# CONFORMAL DIMENSION OF MEASURES AND QUASISYMMETRIC DIMENSION REDUCTION

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Abstract. We prove that the conformal dimension of every locally finite Borel measure is either zero or infinite. The main ingredient is a quasisymmetric dimension-reduction theorem: every full-support probability measure of finite Hausdorff dimension on a separable metric space admits quasisymmetrically equivalent metrics in which its Hausdorff dimension is arbitrarily small. In particular, every locally finite Borel measure on a doubling metric space has conformal dimension zero. We also prove that for every $n \geq 1$ and $p > 0$ there are a metric space $X$, a quasisymmetric homeomorphism $f : [0,1]^n \to X$, and a Borel set $E \subset [0,1]^n$ such that $\dim_{\mathrm{H}} f(E) \leq p$ and $\dim_{\mathrm{H}}([0,1]^n \setminus E) \leq n - 1 + p$. The second bound is sharp up to $p$: if $\dim_{\mathrm{H}} f(E) < 1$, then $\dim_{\mathrm{H}}([0,1]^n \setminus E) \geq n - 1$.

## 1. Introduction

1.1. **Background and main results.** We begin with notation. Let $(X, d)$ be a metric space. For $x \in X$ and $r > 0$, let $B_d(x, r) = \{y \in X : d(x, y) < r\}$ be the open ball centered at $x$ with radius $r$. For $E \subset X$, let $\operatorname{diam}_d E = \sup\{d(x, y) : x, y \in E\}$ be its diameter. For $s \geq 0$, the *$s$-dimensional Hausdorff measure* is

$$\mathcal{H}^s_d(E) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_i (\operatorname{diam}_d U_i)^s : E \subset \bigcup_i U_i,\ \operatorname{diam}_d U_i \leq \delta \right\},$$

where the infimum is over countable covers. The *Hausdorff dimension* of $E$ is

$$\dim^d_{\mathrm{H}} E = \inf\{s \geq 0 : \mathcal{H}^s_d(E) = 0\}.$$

We omit the metric subscript or superscript when it is clear.

Let $(Y, d_Y)$ be another metric space. A homeomorphism $f : X \to Y$ is called *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d(x, a)}{d(x, b)}\right)$$

for all distinct $x, a, b \in X$. A metric $D$ on $X$ is said to be *quasisymmetrically equivalent* to $d$ if the identity map $\mathrm{id}_X : (X, d) \to (X, D)$ is quasisymmetric. The two concepts are equivalent: quasisymmetric maps may be viewed as quasisymmetric changes of metric on their domains. Tukia and Väisälä [TV80] formulated quasisymmetry for maps between general metric spaces, building on a one-dimensional form implicit in Beurling and Ahlfors's study of boundary correspondences for planar quasiconformal maps [BA56].

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A fundamental quasisymmetric invariant is the *conformal dimension* of a metric space, introduced by Pansu [Pan89] and defined by

$$\dim_{\mathrm{C}} X = \inf_f \dim_{\mathrm{H}} f(X),$$

where the infimum is over all quasisymmetric homeomorphisms from $X$ onto metric spaces. This invariant measures how far the Hausdorff dimension of the entire space can be reduced without changing its quasisymmetric geometry.

The space $X$ is called *minimal for conformal dimension* if $\dim_{\mathrm{C}} X = \dim_{\mathrm{H}} X$. A classical theorem of Bishop and Tyson [BT01] states that $[0,1] \times E$ is minimal whenever $E \subset \mathbb{R}^{n-1}$ is compact. Questions concerning the conformal dimension and conformal minimality of sets have been studied in many settings; see, for example, [Kov06, Hak10, Mac11, CP14, Orp17, BHL25, MT26a, MT26b].

The conformal dimension of a space concerns the geometry of the whole set and does not distinguish between different measures on the same support. To study the geometry seen by a measure rather than that of its entire support, Bate and Orponen [BO18] introduced the conformal dimension of a measure.

A Borel measure $\mu$ on $X$ is *locally finite* if every point of $X$ has an open neighborhood of finite $\mu$-measure. For a Borel measure $\mu$ on $(X,d)$, its *Hausdorff dimension* is

$$\dim_{\mathrm{H}}^d \mu = \inf\{\dim_{\mathrm{H}}^d E : E \subset X \text{ is Borel and } \mu(X \setminus E) = 0\}.$$

The *support* of $\mu$, denoted by $S = \operatorname{spt} \mu$, is the set of points every open neighborhood of which has positive $\mu$-measure. We say that $\mu$ has *full support* if $S = X$.

If $f : S \to Y$ is Borel, the *push-forward measure* $f_{\#}\mu$ on $Y$ is defined by $f_{\#}\mu(A) = \mu(f^{-1}(A))$. The *conformal dimension of* $\mu$ is

$$\dim_{\mathrm{C}} \mu = \inf_f \dim_{\mathrm{H}}(f_{\#}\mu), \tag{1.1}$$

where $f$ ranges over all quasisymmetric homeomorphisms from $(S,d)$ onto metric spaces.

Throughout the paper, every measure is assumed to be nonzero, locally finite and to satisfy $\mu(X \setminus S) = 0$. The latter assumption is needed because the quasisymmetric maps in the definition act only on $S$. If $\mu(X \setminus S) > 0$, then $f_{\#}\mu$ is precisely the push-forward of the restricted measure $\mu|_S$; in particular, $f_{\#}\mu(Y) = \mu(S)$. Thus the positive mass outside $S$ has no image, and the resulting conformal dimension describes $\mu|_S$ rather than the original measure $\mu$. Note that when $X$ is separable, every Borel measure satisfies $\mu(X \setminus S) = 0$. Indeed, $X \setminus S$ is covered by open $\mu$-null sets, and a countable base reduces this cover to a countable subcover. The same conclusion holds when $X$ is doubling, since every doubling metric space is separable.

Bate and Orponen showed that the direct measure-theoretic analogue of the Bishop–Tyson phenomenon can fail. For every $n \geq 2$ and $0 < s < n-1$, they constructed a non-Ahlfors-regular compact set $E \subset \mathbb{R}^{n-1}$ such that the restriction of $\mathcal{H}^{1+s}$ to $[0,1] \times E$ has conformal dimension zero [BO18]. Thus the conformal geometry of a support and the conformal geometry of the mass it carries can behave very differently. This led them to pose the following two questions [BO18, Questions 1 and 2].

***Question* 1.** *Do there exist measures with positive and finite conformal dimension?*

***Question* 2.** *Let $C \subset \mathbb{R}$ be the middle-thirds Cantor set of dimension $s = \log 2/\log 3$, and let $\mu$ be the $(1+s)$-dimensional Hausdorff measure on $[0,1] \times C$. Does $\mu$ have conformal dimension $1+s$?*

Our first theorem gives a complete answer to Question 1.

**Theorem 1.1.** *Let $(X, d)$ be a metric space and let $\mu$ be a locally finite Borel measure on $X$. Then $\dim_{\mathrm{C}} \mu \in \{0, +\infty\}$.*

Theorem 1.1 therefore implies that every locally finite Borel measure on a doubling metric space has conformal dimension zero, so Question 2 has a negative answer. A metric space $(X, d)$ is *doubling* if every ball can be covered by a uniformly bounded number of balls of half the radius.

**Corollary 1.2.** *Let $(X, d)$ be a doubling metric space and let $\mu$ be a locally finite Borel measure on $X$. Then $\dim_{\mathrm{C}} \mu = 0$.*

*Proof.* By Assouad's embedding theorem [Hei01, Theorem 12.2], for some $0 < \theta < 1$ the snowflake $(X, d^\theta)$ admits a bi-Lipschitz embedding into $\mathbb{R}^m$ for some $m \geq 1$. Thus there are an injective map $F : X \to \mathbb{R}^m$ and a constant $C \geq 1$ such that $C^{-1} d(x, y)^\theta \leq |F(x) - F(y)| \leq C d(x, y)^\theta$ for all $x, y \in X$. Hence $\dim_{\mathrm{H}}^d X = \theta \dim_{\mathrm{H}}^{d^\theta} X \leq \theta m < \infty$. Writing $S = \operatorname{spt} \mu$, we obtain $\dim_{\mathrm{C}} \mu \leq \dim_{\mathrm{H}}^d \mu \leq \dim_{\mathrm{H}}^d S \leq \dim_{\mathrm{H}}^d X < \infty$, so Theorem 1.1 gives $\dim_{\mathrm{C}} \mu = 0$. □

Corollary 1.2 also has a short direct proof, independent of Theorem 1.1. We collect it in Section 4.

Corollary 1.2 naturally raises the question of what remains true when the ambient space is not doubling. The following two examples show that, without doubling, both values in Theorem 1.1 can occur.

*Example 1:* $\dim_{\mathrm{C}} \mu = \infty$. Let $X$ be an uncountable set with the discrete metric

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

and let $\mu$ be the counting measure on $X$. Then $\mu$ is locally finite, $\operatorname{spt} \mu = X$, and $X$ is neither separable nor doubling. If $h : X \to h(X)$ is a quasisymmetric homeomorphism, then $h_\# \mu$ is the counting measure on $h(X)$, so its only full-measure Borel set is $h(X)$ itself. If $h(X)$ had finite Hausdorff dimension, it would be separable, contradicting the fact that $h$ is a homeomorphism and $X$ is nonseparable. Therefore $\dim_{\mathrm{H}}(h_\# \mu) = \infty$ for every $h$, and $\dim_{\mathrm{C}} \mu = \infty$.

*Example 2:* $\dim_{\mathrm{C}} \mu = 0$. Let

$$X = \prod_{k=1}^{\infty} \{0, \ldots, k\}, \qquad r_0 = 1, \quad r_m = \frac{1}{(m+1)!},$$

and equip $X$ with the ultrametric $d(x, x) = 0$ and $d(x, y) = r_m$ for $x \neq y$, where $m$ is the largest integer such that $x_1 = y_1, \ldots, x_m = y_m$. Let $\mu$ be the product probability measure for which the $k$-th coordinate is uniformly distributed on $\{0, \ldots, k\}$. The metric $d$ induces the product topology, so $X$ is compact, and $\mu$ has full support.

The space $X$ is not doubling. Indeed, every cylinder determined by the first $m-1$ coordinates contains $m + 1$ disjoint cylinders determined by the first $m$ coordinates, so the number of half-radius balls required to cover a ball is not uniformly bounded. Each cylinder of length $m$ has diameter $r_m$ and $\mu$-measure $r_m$. For every $p > 0$, the function $d^{1/p}$ is again an ultrametric that is quasisymmetrically equivalent to $d$. With respect to $d^{1/p}$, a cylinder of length $m$ has diameter $r_m^{1/p}$ and measure $r_m$. This gives $\dim_{\mathrm{H}}^{d^{1/p}}(\mathrm{id}_\# \mu) \leq p$. Letting $p \downarrow 0$ gives $\dim_{\mathrm{C}} \mu = 0$.

