f}) \leq C_1 \cdot R_m^{-1} D_{n+2}(f) \leq 3k^2 C_1 \cdot R_m^{-1}.$$

On the other hand, let $I = W_2 \setminus (\partial W_2 \cup \{w \in W_2 : w \not\sim \partial W_2\})$. We have $I \neq \emptyset$, and

$$\tilde{f}(w \cdot \tau) = f(w), \quad \forall w \in W_n, \tau \in I \cdot W_m$$

by the definition of $\{g_w\}_{w \in W_{n+2}}$. As a consequence, we have,

$$\sum_{v \in W_{n+m+2}} (\tilde{f}(v) - [\tilde{f}]_{W_{n+m+2}})^2 \geq (\# I) \cdot N^m \sum_{w \in W_n} (f(w) - [f]_{W_n})^2 = (\# I) \cdot \lambda_n N^m.$$

Thus, $\lambda_{n+m+2} \geq \left(\sum_{v \in W_{n+m+2}} (\tilde{f}(v) - [\tilde{f}]_{W_{n+m+2}})^2\right) / D_{n+m+2}(\tilde{f}) \geq C_2 \cdot \lambda_n N^m R_m$ for some $C_2 > 0$. \hfill \Box

**Proof of Proposition 3.2** The estimate (3.2) is proved in Lemma A.1. The left hand side of (3.1), $\lambda_n N^m R_m \leq C \cdot \lambda_{n+m}, \forall n, m \geq 1$, will follow immediately from Proposition A.4 and the right hand side of (3.1), $\lambda_{n+m} \leq C \cdot \lambda_n \sigma_m, \forall n, m \geq 1$. So it remains to show the right hand side of (3.1).

Let $f \in l(W_{n+m})$ so that $D_{n+m}(f) = 1$ and $\sum_{v \in W_{n+m}} (f(v) - [f]_{W_{n+m}})^2 = \lambda_{n+m}$. Let $f' \in l(W_n)$ be defined as $f'(w) = [f]_{W_m}, \forall w \in W_n$. Then, by Lemma 3.6, we know that

$$D_n(f') \leq C_2 \cdot N^{-m} \sigma_m D_{n+m}(f) = C_2 \cdot N^{-m} \sigma_m,$$
for some constant $C_2 > 0$. On the other hand, we have
\[
\lambda_{n+m} = \sum_{v \in W_{n+m}} (f(v) - [f]_{W_{n+m}})^2
\]
\[
= \sum_{w \in W_n} \sum_{v \in W_w} (f(v) - f'(w))^2 + N^m \sum_{w \in W_n} (f'(w) - [f']_{W_n})^2
\]
\[
\leq \lambda_m D_{n+m}(f) + N^m \lambda_n D_n(f') \leq \lambda_m + C_2 \cdot \lambda_n \sigma_m.
\]
In addition, by Proposition A.4 and (3.2), we know that
\[
\lambda_{n+m} \geq 2 \cdot \lambda_m, \forall n \geq n'\text{ for some } n' \geq 1.
\]
Thus, the desired inequality holds for large $n$. Clearly, for large $m \geq n'$, the inequality also holds. This finishes the proof. □

Appendix B. Convergence of resistance metrics

This appendix is based on one of the author's previous work [12]. Readers are also invited to read [17] by Croydon and [18] by Croydon, Hambly and Kumagai to see a proof on the weak convergence of associated diffusion processes. Also, see [44] for a general study on $L^2$ spaces.

We consider another setting of $\Gamma$-convergence in this section, which is related to the convergence of resistance metrics. A resistance metric, which is associated with a resistance form (as shown in Theorem 7.2, any form on $K$ satisfying (C1), (C2) is a resistance form), can be defined as follows. See [39] for details.

**Definition B.1.** (a) On a finite set $V$, a bilinear form $(\mathcal{E}, l(V))$ is called a resistance form on $V$ if it takes the form
\[
\mathcal{E}(f,g) = \frac{1}{2} \sum_{x \neq y} c_{x,y}(f(x) - f(y))(g(x) - g(y)), \quad \forall f, g \in l(V),
\]
where $c_{x,y} = c_{y,x} \in [0, \infty)$ is called the conductance between $x, y$, and in addition $\mathcal{E}(f, f) = 0$ if and only if $f \in \text{Constants}$. We use $\mathcal{RF}(V)$ to denote the set of resistance forms on $V$.

(b) Let $X$ be a set. $R \in l((X \times X))$ is called a resistance metric on $X$ if for any finite subset $V$ of $X$, there exists a resistance form $\mathcal{E}_V$ on $V$ such that
\[
R(x, y) = \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}_V(f)} : f \in l(V) \setminus \text{Constants} \right\}, \quad \forall x, y \in V.
\]
Let $\mathcal{RM}(X)$ denote the collection of all resistance metrics on $X$.

We use the notation $\mathcal{M}(X)$ to denote the collection of metrics on $X$, and it is well known $\mathcal{RM}(X) \subset \mathcal{M}(X)$ [39]. With a little abuse of notations, we write $\mathcal{E}(f) := \mathcal{E}(f, f)$ so $\mathcal{E}$ is a quadratic form on $l(V)$. Readers can find detailed discussions about resistance forms and resistance metrics in the monograph [39]. In particular, it was proved that a resistance metric $R$ on $X$ will induce a resistance form $(\mathcal{E}, \mathcal{F})$ on $X$ once $(X, R)$ is a separable metric space. In this way, the concept of resistance form will extend to infinite sets. We will go back to this later.

In this appendix, we will always assume the basic settings as follows.