The proof of Theorem 1.1 is based on the following dimension-reduction theorem.

**Theorem 1.3.** *Let $(X, d)$ be a separable metric space and let $\mu$ be a Borel probability measure with full support on $X$. If $\dim_{\mathrm{H}}^{d} \mu < \infty$, then for every $p > 0$ there is a metric $D$ on $X$, quasisymmetrically equivalent to $d$, such that $\dim_{\mathrm{H}}^{D} \mu \le p$. Consequently, $\dim_{\mathrm{C}} \mu = 0$.*

The proof of Theorem 1.3 adapts the weighted hyperbolic-filling construction of Miller and Tian [MT26a, MT26b] and combines it with a probabilistic choice of weights. In a hyperbolic filling, vertices represent points of $X$ at successively finer scales, horizontal edges join nearby points at the same scale, and vertical edges connect consecutive scales. We choose the weights so that small neighborhoods are inexpensive at most scales while every chain crossing the filling retains a uniform positive cost. The latter bound prevents the metric from collapsing and, together with control of ball diameters, yields quasisymmetry. Our setting introduces two additional features: $X$ need not be compact, so the filling may have countably infinite vertex sets and need not be locally finite; and the metric is defined directly on $X$ from the costs of finite chains, rather than on the boundary of a reweighted graph. Our later application uses the diameter estimate only at $\mu$-almost every point, with the weights chosen by random Borel partitions adapted to $\mu$. These partitions ensure that almost every point lies near a partition boundary at only a controlled proportion of scales, which gives $\dim_{\mathrm{H}}^{D} \mu \le p$.

Write $\mathcal{L}^n$ for $n$-dimensional Lebesgue measure. Note that Corollary 1.2 immediately implies that $\mathcal{L}^n$ has conformal dimension zero for every $n \ge 1$. This is closely related to earlier constructions of singular quasisymmetric maps. Research in this direction began with Tukia's one-dimensional result [Tuk89]: for every $p > 0$ there are a quasisymmetric homeomorphism $f : [0,1] \to [0,1]$ and a set $E \subset [0,1]$ such that

$$\dim_{\mathrm{H}} f(E) \le p, \qquad \dim_{\mathrm{H}}([0,1] \setminus E) \le p. \tag{1.2}$$

Romney [Rom19] later proved that for every $n \ge 2$ and $p > 0$ there are a Borel set $E \subset [0,1]^n$, a metric space $X$, and a quasisymmetric homeomorphism $f : [0,1]^n \to X$ such that

$$\dim_{\mathrm{H}} f(E) \le p, \qquad \mathcal{L}^n([0,1]^n \setminus E) = 0. \tag{1.3}$$

Romney left open the following question [Rom19, Section 1].

***Question* 3.** *To what extent can the Hausdorff dimension of $[0,1]^n \setminus E$ be reduced?*

We can give a sharp quantitative answer by controlling simultaneously the dimension of the image of a set and the dimension of its complement.

**Theorem 1.4.** *Let $n \ge 1$. For every $p > 0$, there exist a metric space $X$, a quasisymmetric homeomorphism $f : [0,1]^n \to X$, and a Borel set $E \subset [0,1]^n$ such that*

$$\dim_{\mathrm{H}} f(E) \le p \quad \textit{and} \quad \dim_{\mathrm{H}}([0,1]^n \setminus E) \le n - 1 + p.$$

**Remark 1.5.** When $n = 1$, the conclusion is Tukia's theorem (1.2). When $n \ge 2$, it refines Romney's theorem (1.3) by replacing the Lebesgue-null conclusion with an explicit Hausdorff-dimension bound. This bound is sharp up to the additive term $p$ when $0 < p < 1$. Indeed, we can further show that (see Proposition 5.5) whenever a homeomorphism $f : [0,1]^n \to X$ and a set $E \subset [0,1]^n$ satisfy $\dim_{\mathrm{H}} f(E) < 1$, one necessarily has

$$\dim_{\mathrm{H}}([0,1]^n \setminus E) \ge n - 1.$$

1.2. **Theorem 1.3 implies Theorem 1.1.** We now deduce Theorem 1.1 from the dimension-reduction theorem, Theorem 1.3.

*Proof of Theorem 1.1.* Let $S = \operatorname{spt} \mu$. Suppose first that $S$ is nonseparable, and let $f : S \to Y$ be any quasisymmetric homeomorphism onto a metric space $Y$. The measure $f_{\#}\mu$ has full

support on $Y$, so every Borel set of full $f_{\#}\mu$-measure is dense in $Y$. If $\dim_{\mathrm{H}}(f_{\#}\mu) < \infty$, there is a full-measure Borel set $A \subset Y$ with $\dim_{\mathrm{H}} A < \infty$. Every metric space of finite Hausdorff dimension is separable, so $A$, and hence its closure $Y$, would be separable. Since $f$ is a homeomorphism, this would make $S$ separable, a contradiction. Therefore $\dim_{\mathrm{C}} \mu = +\infty$ when $S$ is nonseparable.

Assume now that $S$ is separable. Local finiteness gives every point of $S$ an open neighborhood of finite measure. A countable base yields a countable subcover, so $S$ is the union of countably many Borel sets of finite measure. Thus $\mu$ is $\sigma$-finite on $S$. Choose pairwise disjoint Borel sets $F_j$ of finite measure whose union is $S$. Define the Borel function

$$w = \sum_{j \geq 1} \frac{2^{-j}}{1 + \mu(F_j)} \mathbf{1}_{F_j}.$$

Then $w > 0$ on $S$ and $0 < \int_S w \, d\mu < \infty$. Define a probability measure $\nu$ on $S$ by

$$d\nu = \frac{w \, d\mu}{\int_S w \, d\mu}.$$

Because $w$ is strictly positive, $\mu$ and $\nu$ have exactly the same null sets; in particular, $\nu$ has full support. Consequently, for every quasisymmetric homeomorphism $f : S \to Y$,

$$\dim_{\mathrm{H}}(f_{\#}\mu) = \dim_{\mathrm{H}}(f_{\#}\nu) \quad \text{and} \quad \dim_{\mathrm{C}} \mu = \dim_{\mathrm{C}} \nu. \tag{1.4}$$

Suppose that $\dim_{\mathrm{C}} \mu < \infty$. By (1.1) and (1.4), there is a quasisymmetric homeomorphism $f : S \to Y$ such that $\dim_{\mathrm{H}}(f_{\#}\nu) < \infty$. The space $Y$ is separable, and $f_{\#}\nu$ is a probability measure with full support on $Y$. For every $p > 0$, Theorem 1.3 supplies a metric $D_p$ on $Y$ such that $\mathrm{id} : (Y, d_Y) \to (Y, D_p)$ is quasisymmetric and $\dim_{\mathrm{H}}^{D_p}(f_{\#}\nu) \leq p$. Composing this identity map with $f$ and using (1.4) gives $\dim_{\mathrm{C}} \mu \leq p$. Since $p > 0$ is arbitrary, $\dim_{\mathrm{C}} \mu = 0$. □

We organize the paper as follows. Section 2 develops the weighted hyperbolic filling and the associated chain metric. Section 3 uses random Borel partitions to prove Theorem 1.3. Section 4 gives a direct proof of Corollary 1.2 that does not use Theorem 1.1. Section 5 proves Theorem 1.4 and the sharpness statement in Proposition 5.5.

Before ending this section, we note that the full-measure definition of $\dim_{\mathrm{H}}^{d} \mu$ used in this paper is often called the *upper Hausdorff dimension* of $\mu$. One may instead use the *lower Hausdorff dimension*, defined by $\inf\{\dim_{\mathrm{H}}^{d} E : E \subset X \text{ is Borel and } \mu(E) > 0\}$, and define the corresponding conformal dimension. We do not know whether Theorem 1.1 remains valid when conformal dimension is defined using the lower Hausdorff dimension. However, the analogues of Corollary 1.2 and Theorem 1.3 also hold when the lower Hausdorff dimension is used throughout.

## 2. Hyperbolic fillings

This section carries out the hyperbolic-filling construction used in the proof of Theorem 1.3. Here a *hyperbolic filling* means a rooted graph organized into levels. Its vertices represent points of $X$ at successively finer scales. Horizontal edges join nearby points on the same level, and vertical edges join points on consecutive levels. Such multiscale graphs are standard tools in metric geometry; see, for example, [BP03, BS07, CP13, BS18, BBS22, ESS25].

We carry out the construction in three steps. Subsection 2.1 defines the filling and its vertex weights, Subsection 2.2 proves a uniform lower bound for path costs, and Subsection 2.3 transfers the weights to $X$, defines the metric, and proves the required diameter estimate.

2.1. **Separated sets and admissible weights.** This subsection constructs the vertices and edges of the filling and then assigns compatible weights to its vertices.

Let $(X, d)$ be bounded, complete, and separable, with $\operatorname{diam} X < 1$. Set $\alpha_0 = 10^{-3}$ and fix

$$0 < \alpha < \alpha_0.$$

For $r > 0$, a set is *$r$-separated* if distinct points in it are at distance at least $r$, and it is maximal $r$-separated if it is not properly contained in another $r$-separated set. Choose a base point $o \in X$ and set $A_0 = \{o\}$. Inductively choose nested maximal $\alpha^n$-separated sets $A_n$; thus

$$A_0 \subset A_1 \subset A_2 \subset \cdots, \qquad X \subset \bigcup_{x \in A_n} B(x, \alpha^n),$$

and $d(x, y) \geq \alpha^n$ for all distinct $x, y \in A_n$. Inductively extend $A_{n-1}$ to a maximal $\alpha^n$-separated set $A_n$ by Zorn's lemma. Each $A_n$ is countable because $X$ is separable. Put $V_n = A_n \times \{n\}$ and $V = \bigcup_{n \geq 0} V_n$.

Two vertices $u = (x, n)$ and $v = (y, n)$ at the same level are *horizontal neighbors*, written $u \sim v$, if $d(x, y) < 8\alpha^n$; we also declare $u \sim u$. Every $v = (y, n + 1)$ is assigned a *parent* $g(v) = (x, n)$ satisfying $d(x, y) < \alpha^n$. Such a parent need not be unique; we simply choose and fix one for each vertex. The unordered pair $\{v, g(v)\}$ is called a *vertical edge*. The vertical edges form a tree rooted at $(o, 0)$: every vertex has a unique finite parent chain ending at $(o, 0)$. Throughout, $\sim$ refers only to horizontal adjacency. For vertices $u = (x, n)$ and $v = (y, k)$, possibly at different levels, we use the convention $d(u, v) = d(x, y)$.

The choice $\alpha < \alpha_0$ gives the following useful observation. If $n \geq 1$ and

$$u_0 \sim u_1 \sim \cdots \sim u_N, \qquad u_i \in V_n, \qquad N \leq 100,$$

then

$$d(g(u_0), g(u_N)) < 2\alpha^{n-1} + 8N\alpha^n \leq (2 + 800\alpha)\alpha^{n-1} < 8\alpha^{n-1}. \tag{2.1}$$

Thus $g(u_0) \sim g(u_N)$. In particular, the parents of horizontal neighbors are horizontal neighbors.