**Basic settings.**
(a). Let \((B, d)\) be a compact metric space with a compact subset \(A\), and let \(\{A_n\}_{n \geq 1}\) be a sequence of compact subsets in \(B\) such that
\[
\lim_{n \to \infty} \delta(A_n, A) = 0,
\]
where \(\delta\) is the Hausdorff metric on compact subsets of \(B\).

(b). For any topological space \(X\), \(\|\cdot\|_{C(X)}\) denotes the supremum norm on \(C(X)\).

(c). Let \(f_n \in C(A_n)\) for \(n \geq 1\).

We say \(\{f_n\}_{n \geq 1}\) is uniformly bounded if \(\sup_{n \geq 1} \|f_n\|_{C(A_n)} < \infty\).

We say \(\{f_n\}_{n \geq 1}\) is equicontinuous if \(\lim_{\eta \to 0} \sup_{n \geq 1} \{|f_n(x) - f_n(y)| : x, y \in A_n, d(x, y) < \eta\} = 0\).

(d). Let \(f_n \in C(A_n)\) for \(n \geq 1\) and \(f \in l(A)\). We write \(f_n \to f\) if \(f(x) = \lim_{n \to \infty} f_n(x_n)\) for any \(x \in A\) and \(x_n \in A_n, n \geq 1\) such that \(x_n \to x\) as \(n \to \infty\).

**Remark.** We also need to consider the Cartesian product \(B^2 = \{(x, y) : x, y \in B\}\), equipped with the metric \(d_{B^2}(\{x_1, y_1\}, \{x_2, y_2\}) = d(x_1, x_2) + d(y_1, y_2)\) (or the equivalent metric \(\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}\)). Under the basic settings, we immediately have \(A_n^2 \to A^2\).

One important class of functions on \(A^2\) (or \(A_n^2\)) are metrics. In particular, for each metric \(R\) on \(A\), by the triangle inequality, \(\|R(x_1, y_1) - R(x_2, y_2)\| \leq R(x_1, x_2) + R(y_1, y_2)\), so it suffices to study \(\sup_{x, y \in A, d(x, y) < \eta} R(x, y)\) to understand the modulus of continuity of \(R\) on \(A^2\). A same argument works for each metric \(R_n\) on \(A_n\) for each \(n \geq 1\).

In Section 2.1 of [12], one of the authors show that \(f_n \to f\) is a natural extension of the concept of uniform convergence, and there is a version of Arzelà–Ascoli theorem.

**Lemma B.2.** Let \(f_n \in C(A_n)\) for \(n \geq 1\).

(a). If \(\{f_n\}_{n \geq 1}\) is uniformly bounded and equicontinuous, then there is a subsequence \(\{f_{n_l}\}_{l \geq 1}\) and \(f \in C(A)\) such that \(f_{n_l} \to f\).

(b). If \(f_n \to f\) for some \(f \in C(A)\), then \(\{f_n\}_{n \geq 1}\) is uniformly bounded and equicontinuous.

(c). Let \(f \in C(A)\). The following claims (c1), (c2) and (c3) are equivalent:

(c1). \(f_n \to f\).

(c2). Let \(\bar{A} = \{(0) \times A\} \cup \left(\bigcup_{n \geq 1} \left(\frac{1}{n} \times A_n\right)\right)\), with the induced topology from \([0, 1] \times B\). \(\bar{f} \in C(\bar{A})\), where \(\bar{f} \in l(\bar{A})\) is defined by
\[
\bar{f}(t, x) = \begin{cases} f_n(x), & \text{if } t = \frac{1}{n}, \\ f(x), & \text{if } t = 0. \end{cases}
\]

(c3). There exists \(\{g_n\}_{n \geq 1} \cup \{g\} \subset C(B)\) such that \(g_n|_{A_n} = f_n, \forall n \geq 1\), \(g|_{A} = f\) and \(g_n \to g\).

Here \(\Rightarrow\) denotes uniform convergence.

**Proof.** (a). Let \(\{x_m\}_{m \geq 1}\) be a countable dense subset of \(A\). For each \(m \geq 1\), choose a sequence \(\{x_{m,n}\}_{n \geq 1}\) so that \(x_{m,n} \in A_n, \forall n \geq 1\) and \(d(x_{m,n}, x_m) \to 0\) as \(n \to \infty\). Since \(\{f_n\}_{n \geq 1}\) is uniformly bounded, by a diagonalization argument, there is a subsequence \(\{n_l\}_{l \geq 1}\) so that \(\lim_{l \to \infty} f_{n_l}(x_{m,n_l})\) exists for any \(m \geq 1\), and we write \(f(x_m) = \lim_{l \to \infty} f_{n_l}(x_{m,n_l})\).

For \(m, m' \geq 1\), if \(\eta > 0\) and \(d(x_{m',n}, x_m) < \eta\), then \(d(x_{m,n}, x_{m',n}) < \eta\) for any large \(n\), so
\[
|f(x_{m'}) - f(x_m)| = \lim_{l \to \infty} |f_{n_l}(x_{m',n_l}) - f_{n_l}(x_{m,n_l})| 
\leq \sup_{n \geq 1} \{|f_n(x) - f_n(y)| : x, y \in A_n, d(x, y) < \eta\}.
\]
Hence $f$ is uniformly continuous on $\{x_m\}_{m \geq 1}$, and $f$ extends to a continuous function on $A$.

Finally, we check $f_n \to f$. For any $\varepsilon > 0$ and $y_l \in \mathcal{A}_{n_l}, l \geq 1$ such that $y_l \to y \in A$, by the equicontinuity of $\{f_n\}_{n \geq 1}$, we can choose $x_m$ close enough to $y$ and choose $N$ large enough so that $|f_n(y_l) - f(y)| \leq |f_n(y_l) - f_n(x_{m,n_l})| + |f_n(x_{m,n_l}) - f(x_m)| + |f(x_m) - f(y)| < \varepsilon$ for any $l \geq N$.