An assignment $\sigma : V \setminus V_0 \to [0, \infty)$ is called *admissible* if it has the following property. Whenever $u_0 = (x_0, n) \sim \cdots \sim u_N = (x_N, n)$ and $v = (y, n - 1)$ satisfy

$$B(y, \alpha^{n-1}) \cap B(x_0, 4\alpha^n) \neq \varnothing, \qquad (X \setminus B(y, 2\alpha^{n-1})) \cap B(x_N, 4\alpha^n) \neq \varnothing, \tag{2.2}$$

one has

$$\sum_{i=0}^{N} \sigma(u_i) \geq 1. \tag{2.3}$$

Fix $0 < \eta < 1/2$. For $u \in V_n$, $n \geq 1$, define

$$a(u) = \max\left\{\eta, \min\left\{1 - \eta, 2 \sup_{u \sim v \sim w} \sigma(w)\right\}\right\}. \tag{2.4}$$

Thus $\eta \leq a(u) \leq 1 - \eta$.

We next construct a positive number $\pi(u)$ at every vertex. Set $\pi((o, 0)) = 1$. Suppose that $\pi$ has been defined on $V_{n-1}$ and satisfies

$$\eta \leq \frac{\pi(u)}{\pi(v)} \leq \eta^{-1} \quad \text{whenever } u, v \in V_{n-1} \text{ and } u \sim v. \tag{2.5}$$

For $u \in V_n$, put $q(u) = \pi(g(u))a(u)$. Write $v \succ u$ if $v \sim u$ and $q(v) > \eta^{-1} q(u)$. There is no chain $v \succ u \succ w$. Indeed, $v \sim u \sim w$, so (2.1) gives $g(v) \sim g(w)$, whereas such a chain would give

$$\pi(g(v)) > q(v) > \eta^{-2} q(w) \geq \eta^{-1} \pi(g(w)),$$

contrary to (2.5). Define

$$\pi(u) = \begin{cases} q(u), & \text{if no } v \succ u, \\ \eta \sup_{v \succ u} q(v), & \text{otherwise.} \end{cases} \tag{2.6}$$

The next lemma adapts [MT26a, Lemmas 5.2–5.3] to the present setting, where the sets $A_n$ may be infinite. In [MT26a], the corresponding sets are finite, so the relevant suprema are attained; the argument here does not require a maximizing vertex.

**Lemma 2.1.** *For $u, v \in V_n$,*

$$q(u) \le \pi(u) \le \sup_{w \sim u} q(w), \tag{2.7}$$

$$u \sim v \quad \Longrightarrow \quad \eta \le \frac{\pi(u)}{\pi(v)} \le \eta^{-1}. \tag{2.8}$$

*If*

$$\rho(u) = \frac{\pi(u)}{\pi(g(u))}, \tag{2.9}$$

*then*

$$\eta \le a(u) \le \rho(u) \le \sup_{w \sim u} a(w) \le 1 - \eta. \tag{2.10}$$

*Proof.* The first inequality in (2.7) follows from the definition. For the second inequality in (2.7), if the second line of (2.6) applies, then $\pi(u) = \eta \sup_{v \succ u} q(v) \le \sup_{v \sim u} q(v)$; the other case is immediate.

Let $u \sim v$. By symmetry, it is enough to show that $\pi(v) \ge \eta \pi(u)$. Suppose first that $\pi(u) = q(u)$. If $u \succ v$, then

$$\pi(v) = \eta \sup_{w \succ v} q(w) \ge \eta q(u) = \eta \pi(u).$$

If $u \not\succ v$, then $q(u) \le \eta^{-1} q(v)$, and hence

$$\pi(v) \ge q(v) \ge \eta q(u) = \eta \pi(u).$$

Suppose now that $\pi(u) > q(u)$. Then $\pi(u) = \eta \sup_{w \succ u} q(w)$. For every $w \succ u$, the relation $w \sim u \sim v$ and (2.1) give $g(w) \sim g(v)$. Using (2.5) and $\eta \le a(w) \le 1 - \eta$, we obtain

$$q(w) = \pi(g(w)) a(w) \le \pi(g(w)) \le \eta^{-1} \pi(g(v)) \le \eta^{-2} q(v).$$

Taking the supremum over $w \succ u$ gives

$$\pi(u) = \eta \sup_{w \succ u} q(w) \le \eta^{-1} q(v) \le \eta^{-1} \pi(v).$$

This proves (2.8).

Since $q(u) = \pi(g(u)) a(u)$, the lower bounds in (2.10) follow from (2.7). If $\pi(u) = q(u)$, the upper bounds are immediate. Otherwise, for $v \succ u$ we have $g(v) \sim g(u)$, and therefore

$$\eta q(v) = \eta \pi(g(v)) a(v) \le \pi(g(u)) a(v).$$

Taking the supremum proves the upper bounds in (2.10). □

Lemma 2.1 shows that the recursive construction can be continued at every level and yields the following product formula.

For $u \in V_n$ and $0 \le j \le n$, let $g(u)_j \in V_j$ denote the unique level-$j$ vertex on the parent chain from $(o,0)$ to $u$. We call it the *level-$j$ ancestor* of $u$. Thus $g(g(u)_{j+1}) = g(u)_j$, $g(u)_0 = (o,0)$, and $g(u)_n = u$. By (2.9) and (2.10),

$$\pi(u) = \prod_{j=1}^{n} \rho(g(u)_j), \qquad (1-\eta)^{-1} \le \frac{\pi(g(u))}{\pi(u)} \le \eta^{-1}. \tag{2.11}$$

2.2. **The chain lower bound.** We first prove that every horizontal chain crossing the annulus between radii $\alpha^{n-1}$ and $2\alpha^{n-1}$ has total $\rho$-cost at least 1.

**Lemma 2.2.** *Let $v \in V_{n-1}$ and let $u_0 \sim \cdots \sim u_N$ be a level-$n$ chain satisfying* (2.2)*. Then*

$$\sum_{i=1}^{N} \inf\{\rho(w) : w \sim u_{i-1} \text{ or } w \sim u_i\} \ge 1. \tag{2.12}$$

*Proof.* Writing $u_i = (x_i, n)$ and $v = (y, n-1)$, the two conditions in (2.2) give points $\xi_0 \in B(y, \alpha^{n-1}) \cap B(x_0, 4\alpha^n)$ and $\xi_N \in (X \setminus B(y, 2\alpha^{n-1})) \cap B(x_N, 4\alpha^n)$. Thus

$$d(x_0, x_N) \ge d(y, \xi_N) - d(y, \xi_0) - d(\xi_0, x_0) - d(\xi_N, x_N) > (1 - 8\alpha)\alpha^{n-1} > 8\alpha^n,$$

where the last inequality follows from $\alpha < 10^{-3}$. Hence the chain has at least two edges. By (2.10), it is enough to prove

$$\sum_{i=1}^{N} \inf\{a(w) : w \sim u_{i-1} \text{ or } w \sim u_i\} \ge 1. \tag{2.13}$$

If one of the infima in (2.13) equals $1-\eta$, then that term together with any other term contributes at least $(1-\eta)+\eta = 1$. Suppose, therefore, that every infimum is strictly less than $1-\eta$. Fix $i$ and set

$$s_i = \sigma(u_{i-1}) + \sigma(u_i).$$

For every $w$ horizontally adjacent to $u_{i-1}$ or $u_i$, both $u_{i-1}$ and $u_i$ lie in the two-step horizontal neighborhood of $w$. Consequently,

$$2 \sup_{w \sim z \sim z'} \sigma(z') \ge 2 \max\{\sigma(u_{i-1}), \sigma(u_i)\} \ge s_i.$$

Thus $a(w) \ge \min\{1-\eta, s_i\}$. The assumption on the corresponding infimum forces $s_i < 1-\eta$, and hence that infimum is at least $s_i$. Summing in $i$ gives

$$\begin{aligned}
\sum_{i=1}^{N} \inf\{a(w) : w \sim u_{i-1} \text{ or } w \sim u_i\} &\ge \sum_{i=1}^{N} s_i \\
&= \sigma(u_0) + 2\sum_{i=1}^{N-1} \sigma(u_i) + \sigma(u_N) \\
&\ge \sum_{i=0}^{N} \sigma(u_i) \ge 1,
\end{aligned}$$

where the last inequality is (2.3). This proves (2.12). □

For vertices $u \in V_r$ and $v \in V_s$, let $k(u,v)$ be the largest integer $0 \le j \le \min\{r,s\}$ for which their level-$j$ ancestors are horizontal neighbors, and put

$$\Pi(u,v) = \max\{\pi(g(u)_{k(u,v)}), \pi(g(v)_{k(u,v)})\}. \tag{2.14}$$

The next lemma, whose proof follows [MT26a, Lemma 5.4], bounds the ancestor weight $\Pi$ from (2.14) by a constant multiple of every path cost. This estimate is used to construct the chain metric in Subsection 2.3.

**Lemma 2.3.** *There exists $c_\eta > 0$ with the following property. If $P = (u_0, \dots, u_m)$ is a finite path consisting of horizontal and vertical edges, then*

$$\sum_{i=0}^{m} \pi(u_i) \geq c_\eta \Pi(u_0, u_m).$$

*Proof.* For $u \in V_n$, set

$$\pi_*(u) = \inf_{w \sim u} \pi(w).$$

Since $u \sim u$, (2.8) gives

$$\eta\pi(u) \leq \pi_*(u) \leq \pi(u). \tag{2.15}$$

For a horizontal edge $u \sim v$, assign the length

$$\ell(uv) = \min\{\pi_*(u), \pi_*(v)\},$$

and for a vertical edge $g(u)u$, assign the length

$$\ell(g(u)u) = \eta^{-4}\pi_*(u).$$

For a finite path $R = (v_0, \dots, v_M)$, write $\ell(R) = \sum_{i=1}^{M} \ell(v_{i-1}v_i)$.

For each edge of $P$, choose an endpoint at the higher level if the edge is vertical, and either endpoint if it is horizontal. For a horizontal edge, its length is at most the weight of either chosen endpoint by (2.15); for a vertical edge $g(u)u$, its length is at most $\eta^{-4}\pi(u)$. Each occurrence $u_i$ in the path can be chosen only for the edge entering it and the edge leaving it, so each occurrence is chosen at most twice. Therefore

$$\ell(P) \leq 2\eta^{-4} \sum_{i=0}^{m} \pi(u_i), \qquad \sum_{i=0}^{m} \pi(u_i) \geq \frac{\eta^4}{2}\ell(P). \tag{2.16}$$

Let $\Gamma(v)$ be the family of level-$n$ horizontal chains satisfying (2.2) relative to $v \in V_{n-1}$.