(b). is an immediate consequence of (c), so it remains to prove (c).

(c). It is routine to see "(c3)$\Rightarrow$(c1)".

"(c1)$\Rightarrow$(c2)": It suffices to check that $\tilde{f}$ is continuous at $(0, x)$ for each $x \in A$. This can be seen by contradiction. If $\tilde{f}$ is not continuous at $(0, x)$, then we can find $\varepsilon > 0$ and $(t_l, x_l) \to (0, x)$ as $l \to \infty$ such that $|\tilde{f}(t_l, x_l) - \tilde{f}(0, x)| > \varepsilon$. Since $\tilde{f}|_{\{0\} \times A}$ is continuous, we have $t_l \neq 0$ for all large $l$. So by picking a subsequence, we get $n_{t_l} = \frac{1}{t_l^2} \to \infty$ and $x_{t_l} \to x$.

This contradicts (c1).

"(c2)$\Rightarrow$(c3)": $A$ is a closed subset of $[0, 1] \times B$ (by using Basic setting (a), it’s straightforward to check $\forall (t, x) \in ([0, 1] \times B) \setminus \bar{A}$, there is a neighbourhood of $(t, x)$ disjoint with $A$). By the Tietze extension theorem, we have a continuous extension $\tilde{f} \in C([0, 1] \times B)$ of $\tilde{f}$. It suffices to take $g_n(x) = \tilde{f}(\frac{1}{n}, x)$ and $g(x) = \tilde{f}(0, x)$ for any $x \in B, n \geq 1$.

An important consequence of the lemma is that we can create resistance metric at the limit (see [12], Section 2.2). First, we need a lemma to guarantee the limit of resistance metrics is still a resistance metric (if the limit is non-degenerate).

**Lemma B.3.** Let $V$ be a finite set. We assign the topology on $\mathcal{R}_M(V), \mathcal{M}(V)$ and $\mathcal{R}_F(V)$ by naturally embedding them into $\mathbb{R}^{N(N-1)/2}$, where $N = \#V$.

(a). $\mathcal{R}_M(V)$ is a closed subset of $\mathcal{M}(V)$ (both are not completed of course).

(b). The natural correspondence $\mathcal{R}_F(V) \to \mathcal{R}_M(V)$ is a homeomorphism.

**Proof.** We first show the natural correspondence $\mathcal{R}_F(V) \to \mathcal{R}_M(V)$ is continuous. Let $\mathcal{E}_n$ converges to $\mathcal{E}$ in $\mathcal{R}_F(V)$, and let $R_n$ be the resistance metric associated with $\mathcal{E}_n, n \geq 1$ and $R$ be the resistance metric associated with $\mathcal{E}$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

\[
(1 + \varepsilon)^{-1} \mathcal{E}(f) \leq \mathcal{E}_n(f) \leq (1 + \varepsilon) \mathcal{E}(f), \quad \forall f \in l(V).
\]

This is because $\mathcal{E}_n$ converges uniformly to $\mathcal{E}$ on the compact set $M = \{f \in l(V) : \sup_{x,y \in V} |f(x) - f(y)| = 1, \sum_{x \in V} f(x) = 0\}$, and $\inf_{f \in M} \mathcal{E}(f) > 0$. This implies $(1 + \varepsilon)^{-1}R(x, y) \leq R_n(x, y) \leq (1 + \varepsilon)R(x, y)$ for any $x, y \in V$, so $R_n$ converges to $R$ in $\mathcal{R}_M(V)$.

Conversely, let $R_n \in \mathcal{R}_M(V), n \geq 1$ and assume $R_n \to R$ for some $R \in \mathcal{M}(V)$. Let $\mathcal{E}_n \in \mathcal{R}_F(V)$ be the resistance forms associated with $R_n$ for $n \geq 1$. We need to show that $R \in \mathcal{R}_M(V)$ and $\mathcal{E}_n \to \mathcal{E}$ in $\mathcal{R}_F(V)$, where $\mathcal{E}$ is the resistance form associated with $R$. For this purpose, for $n \geq 1$, we write $c_{n,x,y}$ for the conductances of $\mathcal{E}_n$, i.e.

\[
\mathcal{E}_n(f) = \frac{1}{2} \sum_{x \neq y} c_{n,x,y}(f(x) - f(y))^2, \quad \forall f \in l(V).
\]

By choosing a subsequence, we have the limit $c_{x,y} = \lim_{n \to \infty} c_{n,x,y}$ exists for any $x \neq y \in V$, and we define $\mathcal{E}$ by

\[
\mathcal{E}(f) = \frac{1}{2} \sum_{x \neq y} c_{x,y}(f(x) - f(y))^2, \quad \forall f \in l(V).
\]
Then, $\mathcal{E} \in \mathcal{RF}(V)$ because for any $f \in l(V) \setminus \text{constants}$, we have
\[
\mathcal{E}(f) = \lim_{l \to \infty} \mathcal{E}_n(f) \geq \lim_{l \to \infty} \max_{x,y \in V} \frac{(f(x) - f(y))^2}{R_{n_l}(x,y)} = \max_{x,y \in V} \frac{(f(x) - f(y))^2}{R(x,y)} > 0.
\]
By the first part, we know that $R_{n_l}$ converges to some resistance metric on $V$, but we also know that $R_{n_l} \to R$ in $\mathcal{M}(V)$, so $R \in \mathcal{RM}(V)$. This implies (a). To see (b), it suffices to notice that the above argument shows that any subsequence of $\mathcal{E}_n$ has a further subsequence converges to $\mathcal{E}$, so $\mathcal{E}_n \to \mathcal{E}$ in $\mathcal{RF}(V)$.

**Theorem B.4.** Let $R_n \in \mathcal{RM}(A_n)$ for $n \geq 1$, and assume
\[
\psi_1(d(x,y)) \leq R_n(x,y) \leq \psi_2(d(x,y)), \quad \forall n \geq 1, \forall x, y \in A_n,
\]
for some $\psi_1, \psi_2 \in C[0, \infty)$ with $\psi_1(0) = \psi_2(0) = 0$ and $\psi_2(t) \geq \psi_1(t) > 0$ for $t > 0$.

Then, there exists $R \in \mathcal{RM}(A)$ and a subsequence $\{n_l\}_{l \geq 1}$ such that $R_{n_l} \to R$. In addition,
\[
\psi_1(d(x,y)) \leq R(x,y) \leq \psi_2(d(x,y)), \quad \forall x, y \in A.
\]

**Proof.** By Lemma B.2(a), there exists a subsequence $\{n_l\}_{l \geq 1}$ and $R \in C(A^2)$ so that $R_{n_l} \to R$, where $A^2 = \{(x,y) : x \in A, y \in A\}$. For any $(x,y) \in A^2$, we choose $(x_n, y_n) \to (x,y)$ with $(x_n, y_n) \in A^2$, then
\[
R(x,y) = \lim_{l \to \infty} R_{n_l}(x_n, y_n) \geq \lim_{l \to \infty} \psi_1(d(x_n, y_n)) = \psi_1(d(x,y)).
\]
So $R \in \mathcal{M}(A)$. The upper bound $R(x,y) \leq \psi_2(d(x,y))$ follows from a similar argument.

It remains to show that $R \in \mathcal{RM}(A)$. Without loss of generality, we assume that $R_n \to R$. Let $V = \{x_m\}_{m=1}^M$ be a finite subset of $A$. For each $1 \leq m \leq M$, we can choose a sequence $\{x_{m,n}\}_{n \geq 1}$ with $x_{m,n} \in A_n, \forall n \geq 1$, so that $x_{m,n} \to x_m$ as $n \to \infty$. We define $R^{(n)}(x,y) = R_n(x_{m,n}, x_{m,n})$, $\forall 1 \leq m, m' \leq M$. Clearly, for $n$ large enough (so that $x_{m,n} \neq x_{m',n}, \forall m \neq m'$), $R^{(n)} \in \mathcal{RM}(V)$. Then $R^{(n)} \to R$ in $\mathcal{M}(V)$, and so $R \in \mathcal{RM}(A)$ by Lemma B.3(a) and the arbitrariness of $V$.

It is well known (see the monograph [39] and the paper [40] by Kigami) that if $R$ is a resistance metric on $X$ so that $(X,R)$ is a separable metric space, there is a unique bilinear form, named *resistance form* $(\mathcal{E},\mathcal{F})$ on $(X,R)$ such that $\mathcal{F} \subset C(X,R)$, and
\[
R(x,y)^{-1} = \inf \{ \mathcal{E}(f,f) : f(x) = 0, f(y) = 1, f \in \mathcal{F} \}, \quad \forall x, y \in X.
\]
The form is called regular if $\mathcal{F}$ is dense in $C(X,R)$. If $(X,R)$ is compact, then $(\mathcal{E},\mathcal{F})$ is always regular (see Corollary 6.4 of [40]). In addition, if $\mu$ is a Radon measure on $X$ with full support, then $(\mathcal{E},\mathcal{F})$ becomes a regular Dirichlet form on $L^2(X,\mu)$ (see Theorem 9.4 of [40]).

As usual, we still use $\mathcal{E}$ to denote the associated quadratic form with extend real values.

**Definition B.5.** Let $\mathcal{E}_n$ be quadratic forms defined on $C(A_n)$, and $\mathcal{E}$ be a quadratic form defined on $C(A)$. We say $\mathcal{E}_n \Gamma$-converges to $\mathcal{E}$ on $C(B)$ if and only if (a),(b) hold:
(a). If $f_n \Rightarrow f$, where $\{f_n\}_{n \geq 1} \cup \{f\} \subset C(B)$, then
\[
\mathcal{E}(f|A) \leq \liminf_{n \to \infty} \mathcal{E}_n(f_n|A_n).
\]
For each \( f \in C(B) \), there exists a sequence \( \{f_n\}_{n \geq 1} \subset C(B) \) such that \( f_n \Rightarrow f \) and
\[
\mathcal{E}(f|A) = \lim_{n \to \infty} \mathcal{E}_n(f_n|A_n).
\]

**Remark 1.** In the definition above, for a quadratic form \( \mathcal{E} \) on \( C(A) \), we always associate it with a domain \( \mathcal{F} := \{ f \in C(A) : \mathcal{E}(f) < \infty \} \), in other words, \( \mathcal{E}(f) = \infty \) if \( f \notin \mathcal{F} \). We do similarly for \( \mathcal{E}_n \). In the following, we will also say a sequence of resistance forms \((\mathcal{E}_n, \mathcal{F}_n)\) on \( A_n \) \( \Gamma \)-converge to a resistance form \((\mathcal{E}, \mathcal{F})\) on \( A \), which exactly means the \( \Gamma \)-convergence on \( C(B) \) for their associated quadratic forms (with extended real values).

**Remark 2.** \( f_n \Rightarrow f \) can be replaced with \( f_n|A_n \to f|A \) by Lemma B.2(c).

The main result in this appendix is the following theorem stating that the convergence of resistance metrics will result in the \( \Gamma \)-convergence of the resistance forms. We copy the proof from [12] for convenience of readers.