*Claim 1. If $R \in \Gamma(v)$, then $\ell(R) \geq \pi_*(v)$.*

Indeed, consider the initial subchain ending at the first exit from the relevant annulus. For each edge $u_{i-1} \sim u_i$ of this subchain and each $w$ horizontally adjacent to $u_{i-1}$ or $u_i$, the parent $g(w)$ is horizontally adjacent to $v$. Choose $u_j \in \{u_{i-1}, u_i\}$ with $d(u_j, v) < 2\alpha^{n-1}$, and choose $u_k \in \{u_{i-1}, u_i\}$ with $w \sim u_k$. Then

$$d(g(w), v) \leq d(g(w), w) + d(w, u_k) + d(u_k, u_j) + d(u_j, v) < (3 + 16\alpha)\alpha^{n-1} < 8\alpha^{n-1}.$$

Hence

$$\pi(w) = \pi(g(w))\rho(w) \geq \pi_*(v)\rho(w),$$

and therefore

$$\ell(u_{i-1}u_i) \geq \pi_*(v) \inf\{\rho(w) : w \sim u_{i-1} \text{ or } w \sim u_i\}.$$

Summing and applying Lemma 2.2 proves Claim 1. □

*Claim 2. Suppose that a subpath $z_0, z_1, \dots, z_M$ has endpoints in $V_{h-1}$ and all its interior vertices in $V_h$. Then there is a horizontal path $R \subset V_{h-1}$ from $z_0$ to $z_M$ such that*

$$\ell(R) \leq \ell(z_1, \dots, z_M). \tag{2.17}$$

If $z_0 \sim z_M$, take $R = (z_0, z_M)$. Since $g(z_{M-1}) = z_M$, (2.15) and (2.10) give

$$\ell(z_0 z_M) \leq \pi(z_M) \leq \eta^{-1}\pi(z_{M-1}) \leq \eta^{-2}\pi_*(z_{M-1}) \leq \ell(z_{M-1}z_M).$$

Assume $z_0 \not\sim z_M$. Set $i_0 = 1$ and $b_0 = z_0 = g(z_1)$. Whenever $b_r = g(z_{i_r}) \not\sim z_M$, choose the least $i_{r+1} > i_r$ such that

$$(z_{i_r}, \dots, z_{i_{r+1}}) \in \Gamma(b_r),$$

and put $b_{r+1} = g(z_{i_{r+1}})$. This index exists and satisfies $i_{r+1} \le M-1$ because $g(z_{M-1}) = z_M$. Its minimality gives $d(b_r, z_{i_{r+1}-1}) < 2\alpha^{h-1}$, and hence

$$d(b_r, b_{r+1}) \le d(b_r, z_{i_{r+1}-1}) + d(z_{i_{r+1}-1}, z_{i_{r+1}}) + d(z_{i_{r+1}}, b_{r+1}) < (3+8\alpha)\alpha^{h-1} < 8\alpha^{h-1}.$$

Thus $b_r \sim b_{r+1}$, and Claim 1 yields

$$\ell(b_r b_{r+1}) \le \pi_*(b_r) \le \ell(z_{i_r}, \dots, z_{i_{r+1}}). \tag{2.18}$$

The indices increase strictly, so after finitely many steps $b_L \sim z_M$. The selected subchains have disjoint edge sets, and the first case bounds the last edge $b_L z_M$ by $\ell(z_{M-1}z_M)$. Thus $R = (b_0, \dots, b_L, z_M)$ satisfies (2.17) by summing (2.18) over the disjoint subchains. Since $b_0 = z_0$, this proves Claim 2. □

For $n \ge 0$, let $\mathcal{G}_{\le n}$ denote the subgraph induced by $\bigcup_{j=0}^{n} V_j$.

*Claim 3. Suppose that $x \in V_r$, $y \in V_s$, and $r \le s$. Every finite path from $x$ to $y$ can be replaced, without increasing its $\ell$-length, by a path in $\mathcal{G}_{\le r}$ from $x$ to $g(y)_r$. Moreover, if $x, y \in V_n$, $x \not\sim y$, and $g(x) \not\sim g(y)$, then it can be replaced by a path in $\mathcal{G}_{\le n-1}$ from $g(x)$ to $g(y)$.*

To prove the first assertion, let $H$ be the largest level visited by the path. If $H > s$, every maximal subchain of level-$H$ vertices lies in a complete block whose endpoints are in $V_{H-1}$. Apply Claim 2 to all such blocks. The resulting path does not meet $V_H$ and is no longer than the original path. Since the path is finite, iteration reduces it to $\mathcal{G}_{\le s}$.

If $r = s$, this proves the first assertion. Suppose $r < s$. After lowering all complete level-$s$ blocks, the vertices of the path that remain in $V_s$, if any, form the terminal subchain

$$z_0, z_1, \dots, z_M = y, \qquad z_0 \in V_{s-1}, \quad z_1, \dots, z_M \in V_s.$$

Reverse this terminal subchain and temporarily add the parent $g(y)$ at its beginning:

$$g(y),\ y = z_M,\ z_{M-1}, \dots, z_1,\ z_0.$$

This is a complete level-$s$ block. By Claim 2, it can be replaced by a horizontal path $R \subset V_{s-1}$ from $g(y)$ to $z_0$ satisfying

$$\ell(R) \le \ell(y, z_{M-1}, \dots, z_1, z_0).$$

Reverse $R$ and append it to the part of the original path ending at $z_0$. This replaces the terminal level-$s$ subchain by a path ending at $g(y)$, without increasing length. Repeating at levels $s-1, \dots, r+1$ gives a path in $\mathcal{G}_{\le r}$ from $x$ to $g(y)_r$.

For the second assertion, first apply the first assertion so that the path lies in $\mathcal{G}_{\le n}$. If it meets $V_{n-1}$, let $a$ and $b$ be its first and last vertices in $V_{n-1}$. Apply the first assertion to the reverse of the initial segment from $x$ to $a$, to the middle segment from $a$ to $b$, and to the terminal segment from $b$ to $y$. After reversing the first replacement and concatenating, we obtain the required path from $g(x)$ to $g(y)$ in $\mathcal{G}_{\le n-1}$.

It remains to treat the case in which the path never meets $V_{n-1}$. It is then a horizontal chain

$$x = x_0 \sim x_1 \sim \cdots \sim x_M = y$$

in $V_n$. Put $p_i = g(x_i)$. Starting with $j_0 = 0$, choose $j_{q+1} > j_q$ minimally so that

$$(x_{j_q}, \dots, x_{j_{q+1}}) \in \Gamma(p_{j_q}),$$

and stop when the remaining chain contains no such crossing. Since $p_0 \not\sim p_M$, at least one crossing is selected. The argument leading to (2.18) gives, for every selected crossing,

$$p_{j_q} \sim p_{j_{q+1}}, \qquad \ell(p_{j_q} p_{j_{q+1}}) \le \ell(x_{j_q}, \dots, x_{j_{q+1}}).$$

Let $0 = j_0 < \cdots < j_L$ be the selected indices. The minimality of the last crossing gives

$$d(p_{j_{L-1}}, x_{j_L - 1}) < 2\alpha^{n-1},$$

whereas the absence of a crossing after $j_L$ gives $d(p_{j_L}, x_M) < 2\alpha^{n-1}$. Therefore

$$\begin{aligned} d(p_{j_{L-1}}, p_M) &\le d(p_{j_{L-1}}, x_{j_L - 1}) + d(x_{j_L - 1}, x_{j_L}) + d(x_{j_L}, p_{j_L}) + d(p_{j_L}, x_M) + d(x_M, p_M) \\ &< 2\alpha^{n-1} + 8\alpha^n + \alpha^{n-1} + 2\alpha^{n-1} + \alpha^{n-1} \\ &= (6 + 8\alpha)\alpha^{n-1} < 8\alpha^{n-1}. \end{aligned}$$

Thus $p_{j_{L-1}} \sim p_M$. By Claim 1, the last selected subchain has length at least $\pi_*(p_{j_{L-1}})$, which is at least the length of the last horizontal edge:

$$\ell(p_{j_{L-1}} p_M) \le \pi_*(p_{j_{L-1}}) \le \ell(x_{j_{L-1}}, \dots, x_{j_L}).$$

The selected subchains have disjoint edge sets, so

$$p_0 \sim p_{j_1} \sim \cdots \sim p_{j_{L-1}} \sim p_M$$

is no longer than the original chain. This proves Claim 3. $\square$

We now finish the proof. The case $m = 0$ is immediate, so assume $m \ge 1$. After interchanging the endpoints if necessary, write $u_0 \in V_r$, $u_m \in V_s$, with $r \le s$, and put $k = k(u_0, u_m)$. Claim 3 first gives a path no longer than $P$ from $u_0$ to $g(u_m)_r$ in $\mathcal{G}_{\le r}$.

Suppose $k < r$. Repeatedly apply the second assertion of Claim 3 until the endpoints have reached level $k + 1$. We obtain a path $R$ no longer than $P$, with endpoints

$$x = g(u_0)_{k+1}, \qquad y = g(u_m)_{k+1},$$

such that $x \not\sim y$ and $g(x) \sim g(y)$. By Lemma 2.1 and (2.15), every edge incident to a vertex $z$ has length at least $\eta^2 \pi(z)$. Indeed, if $zw$ is horizontal, then

$$\pi_*(z) \ge \eta \pi(z), \qquad \pi_*(w) \ge \eta \pi(w) \ge \eta^2 \pi(z),$$

and therefore $\ell(zw) \ge \eta^2 \pi(z)$. If $w = g(z)$, then

$$\ell(zw) = \eta^{-4} \pi_*(z) \ge \eta^{-3} \pi(z) \ge \eta^2 \pi(z).$$

Finally, if $g(w) = z$, then

$$\ell(zw) = \eta^{-4} \pi_*(w) \ge \eta^{-3} \pi(w) \ge \eta^{-2} \pi(z) \ge \eta^2 \pi(z).$$

Applying this estimate to the first edge of $R$ and using Lemma 2.1 gives

$$\begin{aligned} \ell(P) \ge \ell(R) &\ge \eta^2 \pi(x) \ge \eta^3 \pi(g(x)) \\ &\ge \eta^4 \max\{\pi(g(x)), \pi(g(y))\} = \eta^4 \Pi(u_0, u_m). \end{aligned}$$

If $k = r$, applying the same estimate to the first edge of the original path and using $u_0 \sim g(u_m)_r$ together with Lemma 2.1 gives

$$\ell(P) \ge \eta^2 \pi(u_0) \ge \eta^3 \max\{\pi(u_0), \pi(g(u_m)_r)\} = \eta^3 \Pi(u_0, u_m).$$

This is stronger than the bound required in the case $k < r$. Combining both cases with (2.16) gives

$$\sum_{i=0}^{m} \pi(u_i) \ge \frac{\eta^8}{2} \Pi(u_0, u_m).$$

Thus the lemma holds with $c_\eta = \eta^8/2$. □

2.3. **The chain metric.** We now convert the vertex weights into a scale function on $X$ and then take the associated intrinsic chain metric. Lemma 2.3 provides the lower bound that prevents this metric from degenerating.