**Theorem B.6.** Let \( R_n \in \mathcal{RM}(A_n) \cap C(A_n) \) for \( n \geq 1 \), \( R \in \mathcal{RM}(A) \cap C(A^2) \), and assume \( R_n \Rightarrow R \). Let \((\mathcal{E}_n, \mathcal{F}_n)\) be the resistance form associated with \( R_n \) for \( n \geq 1 \), and let \((\mathcal{E}, \mathcal{F})\) be the resistance form associated with \( R \). Then, we have \((\mathcal{E}_n, \mathcal{F}_n)\) \( \Gamma \)-converges to \((\mathcal{E}, \mathcal{F})\) on \( C(B) \).

**Proof.** We need to prove (a), (b) in Definition B.5 (a). By the remark above, we let \( f_n \in C(A_n), f \in C(A) \) and \( f_n \Rightarrow f \) in the proof. Notice that \( R \) induces the same topology as \( d \) on \( A \), so \((A, R)\) is separable. Let \( V_1 \subset V_2 \subset \cdots \) be a nested sequence of finite subsets of \( A \), and assume that \( V = \bigcup_{m \geq 1} V_m \) is dense in \( A \). Then, by Theorem 2.3.7 in [39], we have
\[
\mathcal{E}(f) = \sup_{m \geq 1} [\mathcal{E}]_{V_m}(f|V_m) = \lim_{m \to \infty} [\mathcal{E}]_{V_m}(f|V_m), \quad \forall f \in C(A),
\]
where \([\mathcal{E}]_{V_m}\) denotes the trace of \( \mathcal{E} \) on \( V_m \). For each \( n \geq 1 \), we can find a nested sequence of finite set \( V_{1,n} \subset V_{2,n} \subset \cdots \) such that \( \delta(V_{m,n}, V_m) \to 0 \) as \( n \to \infty \). In addition, for each \( m \geq 1 \), we require that \#\( V_{m,n} = \#V_m \) for \( n \) large enough (to achieve this, we can choose a converging sequence \( x_n \to x \) with \( x_n \in A_n \) for each \( x \in V \)). Then, by Lemma B.3(b),
\[
[\mathcal{E}]_{V_{m,n}}(f|V_{m,n}) = \lim_{n \to \infty} [\mathcal{E}]_{V_{m,n}}(f_n|V_{m,n}). \tag{B.1}
\]
This implies
\[
\mathcal{E}(f) = \lim_{m \to \infty} [\mathcal{E}]_{V_m}(f|V_m) \leq \liminf_{m \to \infty} \sup_{m \geq 1} [\mathcal{E}]_{V_{m,n}}(f_n|V_{m,n}) = \liminf_{n \to \infty} \mathcal{E}_n(f_n).
\]

(b). It suffices to consider \( f \in \mathcal{F} \). Let \( V_m \) and \( V_{m,n}, m, n \geq 1 \) be the same nest sequences in the proof of (a), and let \( f'_n \in C(A_n), n \geq 1 \) so that \( f'_n \Rightarrow f \). Let
\[
m_n = \max \{ 1 \leq m \leq n : [\mathcal{E}]_{V_m}(f|V_m) \geq [\mathcal{E}]_{V_{m,n}}(f'_n|V_{m,n}) - \frac{1}{m - 1} \}.
\]
In particular, we set \( \frac{1}{0} = +\infty \), so the maximum makes senses. In addition, by (B.1), \( m_n \to \infty \). It always holds
\[
[\mathcal{E}]_{V_{m,n}}(f|V_{m,n}) \geq [\mathcal{E}]_{V_{m,n}}(f'_n|V_{m,n}) \geq [\mathcal{E}]_{V_{m,n}}(f'_n|V_{m,n}) - \frac{1}{m_n - 1}.
\]
Let $f_n$ be the harmonic extension of $f'_n|_{V_{mn,n}}$ on $A_n$ with respect to $(E_n, F_n)$ for $n \geq 1$. Then

$$E(f) = \lim_{n \to \infty} [E]_{V_n}(f|_{V_n}) \geq \limsup_{n \to \infty} |E_n|_{V_{mn,n}}(f'_n|_{V_{mn,n}}) = \limsup_{n \to \infty} E_n(f_n).$$

Finally, we show $f_n \Rightarrow f$. One can see that $\{f_n\}_{n \geq 1}$ is equicontinuous, since $\{R_n\}_{n \geq 1}$ is equicontinuous by Lemma B.2 (b) and $\{E_n(f_n)\}_{n \geq 1}$ is uniformly bounded. In addition, by the construction of $f_n$, for each $y \in V_n$, there is a sequence $y_n \in A_n, n \geq 1$ such that $f_n(y_n) \to f(y)$ as $n \to \infty$. Thus, for any $x_n \in A_n, n \geq 1$ such that $x_n \to x \in A$, and any $\varepsilon > 0$, we pick $y \in V_n$ close enough to $x$ so that for any $n$ large enough $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(y_n)| + |f_n(y_n) - f(y)| + |f(y) - f(x)| < \varepsilon$ by equicontinuity of $\{f_n\}_{n \geq 1}$. \hfill \square

APPENDIX C. WEAK CONVERGENCE

Following Appendix B. We still consider convergence of resistance metrics in this last appendix, but focus on the probability side. This appendix will be based on the tightness property of the processes shown in [17] by Croydon, and an analytic proof of convergence of resolvent kernels in [12] by one of the authors. Readers can find a different proof of convergence of finite dimensional distributions, and a story about non-compact spaces in [17]. Also read [18] by Croydon, Hambly and Kumagai for a study of local times.