For distinct $x, y \in X$, define

$$\kappa(x,y) = \max\{n \geq 0 : \text{some } (z,n) \in V_n \text{ satisfies } x, y \in B(z, 2\alpha^n)\}, \tag{2.19}$$

$$Q(x,y) = \sup\{\pi((z,n)) : (z,n) \in V_{\kappa(x,y)},\ x, y \in B(z, 2\alpha^{\kappa(x,y)})\}. \tag{2.20}$$

The set of levels in (2.19) is nonempty and finite when $x \neq y$. Define

$$D_\sigma(x,y) = \inf_{x=x_0,\dots,x_m=y} \sum_{i=1}^{m} Q(x_{i-1}, x_i), \qquad D_\sigma(x,x) = 0. \tag{2.21}$$

The next proposition constructs the metric on $X$ and proves the diameter estimate used in Section 3. It adapts [MT26b, Lemma 2.3] to the present filling, where $V_n$ need not be finite. Unlike the compact setting of [MT26a, Corollary 5.6], our application uses the estimate only at $\mu$-almost every point, with the weights chosen by random partitions.

**Proposition 2.4.** *The function $D_\sigma$ is a metric, and* $\mathrm{id} : (X,d) \to (X, D_\sigma)$ *is quasisymmetric. Moreover, there is a constant $C > 0$, depending only on $\alpha$ and $\eta$, such that, for every $x \in X$ and $n \geq 1$,*

$$\operatorname{diam}_{D_\sigma} B_d(x, \tfrac{1}{4}\alpha^n) \leq C \prod_{j=1}^{n} \tau_j(x), \tag{2.22}$$

*where*

$$\tau_j(x) = \begin{cases} \eta, & \sigma(z,j) = 0 \text{ for every } z \in A_j \text{ with } d(x,z) \leq 30\alpha^j, \\ 1-\eta, & \text{otherwise.} \end{cases} \tag{2.23}$$

*Proof. Claim 1. The function $D_\sigma$ is a metric. Moreover, there is a constant $C_1 \geq 1$, depending only on $\eta$, such that*

$$C_1^{-1} Q(x,y) \leq D_\sigma(x,y) \leq Q(x,y) \qquad (x, y \in X,\ x \neq y). \tag{2.24}$$

We first prove the following elementary scale estimate:

$$\alpha^{\kappa(x,y)+1} < d(x,y) < 4\alpha^{\kappa(x,y)}. \tag{2.25}$$

The upper bound follows from the definition of $\kappa(x,y)$. For the lower bound, put $\kappa = \kappa(x,y)$. If $d(x,y) \leq \alpha^{\kappa+1}$, choose $z' \in A_{\kappa+1}$ with $d(x,z') < \alpha^{\kappa+1}$. Then $x, y \in B(z', 2\alpha^{\kappa+1})$, contradicting the maximality of $\kappa$.

The upper bound in (2.24) follows by using the one-edge chain $(x,y)$ in (2.21). For the lower bound, fix a finite chain $x = x_0, \dots, x_m = y$ with distinct consecutive points, and put $\kappa_i = \kappa(x_{i-1}, x_i)$. Choose

$$L > \max\{\kappa(x,y), \kappa_1, \dots, \kappa_m\}.$$

For each $i$, choose $\xi_i \in A_L$ with $d(x_i, \xi_i) < \alpha^L$, and set $u_i = (\xi_i, L)$. Also choose a vertex $z_i \in V_{\kappa_i}$ occurring in (2.20). Let $a_i^- = g(u_{i-1})_{\kappa_i}$ and $a_i^+ = g(u_i)_{\kappa_i}$. Since $z_i$ and $u_{i-1}$ are both

close to $x_{i-1}$,

$$d(a_i^-, z_i) < \sum_{j=\kappa_i}^{L-1} \alpha^j + \alpha^L + 2\alpha^{\kappa_i} < \left(\frac{1}{1-\alpha} + \alpha + 2\right)\alpha^{\kappa_i} < 4\alpha^{\kappa_i}.$$

The same estimate holds for $a_i^+$. Thus $a_i^- \sim z_i \sim a_i^+$.

Construct a graph path $P_i$ from $u_{i-1}$ to $u_i$ by following the parent chains to $a_i^-$ and $a_i^+$ and joining these ancestors through $z_i$. Equations (2.8) and (2.11) give

$$\sum_{v \text{ on } P_i} \pi(v) \le C_2 \pi(z_i), \qquad C_2 = 1 + 2\eta^{-2}.$$

Indeed, (2.11) shows that the weight decreases by a factor of at most $1-\eta$ at each step along either vertical leg. Hence its total weight is at most $\pi(a_i^\pm)\sum_{j=0}^{\infty}(1-\eta)^j = \eta^{-1}\pi(a_i^\pm) \le \eta^{-2}\pi(z_i)$. Since $\pi(z_i) \le Q(x_{i-1}, x_i)$, concatenating the paths $P_i$ and applying Lemma 2.3 yields

$$\sum_{i=1}^{m} Q(x_{i-1}, x_i) \ge C_2^{-1} c_\eta \Pi(u_0, u_m). \tag{2.26}$$

It remains to compare $\Pi(u_0, u_m)$ with $Q(x, y)$. Put $\kappa = \kappa(x, y)$ and choose a vertex $z \in V_\kappa$ occurring in (2.20) such that $\pi(z) > Q(x, y)/2$. Repeating the ancestor-center estimate used for $a_i^-$ with $\kappa_i$ replaced by $\kappa$ shows that $g(u_0)_\kappa \sim z \sim g(u_m)_\kappa$. Hence, if $h = k(u_0, u_m)$, then

$$h \ge \max\{\kappa - 1, 0\}, \tag{2.27}$$

because, when $\kappa \ge 1$, applying (2.1) to $g(u_0)_\kappa \sim z \sim g(u_m)_\kappa$ gives $g(u_0)_{\kappa-1} \sim g(u_m)_{\kappa-1}$; when $\kappa = 0$, the bound is immediate.

Conversely, the level-$h$ ancestors of $u_0$ and $u_m$ are horizontal neighbors and hence less than $8\alpha^h$ apart, while $x$ and $y$ are each within $\alpha^h/(1-\alpha)$ of the corresponding ancestor; hence the triangle inequality gives

$$d(x, y) < \left(8 + \frac{2}{1-\alpha}\right)\alpha^h < 12\alpha^h.$$

Together with (2.25), this forces

$$h \le \kappa + 1. \tag{2.28}$$

Indeed, if $h \ge \kappa + 2$, then $12\alpha^h \le 12\alpha^{\kappa+2} < \alpha^{\kappa+1}$ because $\alpha < 10^{-3}$, a contradiction. From (2.27)–(2.28) and (2.11),

$$\pi(g(u_0)_h) \ge \eta\pi(g(u_0)_\kappa).$$

Since $g(u_0)_\kappa \sim z$, it follows that

$$\Pi(u_0, u_m) \ge \pi(g(u_0)_h) \ge \eta^2\pi(z) > \frac{\eta^2}{2} Q(x, y).$$

Combining this with (2.26) gives

$$\sum_{i=1}^{m} Q(x_{i-1}, x_i) \ge \frac{c_\eta \eta^2}{2C_2} Q(x, y).$$

Taking the infimum over all chains proves (2.24) with $C_1 = 2C_2/(c_\eta\eta^2)$. In particular, $D_\sigma(x, y) > 0$ when $x \ne y$.

Symmetry follows from the symmetry of $Q$, and concatenating chains gives the triangle inequality. Thus $D_\sigma$ is a metric, and Claim 1 follows. $\square$

*Claim 2. The identity* $\mathrm{id} : (X,d) \to (X, D_\sigma)$ *is quasisymmetric.*

Let $x, a, b$ be distinct, and write $\kappa_a = \kappa(x,a)$ and $\kappa_b = \kappa(x,b)$. Choose a vertex $z \in V_{\kappa_a}$ occurring in the definition of $Q(x,a)$ with $\pi(z) > Q(x,a)/2$, and choose any vertex $z' \in V_{\kappa_b}$ occurring in the definition of $Q(x,b)$. Then

$$\frac{Q(x,a)}{Q(x,b)} < 2\frac{\pi(z)}{\pi(z')}. \tag{2.29}$$

If $\kappa_a \geq \kappa_b$, then

$$\begin{aligned} d(g(z)_{\kappa_b}, z') &\leq d(g(z)_{\kappa_b}, z) + d(z,x) + d(x,z') \\ &< \sum_{j=\kappa_b}^{\kappa_a - 1} \alpha^j + 2\alpha^{\kappa_a} + 2\alpha^{\kappa_b} \\ &< \left(\frac{1}{1-\alpha} + 4\right)\alpha^{\kappa_b} < 8\alpha^{\kappa_b}. \end{aligned}$$

Thus $g(z)_{\kappa_b} \sim z'$. If $\kappa_a < \kappa_b$, the same calculation gives $z \sim g(z')_{\kappa_a}$. Equations (2.8), (2.29), and (2.11) now imply

$$\frac{Q(x,a)}{Q(x,b)} \leq 2\eta^{-1} \begin{cases} (1-\eta)^{\kappa_a - \kappa_b}, & \kappa_a \geq \kappa_b, \\ \eta^{-(\kappa_b - \kappa_a)}, & \kappa_a < \kappa_b. \end{cases} \tag{2.30}$$

Suppose that $d(x,a) \leq t\, d(x,b)$, and set

$$\beta = \frac{\log(1/(1-\eta))}{\log(1/\alpha)}, \qquad \gamma = \frac{\log(1/\eta)}{\log(1/\alpha)}.$$

By (2.25),

$$\alpha^{\kappa_a - \kappa_b} < \frac{4t}{\alpha}.$$

Consequently,

$$(1-\eta)^{\kappa_a - \kappa_b} \leq \left(\frac{4}{\alpha}\right)^{\beta} t^{\beta} \quad \text{if } \kappa_a \geq \kappa_b,$$

whereas

$$\eta^{-(\kappa_b - \kappa_a)} \leq \left(\frac{4}{\alpha}\right)^{\gamma} t^{\gamma} \quad \text{if } \kappa_a < \kappa_b.$$

Using Claim 1 and (2.30), we obtain

$$\frac{D_\sigma(x,a)}{D_\sigma(x,b)} \leq C_3(t^\beta + t^\gamma),$$

where

$$C_3 = 2C_1\eta^{-1} \max\left\{\left(\frac{4}{\alpha}\right)^{\beta}, \left(\frac{4}{\alpha}\right)^{\gamma}\right\}.$$

Since $t \mapsto C_3(t^\beta + t^\gamma)$ is a homeomorphism of $[0,\infty)$ onto itself, Claim 2 follows. □