Basic settings.

We take all the (a),(b),(c),(d) of Basic settings in Appendix B. In addition, we assume:

(e). There is a sequence of resistance metrics $R_n \in C(A_{n}^2) \cap \mathcal{RM}(A_n), n \geq 1$ and $R \in C(A^2) \cap \mathcal{RM}(A)$. Let $(E_n, F_n)$ be the resistance form associated with $R_n$ for $n \geq 1$; let $(E, F)$ be the resistance form associated with $R$. Assume $R_n \Rightarrow R$.

(f). Let $\mu_n$ be a sequence of Radon measures supported on $A_n$, $n \geq 1$ and $\mu$ be a Radon measure supported on $A$. Assume $\mu_n \Rightarrow \mu$, where $\Rightarrow$ refers to weak convergence.

By Theorem 9.4 of [10], since $(A, R)$ is compact, $(E, F)$ is a regular Dirichlet form on $L^2(A, \mu)$, so by Theorem 7.2.1 of [24] there is a Hunt process $M = (\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ associated with $(E, F)$ on $A$. In addition, we have

i). The process is unique by Theorem 4.2.8 of [24], noticing that each point has positive capacity in our setting;

ii). $M$ is Feller: let $p_t(\cdot, \cdot)$ be the Markov kernel associated with the process, i.e. $p_t(x, B) = \mathbb{P}_x(X_t \in B)$. Then $p_t f \in C(A)$ for any $f \in C(A)$ and $p_t f \rightharpoonup f$ as $t \to 0$, where $p_t f(x) = \int_A p_t(x, dy) f(y)$. In fact, this can be verified easily for $f \in F$, then extends to any $f \in C(A)$ noticing that $F$ is dense in $C(A)$.

By a same reason, for $n \geq 1$, we have a unique Feller process $M_n = (\Omega_n, \mathcal{M}_n, X^{(n)}_t, \mathbb{P}^{(n)}_x)$ associated with $(E_n, F_n)$. We denote the associated transition kernel by $p_t^{(n)}(\cdot, \cdot)$, and write $p_t^{(n)} f(x) = \int_{A_n} p_t^{(n)}(x, dy) f(y)$. The main result in this appendix is the following theorem concerning weak convergence.

**Theorem C.1.** Assume all the basic settings. Let $x_n \in A_n$ and $x \in A$. If $x_n \to x$ in $(B, d)$, then

$$\mathbb{P}^{(n)}_{x_n}((X^{(n)}_t)_{t \geq 0} \in \cdot) \Rightarrow \mathbb{P}_x((X_t)_{t \geq 0} \in \cdot),$$
where the weak convergence is in the sense of probability measures on $D(\mathbb{R}_+ B)$ (that is, the space of cadlag processes on $(B, d)$, equipped with the usual Skorohod $J_1$-topology).

**Remark.** Theorem [C.1] is almost same as Theorem 1.2 of [17], with additional assumption about compactness. However, since our assumption is a little different here: in [17] and [18], it is assumed that $(A_n, R_n)$ and $(A, R)$ are embedded isometrically in some space $M$ by Gromov-Hausdorff vague convergence, so we can not use it directly, noticing that our basic setting only tells us the Gromov-Hausdorff convergence of $(A, R)$ by Gromov-Hausdorff vague convergence, so we can not use it directly, noticing that our basic setting only tells us the Gromov-Hausdorff convergence of $(A, R)$, while the Gromov-Hausdorff convergence of $(A_n, R_n)$ to $(A, R)$ is not easy to prove (see Section 2.5 of [12] for a proof). However, an essentially same idea leads to Theorem C.1. In this appendix, for convenience of readers, we sketch the proof, with the probability arguments about convergence of kernels replaced by purely analytic ones. The proof of Theorem C.1 relies on two ingredients: tightness and convergence of transition kernels (finite dimensional distributions). We refer to Lemma 4.3 of [17] for a neat result on tightness.

**Lemma C.2** ([17]). Let $(A, R)$ be a compact metric space, where $R \in \mathcal{RM}(A)$; let $\mu$ be a Radon measure supported on $A$. Then, for any $\varepsilon > 0$ and $0 < \eta \leq \varepsilon / 8$,

$$
\sup_{x \in A} \left( \sup_{s \leq t} R(x, X_s) < \varepsilon \right) \leq \frac{32 N_R(A, \varepsilon/4)}{\varepsilon} \cdot \left( \eta + \frac{t}{\inf_{x \in A} \mu(B_R(x, \eta))} \right),
$$

where $N_R(A, \varepsilon)$ is the minimal size of an $\varepsilon$-net of $(A, R)$, and $B_R(x, \eta)$ is a ball centered at $x$ with respect to $R$.

The tightness of the processes follows immediately from Lemma C.2.

Before proceeding, we point out an easy consequence of Lemma B.2 (c): “(c1)⇒(c3)”.

Assuming the basic settings, if $f_n \rightarrow f$, where $f_n \in C(A_n)$, $n \geq 1$ and $f \in C(A)$, then

$$
\int_{A_n} f_n(x) \mu_n(dx) \rightarrow \int_{A_n} f(x) \mu(dx). \quad (C.1)
$$

In fact, we have $\{g_n\}_{n \geq 1} \cup \{g\} \subset C(B)$ such that $g_n|_{A_n} = f_n, \forall n \geq 1, g|_{A} = f$ and $g_n \Rightarrow g$. Then

$$
\int_{A_n} f_n(x) \mu_n(dx) = \int_{A_n} g(x) \mu_n(dx) + \int_{A_n} (g_n - g)(x) \mu_n(dx) \rightarrow \int_{A} g(x) \mu(dx),
$$

noticing

$$
\lim_{n \rightarrow \infty} \mu_n(B) = \mu(B) < \infty.
$$

In particular, an easy application of (C.1) leads to $f_n \mu_n \Rightarrow f \mu$.