*Claim 3. For every* $x \in X$ *and* $n \geq 1$, (2.22) *holds.*

Choose $u = (z,n) \in V_n$ with $d(x,z) < \alpha^n$. If $y, y' \in B_d(x, \alpha^n/4)$ are distinct, then

$$d(y,z) < \tfrac{5}{4}\alpha^n < 2\alpha^n,$$

and the same holds for $y'$. Hence $\kappa(y,y') \ge n$. Put $\kappa = \kappa(y,y')$, and let $\widetilde{u} = (w,\kappa)$ be any vertex occurring in the definition of $Q(y,y')$. Then

$$\begin{aligned} d(g(\widetilde{u})_n, u) &\le d(g(\widetilde{u})_n, w) + d(w,y) + d(y,x) + d(x,z) \\ &< \sum_{\ell=n}^{\kappa-1} \alpha^\ell + 2\alpha^\kappa + \tfrac{1}{4}\alpha^n + \alpha^n \\ &< \left(\frac{1}{1-\alpha} + \frac{13}{4}\right)\alpha^n < 8\alpha^n. \end{aligned}$$

Thus $g(\widetilde{u})_n \sim u$. By (2.8) and (2.11),

$$\pi(\widetilde{u}) \le (1-\eta)^{\kappa-n}\pi(g(\widetilde{u})_n) \le \eta^{-1}\pi(u).$$

Taking the supremum over all such $\widetilde{u}$ and using the upper bound in (2.24), we obtain

$$\operatorname{diam}_{D_\sigma} B_d(x, \tfrac{1}{4}\alpha^n) \le \eta^{-1}\pi(u). \tag{2.31}$$

For $1 \le j \le n$, the parent displacement estimates give

$$d(x, g(u)_j) < \sum_{\ell=j}^{n} \alpha^\ell < \frac{\alpha^j}{1-\alpha} < 2\alpha^j.$$

Suppose the first case of (2.23) holds at level $j$. If $g(u)_j \sim v \sim v_1 \sim v_2$, then

$$d(x, v_2) < 2\alpha^j + 3 \cdot 8\alpha^j = 26\alpha^j < 30\alpha^j.$$

Therefore $\sigma(v_2) = 0$. It follows from (2.4) that $a(v) = \eta$ for every $v \sim g(u)_j$, and hence

$$\rho(g(u)_j) = \eta = \tau_j(x)$$

by (2.10). In the second case of (2.23), the same bounds give $\rho(g(u)_j) \le 1 - \eta = \tau_j(x)$. Thus

$$\pi(u) = \prod_{j=1}^{n} \rho(g(u)_j) \le \prod_{j=1}^{n} \tau_j(x).$$

Combining this with (2.31) proves Claim 3 with $C = \eta^{-1}$. □

The proposition follows from Claims 1–3. □

## 3. Proof of the dimension-reduction theorem

In this section we prove Theorem 1.3. Subsection 3.1 relates Hausdorff dimension to local dimension and constructs random partitions with an explicit bound on the probability that a ball is not contained in the partition cell containing its center. Subsection 3.2 uses these partitions to choose the filling weights when the metric space $(X,d)$ is bounded and complete. Subsection 3.3 then removes the assumptions of boundedness and completeness.

### 3.1. **Local dimension and random partitions.**

This subsection proves two measure-theoretic lemmas used later, one concerning local dimension and the other concerning random partitions.

For a finite measure $\mu$ on $(X,d)$, the *lower and upper local dimensions* of $\mu$ at $x \in X$ are defined by

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\downarrow 0} \frac{\log \mu(B_d(x,r))}{\log r}, \qquad \overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r\downarrow 0} \frac{\log \mu(B_d(x,r))}{\log r}.$$

If these two quantities agree, their common value is called the *local dimension* of $\mu$ at $x$. These definitions are needed only on $\operatorname{spt}\mu$, which has full measure; their values off the support are irrelevant. We use the following relation between the lower local dimension and the Hausdorff dimension of a measure.

**Lemma 3.1.** *If $\mu$ is a finite measure on a separable metric space $(X,d)$, then*

$$\dim_{\mathrm{H}}^{d}\mu = \operatorname*{ess\,sup}_{x\in X} \underline{\dim}_{\mathrm{loc}}\mu(x), \tag{3.1}$$

*where the essential supremum is taken with respect to $\mu$.*

*Proof.* For measures on Euclidean spaces, (3.1) is standard; see [Fal97, Proposition 10.3]. The same proof works in a separable metric space. For $s$ above the right-hand side of (3.1), applying the metric $5r$-covering lemma [Hei01, Theorem 1.2] to balls satisfying $\mu(B(x,r)) \geq r^s$ gives the upper bound. For $s$ below the right-hand side, the definition of lower local dimension gives a set of positive measure on which $\mu(B(x,r)) \leq r^s$ uniformly for all sufficiently small $r$; the definition of Hausdorff measure then gives the lower bound. □

The following lemma constructs a random Borel partition of a separable metric space equipped with a probability measure that has full support. A *Borel partition* is a pairwise disjoint collection of Borel sets whose union is the whole space; a *random Borel partition* is a partition-valued random variable. We write $\mathcal{P}(x)$ for the cell containing $x$ and $\mathbb{P}$ for probability with respect to this random choice.

**Lemma 3.2.** *Let $(X,d)$ be a separable metric space and let $\mu$ be a Borel probability measure with full support. For every $\Delta > 0$ there is a random Borel partition $\mathcal{P}$ of $X$ whose cells have diameter at most $\Delta$ and such that, for $0 < t \leq \Delta/8$ and $x \in X$,*

$$\mathbb{P}[B_d(x,t) \not\subset \mathcal{P}(x)] \leq \frac{8t}{\Delta} \log \frac{\mu(B_d(x,\Delta))}{\mu(B_d(x,\Delta/8))}. \tag{3.2}$$

*Proof.* Choose $R$ uniformly from $[\Delta/4, \Delta/2]$. Independently of $R$, let $Z_1, Z_2, \ldots$ be independent samples with law $\mu$.

We first verify that the sequence $\{Z_i : i \geq 1\}$ is almost surely dense in $X$. Let $\{U_j : j \geq 1\}$ be a countable base consisting of nonempty open sets. Since $\mu$ has full support, $\mu(U_j) > 0$ for every $j \geq 1$. By the independence of the samples,

$$\begin{aligned}\mathbb{P}\big[Z_i \notin U_j \text{ for every } i \geq 1\big] &= \lim_{N\to\infty} \mathbb{P}\big[Z_1, \ldots, Z_N \notin U_j\big] \\ &= \lim_{N\to\infty} (1-\mu(U_j))^N = 0.\end{aligned}$$

It follows that, with probability one, every $U_j$ contains at least one sample $Z_i$. Thus $\{Z_i : i \geq 1\}$ is dense in $X$ almost surely.

Fix such a dense sequence $\{Z_i : i \geq 1\}$. For $x \in X$, define $I_R(x) = \min\{i \geq 1 : d(x, Z_i) < R\}$. For each $i \geq 1$, define $P_i = \{x \in X : I_R(x) = i\}$. Equivalently,

$$P_i = B(Z_i, R) \setminus \bigcup_{\ell=1}^{i-1} B(Z_\ell, R).$$

Thus every $P_i$ is Borel. Moreover, the sets $P_i$ are pairwise disjoint and their union is $X$. Define

$$\mathcal{P} = \{P_i : i \geq 1,\ P_i \neq \varnothing\}.$$

Consequently, $\mathcal{P}$ is a Borel partition of $X$ whose cells have diameter at most $\Delta$.

For $x \in X$, write $\mathcal{P}(x) = P_{I_R(x)}$ for the unique cell containing $x$. Fix $x$ and $t \le \Delta/8$, put $m(r) = \mu(B(x,r))$, and let $j = I_{R+t}(x)$. Suppose that $Z_j \in B(x, R-t)$. For every $y \in B(x,t)$,

$$d(y, Z_j) \le d(y,x) + d(x, Z_j) < t + (R-t) = R.$$

Thus $y \in B(Z_j, R)$. On the other hand, if $i < j$, then $Z_i \notin B(x, R+t)$. Therefore

$$d(y, Z_i) \ge d(x, Z_i) - d(x,y) > R + t - t = R,$$

so $y \notin B(Z_i, R)$. Hence $I_R(y) = j$ for every $y \in B(x,t)$. In particular, all points of $B(x,t)$ belong to the same partition cell as $x$, and $B(x,t) \subset \mathcal{P}(x)$. Note that

$$\begin{aligned}\mathbb{P}[Z_j \in B(x, R-t) \mid R] &= \sum_{i=1}^{\infty} \mathbb{P}\big[Z_1, \dots, Z_{i-1} \notin B(x, R+t), Z_i \in B(x, R-t) \mid R\big] \\ &= \sum_{i=1}^{\infty} (1 - m(R+t))^{i-1} m(R-t) = \frac{m(R-t)}{m(R+t)}.\end{aligned}$$

Therefore

$$\mathbb{P}[B(x,t) \not\subset \mathcal{P}(x) \mid R] \le 1 - \frac{m(R-t)}{m(R+t)} \le \log \frac{m(R+t)}{m(R-t)}.$$

If $F(r) = \log m(r)$, monotonicity gives

$$\begin{aligned}\int_{\Delta/4}^{\Delta/2} [F(r+t) - F(r-t)]\, dr &\le 2t[F(\Delta/2 + t) - F(\Delta/4 - t)] \\ &\le 2t \log \frac{m(\Delta)}{m(\Delta/8)}.\end{aligned}$$

Multiplication by the density $4/\Delta$ of $R$ proves (3.2). □

3.2. **The bounded complete case.** In this subsection we prove Theorem 1.3 when $(X,d)$ is bounded and complete. The main point is to arrange that most scales receive the smaller factor $\eta$, while the exceptional scales have sufficiently small upper density.

*Proof of Theorem 1.3 in the bounded complete case.* Assume first that $X$ is bounded and complete, and rescale so that $\operatorname{diam} X < 1$. Set $s_\mu = \dim_{\mathrm{H}}^d \mu < \infty$. Fix $p > 0$, $\varepsilon > 0$, and a constant $K > 40$. Choose a large integer $M$ such that, with $\alpha = M^{-1} < \alpha_0$, where $\alpha_0$ is the constant fixed in Subsection 2.1,

$$M \ge 32K, \qquad \delta_M := \frac{32K(s_\mu + \varepsilon)\log M}{M} < \frac{1}{2}. \tag{3.3}$$

At every level $n$, independently apply Lemma 3.2 with

$$\Delta_n = \frac{\alpha^{n-1}}{4} \tag{3.4}$$

and denote the partition by $\mathcal{P}_n$. For each vertex $(z,n) \in V_n$, define

$$\sigma(z,n) = \mathbf{1}_{\{B(z, 9\alpha^n) \not\subset \mathcal{P}_n(z)\}}. \tag{3.5}$$

This assignment is admissible. Indeed, suppose that $u_i = (x_i, n) \in V_n$ satisfies (2.2) and that $\sigma(u_i) = 0$ for every $0 \le i \le N$. Since $d(x_{i-1}, x_i) < 8\alpha^n < 9\alpha^n$, we have

$$x_i \in B(x_{i-1}, 9\alpha^n) \subset \mathcal{P}_n(x_{i-1}),$$

and hence $\mathcal{P}_n(x_i) = \mathcal{P}_n(x_{i-1})$. Thus all $x_i$ belong to one cell, so

$$d(x_0, x_N) \le \Delta_n = \frac{\alpha^{n-1}}{4}.$$

On the other hand, (2.2) gives

$$d(x_0, x_N) > \alpha^{n-1} - 8\alpha^n = \left(1 - \frac{8}{M}\right)\alpha^{n-1} > \frac{\alpha^{n-1}}{4}$$

The last inequality follows from (3.3). This is a contradiction.