**Proposition C.3.** Assume all the basic settings. For any sequence $x_n, n \geq 1$ with $x_n \in A_n$, the laws of $X^{(n)}$ under $\mathbb{P}_{x_n}, n \geq 1$ are tight in $D(\mathbb{R}_+ B)$.

**Proof.** We can use the same idea as that in Lemma 4.4 and Proposition 4.1 of [17].

First, we let $\phi(s) = \sup\{R_n(x, y) : n \geq 1, x, y \in A_n\}$ and $d(x, y) \leq s\}$. Then, $\phi(s)$ is continuous at 0 by equicontinuity of $R_n$ (by Lemma B.2), and $R_n(x, y) \leq \phi(d(x, y))$ for any $n \geq 1$ and $x, y \in A_n$. Now, fix any $\varepsilon > 0$, we can find $s > 0$ such that $\phi(2s) \leq \varepsilon$, so

$$
N_{R_n}(A_n, \varepsilon) \leq N_d(B, s) < \infty, \quad \forall n \geq 1, \quad (C.2)
$$

where $N_d(B, s)$ is the minimal size of an $s$-net of $(B, d)$. In fact, given an $s$-net $\{x_1, x_2, \cdots, x_m\}$ of $(A, d)$, we can simply pick a point in $B_d(x_j, s) \cap A_n$ for each $j$ (if the intersection is non-empty) to form a $2s$-net of $(A_n, d)$, hence an $\varepsilon$-net of $(A_n, R_n)$.

Next, for $0 < \eta \leq \varepsilon / 8$, $n \geq 1$ define $g_n(x, y) = (1 - \eta^{-1} R_n(x, y)) \lor 0$; define $g(x, y) = (1 - \eta^{-1} R(x, y)) \lor 0$. Then since $R_n \rightarrow R$, we have $g_n \rightarrow g$. For $n \geq 1$, let $f_n(x) = \sup_{s \leq t} R_n(x, x_s) \leq \varepsilon / 8$.

where the weak convergence is in the sense of probability measures on $D(\mathbb{R}_+ B)$ (that is, the space of cadlag processes on $(B, d)$, equipped with the usual Skorohod $J_1$-topology).
Lemma C.5. With the same conditions as in Definition C.4, there exists $C > 0$ depending only on $\alpha$, $\mu(A)$ and $\text{diam}_R(A) = \sup_{x,y \in A} R(x,y)$ such that

$$\left\{ \begin{array}{l} \|u_\alpha\|_{C(A^2)} \leq C; \\
|u_\alpha(x_1,y_1) - u_\alpha(x_2,y_2)| \leq C \left( R(x_1,x_2)^{1/2} + R(y_1,y_2)^{1/2} \right), \quad \forall x_i, y_i \in A, i = 1, 2. \end{array} \right.$$ 

Proof. By definition, we have $\mathcal{E}(u_\alpha(x,\cdot)) \leq \mathcal{E}(u_\alpha(y,\cdot)) = u_\alpha(x,x)$, where $(\mathcal{E}, \mathcal{F})$ is the resistance form associated with $R$. As a consequence, we have

$$|u_\alpha(x,y_1) - u_\alpha(x,y_2)|^2 \leq R(y_1,y_2) u_\alpha(x,x).$$

Combining the above estimate with the fact that $\int_A u_\alpha(x,y)\mu(dy) = \alpha^{-1}$, we have

$$u_\alpha(x,x) - (u_\alpha(x,x))^{1/2} \text{diam}_R^{1/2}(A) \leq \left( \alpha \mu(A) \right)^{-1}.$$ 

This shows that $u_\alpha(x,x) \leq C'$ for some $C' > 0$ depending only on $\text{diam}_R(A)$, $\alpha$ and $\mu(A)$. \(\Box\)
Proposition C.6. Assume all the basic settings. Let $\alpha > 0$, then
\[
 u_{\alpha,n} \rightharpoonup u_{\alpha},
\]
where $u_{\alpha,n}$ is the resolvent kernel associated with $R_n, \mu_n$ for each $n \geq 1$, and $u_{\alpha}$ is the resolvent kernel associated with $R, \mu$.

Proof. We follow a similar idea as the proof of Theorem 2.4.1 in [48]. Let $(E_n, F_n)$ be the resistance form associated with $R_n$ for $n \geq 1$, and let $(E, F)$ be the resistance form associated with $R$. Then, $u_{\alpha,n}(x, \cdot)$ is the minimizer of $E_{\alpha,n}(f) - 2f(x)$ for $n \geq 1$, $x \in A_n$; $u_{\alpha}(x, \cdot)$ is the minimizer of $E_{\alpha}(f) - 2f(x)$, for $x \in A$.