As in the proof of Lemma 3.2, let $R_n$ and $Z_{n,1}, Z_{n,2}, \dots$ be the random radius and samples used to construct $\mathcal{P}_n$. For $x \in X$, let $I_n(x)$ be the first $i$ such that $Z_{n,i} \in B(x, R_n + K\alpha^n)$ and set

$$J_n(x) = \mathbf{1}_{\{d(x, Z_{n,I_n(x)}) \geq R_n - K\alpha^n\}}.$$

The events $\{I_n(x) = i\}$ are Borel in the sample variables and $x$, so $J_n(x)$ is jointly measurable. It is related to the factor $\tau_n(x)$ in (2.23) through the implication

$$J_n(x) = 0 \quad \Longrightarrow \quad \tau_n(x) = \eta. \tag{3.6}$$

Indeed, if $J_n(x) = 0$, then the proof of Lemma 3.2 gives $B(x, K\alpha^n) \subset \mathcal{P}_n(x)$. If $(z, n) \in V_n$ satisfies $d(x, z) \leq 30\alpha^n$, then, since $K > 40$,

$$B(z, 9\alpha^n) \subset B(x, 39\alpha^n) \subset B(x, K\alpha^n) \subset \mathcal{P}_n(x).$$

In particular, $z \in \mathcal{P}_n(x)$, so $\mathcal{P}_n(z) = \mathcal{P}_n(x)$ and $B(z, 9\alpha^n) \subset \mathcal{P}_n(z)$. It follows from (3.5) that

$$\sigma(z, n) = 0 \qquad \text{whenever } d(x, z) \leq 30\alpha^n.$$

Hence (3.6) holds, and therefore

$$\#\{1 \leq j \leq n : \tau_j(x) = \eta\} \geq n - \sum_{j=1}^{n} J_j(x). \tag{3.7}$$

Writing $\mathbb{E}$ for expectation over all the random partitions, note that $M \geq 32K$ gives $K\alpha^n \leq \Delta_n/8$. The calculation in the proof of Lemma 3.2, with $t = K\alpha^n$, gives

$$p_n(x) := \mathbb{E}J_n(x) \leq \frac{32K}{M} \log \frac{\mu(B(x, \Delta_n))}{\mu(B(x, \Delta_n/8))}. \tag{3.8}$$

By (3.4), $\Delta_{n+1} = \Delta_n/M \leq \Delta_n/8$. Summing (3.8) and telescoping therefore give

$$\sum_{j=1}^{n} p_j(x) \leq \frac{32K}{M} \log \frac{\mu(B(x, \Delta_1))}{\mu(B(x, \Delta_{n+1}))}. \tag{3.9}$$

For fixed $x$, the variables $J_n(x)$ are independent and bounded. Writing Var for variance, we have $\sum_n \mathrm{Var}(J_n(x))/n^2 < \infty$, so Kolmogorov's strong law gives

$$\frac{1}{n}\sum_{j=1}^{n}[J_j(x) - p_j(x)] \longrightarrow 0$$

with probability one. By joint measurability and Fubini's theorem, there is a deterministic choice of all partitions for which

$$\frac{1}{n}\sum_{j=1}^{n}[J_j(x) - p_j(x)] \longrightarrow 0 \quad \text{for } \mu\text{-almost every } x. \tag{3.10}$$

Fix such an $x$ with $\underline{\dim}_{\mathrm{loc}}\mu(x) \leq s_\mu$, as holds almost everywhere by Lemma 3.1. There are arbitrarily small radii $r$ for which $\mu(B(x, r)) \geq r^{s_\mu + \varepsilon}$. For each such $r$, choose $n$ so that $\frac{1}{4}\alpha^{n+1} < r \leq \frac{1}{4}\alpha^n$. Hence there are arbitrarily large $n$ such that

$$\mu(B(x, \tfrac{1}{4}\alpha^n)) \geq \alpha^{s_\mu + \varepsilon}(\tfrac{1}{4}\alpha^n)^{s_\mu + \varepsilon}. \tag{3.11}$$

Since $\frac{1}{4}\alpha^n = \Delta_{n+1}$, equations (3.9) and (3.10) give, along this subsequence, with $o(1)$ denoting a quantity that tends to zero,

$$\frac{1}{n}\sum_{j=1}^{n} J_j(x) \le \delta_M + o(1). \tag{3.12}$$

Apply Proposition 2.4. By (3.7) and (3.12), at least $(1-\delta_M - o(1))n$ scales contribute the factor $\eta$, and every remaining scale contributes at most $1-\eta < 1$. Proposition 2.4 gives

$$B_d(x, \tfrac{1}{4}\alpha^n) \subset B_{D_\sigma}\Big(x, 2C\eta^{(1-\delta_M - o(1))n}\Big) \tag{3.13}$$

along the same subsequence. Write $\underline{\dim}_{\mathrm{loc}}^{D_\sigma}\mu$ for the lower local dimension computed using $D_\sigma$-balls. Combining (3.11) and (3.13) yields

$$\underline{\dim}_{\mathrm{loc}}^{D_\sigma}\mu(x) \le \frac{(s_\mu + \varepsilon)\log M}{(1-\delta_M)\log(1/\eta)} \quad \text{for } \mu\text{-almost every } x. \tag{3.14}$$

Lemma 3.1, applied to (3.14), gives the same upper bound for $\dim_{\mathrm{H}}^{D_\sigma}\mu$. After $M$ has been fixed, $\eta$ may be chosen arbitrarily small, so the right side is less than the prescribed $p$. □

3.3. **The general case.** The preceding subsection proves the theorem for bounded complete spaces. We now pass first to complete unbounded spaces by *sphericalization*, which adds one point at infinity, and then to incomplete spaces by taking a completion.

For positive quantities $A$ and $B$, we write $A \asymp B$ if there is a constant $C \ge 1$, independent of the relevant variables, such that $C^{-1}B \le A \le CB$. Two metrics $d_1$ and $d_2$ are *bi-Lipschitz equivalent* on a set $E$ if $d_1(x,y) \asymp d_2(x,y)$ for all $x, y \in E$.

**Lemma 3.3.** *Let $(X,d)$ be complete, unbounded, and separable, and fix $o \in X$. There is a bounded complete metric $\widehat{d}$ on $\widehat{X} = X \cup \{\infty\}$ satisfying*

$$\widehat{d}(x,y) \asymp \frac{d(x,y)}{(1+d(x,o))(1+d(y,o))}, \tag{3.15}$$

$$\widehat{d}(x,\infty) \asymp \frac{1}{1+d(x,o)}. \tag{3.16}$$

*Suppose $\widehat{D}$ is another metric on $\widehat{X}$ for which* $\mathrm{id} : (\widehat{X},\widehat{d}) \to (\widehat{X},\widehat{D})$ *is quasisymmetric. Then there exists a metric $D$ on $X$ such that*

$$D(x,y) \asymp \frac{\widehat{D}(x,y)}{\widehat{D}(x,\infty)\widehat{D}(y,\infty)}, \qquad x,y \in X, \tag{3.17}$$

*and* $\mathrm{id} : (X,d) \longrightarrow (X,D)$ *is quasisymmetric. Moreover, $D$ and $\widehat{D}$ are bi-Lipschitz equivalent on every set $E \subset X$ on which $\widehat{D}(\,\cdot\,,\infty)$ is bounded away from zero.*

*Proof.* For $x \in X$, let $s(x) = 1 + d(x,o)$. Write $s(\infty) = \infty$ and $1/s(\infty) = 0$. Define a symmetric kernel $\widehat{q}$ on $\widehat{X} = X \cup \{\infty\}$ by

$$\widehat{q}(x,y) = \frac{d(x,y)}{s(x)s(y)}, \qquad \widehat{q}(x,\infty) = \widehat{q}(\infty,x) = \frac{1}{s(x)}, \qquad \widehat{q}(\infty,\infty) = 0$$

for $x, y \in X$. For $x, y \in \widehat{X}$, define

$$\widehat{d}(x,y) = \inf\left\{\sum_{i=1}^{N} \widehat{q}(x_{i-1},x_i) : N \ge 1,\ x_0 = x,\ x_N = y,\ x_i \in \widehat{X}\right\}. \tag{3.18}$$

Thus (3.18) is the infimum of the $\widehat{q}$-lengths of all finite chains in $\widehat{X}$ joining $x$ to $y$. To prove (3.15)–(3.16), it suffices to verify that, for $x, y \in \widehat{X}$,

$$\widehat{d}(x,y) \geq \frac{1}{4}\widehat{q}(x,y).$$

If $x = y$, this inequality is trivial. If $x \neq y$, symmetry allows us to assume $s(x) \leq s(y)$, and then $x \in X$. Consider a chain from $x$ to $y$. Since

$$\left|\frac{1}{s(u)} - \frac{1}{s(v)}\right| \leq \widehat{q}(u,v), \tag{3.19}$$

if the chain first reaches a point $z$ with $s(z) \geq 2s(x)$, then (3.19) gives

$$\sum_i \widehat{q}(x_{i-1}, x_i) \geq \frac{1}{s(x)} - \frac{1}{s(z)} \geq \frac{1}{2s(x)} \geq \frac{1}{4}\widehat{q}(x,y).$$

Here we used $\widehat{q}(x,y) \leq 2/s(x)$. If the chain never reaches such a point, none of its vertices is $\infty$, all denominators are at most $4s(x)^2$, and the triangle inequality gives

$$\sum_i \widehat{q}(x_{i-1}, x_i) \geq \frac{d(x,y)}{4s(x)^2} \geq \frac{1}{4}\widehat{q}(x,y).$$

Taking the infimum over all chains yields $\widehat{d}(x,y) \geq \frac{1}{4}\widehat{q}(x,y)$. The definition gives symmetry and the triangle inequality, while the preceding lower bound separates distinct points; hence $\widehat{d}$ is a metric. It is bounded because $\widehat{q} \leq 2$. To check completeness, let $(x_j)$ be a $\widehat{d}$-Cauchy sequence. If $s(x_j)$ is unbounded along a subsequence, (3.16) and the Cauchy property imply $x_j \to \infty$. Otherwise $s(x_j)$ is eventually bounded, and (3.15) shows that $(x_j)$ is $d$-Cauchy. It then converges to some $x \in X$ because $(X,d)$ is complete. Since $\widehat{d}(x_j, x) \leq \widehat{q}(x_j, x) \leq d(x_j, x)$, it also converges to $x$ in $\widehat{d}$. Thus $(\widehat{X}, \widehat{d})$ is a bounded complete metric space.