By Lemma [C.5], the sequence $\{u_{\alpha,n}\}_{n \geq 1}$ is uniformly bounded and equicontinuous, so there is a subsequence $n_l \to \infty$ and $u' \in C(A^2)$ such that $u_{\alpha,n_l} \rightharpoonup u'$. Let $x_{n_l} \in A_{n_l}, l \geq 1$ and $x_{n_l} \to x \in A$. By Theorem [B.6] and (C.1), we have
\[
 E_{\alpha}(u'(x, \cdot)) - 2u'(x, x) \leq \liminf_{l \to \infty} E_{\alpha,n_l}(u_{\alpha,n_l}(x_{n_l}, \cdot)) - 2u_{\alpha,n_l}(x_{n_l}, x_{n_l}).
\]

On the other hand, still by Theorem [B.6] and (C.1), there is a sequence $f_n \rightharpoonup u_{\alpha}(x, \cdot)$ such that
\[
 E_{\alpha}(u_{\alpha}(x, \cdot)) = \lim_{n \to \infty} E_{\alpha,n}(f_n).
\]
Thus,
\[
 E_{\alpha}(u_{\alpha}(x, \cdot)) - 2u_{\alpha}(x, x) = \lim_{n \to \infty} E_{\alpha,n}(f_n) - 2f_n(x_n)
\geq \liminf_{l \to \infty} E_{\alpha,n_l}(u_{\alpha,n_l}(x_{n_l}, \cdot)) - 2u_{\alpha,n_l}(x_{n_l}, x_{n_l})
\geq E_{\alpha}(u'(x, \cdot)) - 2u'(x, x),
\]
where the second inequality follows from the minimizer property of $u_{\alpha,n_l}(x_{n_l}, \cdot)$. As $x \in A$ is arbitrary, this implies that $u' = u_{\alpha}$ from the minimizer property of $u_{\alpha}(x, \cdot)$. The argument works for any subsequence of $u_{\alpha,n}$, so we finally see $u_{\alpha,n} \rightharpoonup u_{\alpha}$, since otherwise we can find a subsequence $\{n_l\}_{l \geq 1}$, $(x_l, y_l) \in A^2_{n_l}$ and $\varepsilon > 0$ such that $(x_l, y_l) \to (x, y)$ and $|u_{\alpha,n_l}(x_l, y_l) - u_{\alpha}(x, y)| > \varepsilon, \forall l \geq 1$.

Theorem [C.1] follows from Proposition [C.3] and [C.6] which is essentially the same as Proposition 6.5 of [3]. For convenience of readers, we reproduce the proof here.

Proof of Theorem [C.1]. Let $x_n \to x$, with $x_n \in A_n, n \geq 1$ and $x \in A$. By Proposition [C.3] there is a subsequence $n_l, l \geq 1$ so that $P_{x_{n_l}}((X_{t_{n_l}})_{t \geq 0}) \Rightarrow P'((X'_t)_{t \geq 0})$, where the later is understood as a probability measure on $D(\mathbb{R}_+, B)$. By using Proposition [C.6] and (C.1), (C.4), we have for any $\alpha > 0$, $f \in C(B)$:
\[
 \int_0^\infty e^{-\alpha t}E'[f(X'_t)]dt = E'[\int_0^\infty e^{-\alpha t}f(X'_t)dt] = \lim_{l \to \infty} E_{x_{n_l}}[\int_0^\infty e^{-\alpha t} f(X_{t_{n_l}})dt]
\]
\[
 = \lim_{l \to \infty} \int_{A_n} u_{\alpha,n_l}(x_{n_l}, y)f(y)\mu_{n_l}(dy) = \int_A u_{\alpha}(x, y)f(y)\mu(dy) = \int_0^\infty e^{-\alpha t}E_x[f(X_t)]dt.
\]
Since both $X_t, X'_t$ are in $D(\mathbb{R}_+, B)$, and the fact that Laplace transformation determines the function a.e, we see that $E_x[f(X_t)] = E'[f(X'_t)]$ for each $t > 0$ and any $f \in C(B)$. Hence,
\[
 P'(X'_t \in \cdot) = P_x(X_t \in \cdot), \quad \forall t > 0.
\]
In other words, \( p_t^{(n)}(x_n, \cdot) \Rightarrow p_t(x, \cdot) \). Since the argument works for any subsequence, and the space of probability measures on \( K \) forms a separable complete metric space with Prohorov metric (see Theorem 1.7, Chapter 3 of [21]), we have \( p_t^{(n)}(x_n, \cdot) \Rightarrow p_t(x, \cdot) \).

As an immediate consequence, one can check that \( P_t^{(n)} f \refrightarrow P_t f, \forall t \geq 0, f \in C(B) \) by using (C.1). So, we can easily extend the result to finite dimensional cases by using the Markov property. For \( t_1 < t_2 \) and \( f, g \in C(B) \), we have
\[
E_{x_n}(f(X_{t_1}^{(n)})g(X_{t_2}^{(n)})) = P_{t_1}^{(n)}(f \cdot P_{t_2-t_1}^{(n)} g)(x_n) \rightarrow P_{t_1}(f \cdot P_{t_2-t_1} g)(x) = E_x(f(X_{t_1})g(X_{t_2})),
\]
where we use the fact \( f \cdot P_{t_2-t_1} g \refrightarrow f \cdot P_{t_2-t_1} g \), (C.1) and (C.4). The same argument works for longer sequences \( t_1 < t_2 < \cdots < t_j \), so finite dimensional distributions converge. This, together with tightness (Proposition C.3), implies Theorem C.1 by Theorem 7.8, Chapter 3 of [21].

Finally, noticing that by Theorem 4.5.1 of [24], the form \((\mathcal{E}, \mathcal{F})\) is local if and only if the associated Hunt process has continuous sample paths (for q.e. starting points). In addition, by Theorem 10.2 of [21] the weak limit \( X_t \) is a.s. continuous if \( X_t^{(n)} \) is a.s. continuous for \( n \geq 1 \), so there is an immediate corollary of Theorem C.1 concerning the local property. Readers can also find a purely analytic proof of a slightly stronger version in [12].

**Corollary C.7.** Assume all the basic settings. If \((\mathcal{E}_n, \mathcal{F}_n)\) is local for \( n \geq 1 \), then \((\mathcal{E}, \mathcal{F})\) is local.

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