For the reverse construction, let

$$r(x) = \widehat{D}(x, \infty), \qquad i(x,y) = \frac{\widehat{D}(x,y)}{r(x)r(y)}, \qquad x, y \in X,$$

and define $D$ to be the chain metric induced by $i$. The triangle inequality for $\widehat{D}$ gives

$$\left|\frac{1}{r(u)} - \frac{1}{r(v)}\right| \leq i(u,v).$$

Repeating the preceding chain argument, with $r$ in place of $s$, gives

$$\frac{1}{4}i(x,y) \leq D(x,y) \leq i(x,y),$$

and hence proves (3.17). By its definition, $D$ is symmetric and satisfies the triangle inequality, while the lower bound separates distinct points; hence $D$ is a metric.

We next prove that $\mathrm{id} : (X,d) \longrightarrow (X,D)$ is quasisymmetric. By [Bon11, Proposition 3.1(ii)], every quasisymmetric map satisfies the following four-point inequality for some homeomorphism $\Theta : [0,\infty) \to [0,\infty)$ such that

$$\frac{\widehat{D}(x,a)\widehat{D}(b,\infty)}{\widehat{D}(x,b)\widehat{D}(a,\infty)} \leq \Theta\left(\frac{\widehat{d}(x,a)\widehat{d}(b,\infty)}{\widehat{d}(x,b)\widehat{d}(a,\infty)}\right)$$

for all distinct $x, a, b \in X$.

By the comparison $i/4 \le D \le i$,

$$\frac{D(x,a)}{D(x,b)} \le 4\frac{i(x,a)}{i(x,b)} = 4\frac{\widehat{D}(x,a)\widehat{D}(b,\infty)}{\widehat{D}(x,b)\widehat{D}(a,\infty)}.$$

Moreover, the estimates $\widehat{q}/4 \le \widehat{d} \le \widehat{q}$ give

$$\frac{\widehat{d}(x,a)\widehat{d}(b,\infty)}{\widehat{d}(x,b)\widehat{d}(a,\infty)} \le 16\frac{\widehat{q}(x,a)\widehat{q}(b,\infty)}{\widehat{q}(x,b)\widehat{q}(a,\infty)} = 16\frac{d(x,a)}{d(x,b)}.$$

Consequently,

$$\frac{D(x,a)}{D(x,b)} \le 4\Theta\left(16\frac{d(x,a)}{d(x,b)}\right).$$

Since $t \mapsto 4\Theta(16t)$ is a homeomorphism of $[0,\infty)$ onto itself, the identity $\mathrm{id} : (X,d) \to (X,D)$ is quasisymmetric.

Since $\widehat{d}$ is bounded and the identity from $\widehat{d}$ to $\widehat{D}$ is quasisymmetric, $\widehat{D}$ is bounded. Hence there is $C < \infty$ such that $\widehat{D}(x,\infty) \le C$ for every $x \in X$. Finally, suppose that

$$0 < c \le \widehat{D}(x,\infty), \qquad x \in E.$$

For $x, y \in E$, the definition of $i$ gives

$$C^{-2}\widehat{D}(x,y) \le i(x,y) \le c^{-2}\widehat{D}(x,y).$$

Combining this with $i/4 \le D \le i$, we obtain

$$\frac{1}{4C^2}\widehat{D}(x,y) \le D(x,y) \le \frac{1}{c^2}\widehat{D}(x,y), \qquad x,y \in E.$$

Hence $D$ and $\widehat{D}$ are bi-Lipschitz equivalent on $E$.

□

*Proof of Theorem 1.3 in the general case.* Assume now that $(X,d)$ is complete and unbounded. Let $\widehat{X} = X \cup \{\infty\}$ and let $\widehat{d}$ be the metric given by Lemma 3.3. Extend $\mu$ to $\widehat{X}$ by setting $\widehat{\mu}(A) = \mu(A \cap X)$. The measure $\widehat{\mu}$ has full support. Indeed, $\widehat{d}$ and $d$ induce the same topology on $X$, while every $\widehat{d}$-neighborhood of $\infty$ contains the complement of a bounded $d$-ball. Since $X$ is unbounded and $\mu$ has full support, such a neighborhood has positive measure.

On every bounded $d$-ball, the metrics $d$ and $\widehat{d}$ are bi-Lipschitz equivalent by (3.15). Since $X$ is the countable union of bounded balls, the countable-union formula shows that $\dim_{\mathrm{H}}^{\widehat{d}} E = \dim_{\mathrm{H}}^{d} E$ for every $E \subset X$. Moreover, $\widehat{\mu}(\{\infty\}) = 0$. Therefore

$$\dim_{\mathrm{H}}^{\widehat{d}} \widehat{\mu} = \dim_{\mathrm{H}}^{d} \mu < \infty.$$

The bounded complete case therefore provides a metric $\widehat{D}$ on $\widehat{X}$ such that $\mathrm{id} : (\widehat{X},\widehat{d}) \longrightarrow (\widehat{X},\widehat{D})$ is quasisymmetric and $\dim_{\mathrm{H}}^{\widehat{D}} \widehat{\mu} \le p$.

Let $D$ be the metric associated with $\widehat{D}$ by Lemma 3.3. Then $\mathrm{id} : (X,d) \longrightarrow (X,D)$ is quasisymmetric. For $m \ge 1$, set

$$E_m = \left\{x \in X : m^{-1} \le \widehat{D}(x,\infty) \le m\right\}.$$

The sets $E_m$ cover $X$, and $D$ and $\widehat{D}$ are bi-Lipschitz equivalent on each $E_m$.

Fix $\varepsilon > 0$ and choose a full $\widehat{\mu}$-measure set $A \subset \widehat{X}$ such that $\dim_{\mathrm{H}}^{\widehat{D}} A \le p + \varepsilon$. Then

$$A \cap X = \bigcup_{m=1}^{\infty} (A \cap E_m),$$

and bi-Lipschitz invariance, followed by the same countable-union formula, gives

$$\dim_{\mathrm{H}}^{D}(A \cap X) = \sup_{m \ge 1} \dim_{\mathrm{H}}^{D}(A \cap E_m) \le p + \varepsilon.$$

Since $A \cap X$ has full $\mu$-measure, it follows that $\dim_{\mathrm{H}}^{D} \mu \le p + \varepsilon$. Letting $\varepsilon \to 0$ yields

$$\dim_{\mathrm{H}}^{D} \mu \le p.$$

Finally, suppose that $(X, d)$ is not complete. Let $(\overline{X}, \overline{d})$ be its completion, and let $\iota : X \to \overline{X}$ be the isometric inclusion. Define $\overline{\mu} = \iota_{\#}\mu$. Then $\overline{X}$ is separable and $\overline{\mu}$ has full support. We also have $\dim_{\mathrm{H}}^{\overline{d}} \overline{\mu} \le \dim_{\mathrm{H}}^{d} \mu < \infty$. To see this, fix $s > \dim_{\mathrm{H}}^{d} \mu$ and choose a Borel set $A \subset X$ such that $\mu(A) = 1$ and $\dim_{\mathrm{H}}^{d} A < s$. Choose $t$ with $\dim_{\mathrm{H}}^{d} A < t < s$. Then $\mathcal{H}_{\overline{d}}^{t}(\iota(A)) = 0$. By Borel regularity, there is a Borel set $\overline{A} \subset \overline{X}$ containing $\iota(A)$ such that $\mathcal{H}_{\overline{d}}^{t}(\overline{A}) = 0$. Thus $\overline{\mu}(\overline{A}) = 1$, $\dim_{\mathrm{H}}^{\overline{d}} \overline{A} < s$, and $\dim_{\mathrm{H}}^{\overline{d}} \overline{\mu} \le s$. Since $s > \dim_{\mathrm{H}}^{d} \mu$ is arbitrary, $\dim_{\mathrm{H}}^{\overline{d}} \overline{\mu} \le \dim_{\mathrm{H}}^{d} \mu$.

Apply the complete space case to $(\overline{X}, \overline{d}, \overline{\mu})$. We obtain a metric $\overline{D}$ on $\overline{X}$ such that $\mathrm{id} : (\overline{X}, \overline{d}) \longrightarrow (\overline{X}, \overline{D})$ is quasisymmetric and $\dim_{\mathrm{H}}^{\overline{D}} \overline{\mu} \le p$. Let $D$ be the restriction of $\overline{D}$ to $X$. The restriction of a quasisymmetric map is quasisymmetric, so

$$\mathrm{id} : (X, d) \longrightarrow (X, D)$$

is quasisymmetric. Moreover, intersecting any full $\overline{\mu}$-measure Borel set in $\overline{X}$ with $X$ gives a full $\mu$-measure Borel set in $X$. Therefore

$$\dim_{\mathrm{H}}^{D} \mu \le \dim_{\mathrm{H}}^{\overline{D}} \overline{\mu} \le p.$$

This completes the proof. □

## 4. An alternative proof of Corollary 1.2

This section gives an independent proof of Corollary 1.2. It combines Assouad's embedding with Romney's dimension reduction for Lebesgue measure.

**Lemma 4.1.** *Let $n \ge 1$, let $(S, d)$ be a metric space, let $\mu$ be a $\sigma$-finite Borel measure on $S$, let $\varphi : S \to \mathbb{R}^n$ be a Borel map, and let $U \subset \mathbb{R}^n$ be a Borel set with $\mathcal{L}^n(U) = 0$. Then for $\mathcal{L}^n$-almost every $t \in [0,1]^n$,*

$$\mu\{x \in S : \ \varphi(x) + t \in U\} = 0.$$

*Proof.* Choose Borel sets $S_j \subset S$ such that $S = \bigcup_{j=1}^{\infty} S_j$ and $\mu(S_j) < \infty$ for every $j$.

Fix $j \in \mathbb{N}$. By Tonelli's theorem,

$$\begin{aligned}
\int_{[0,1]^n} \mu\{x \in S_j : \ \varphi(x) + t \in U\}\, dt &= \int_{[0,1]^n} \int_{S_j} \mathbf{1}_U(\varphi(x) + t)\, d\mu(x)\, dt \\
&= \int_{S_j} \int_{[0,1]^n} \mathbf{1}_U(\varphi(x) + t)\, dt\, d\mu(x) \\
&= \int_{S_j} \mathcal{L}^n\big([0,1]^n \cap (U - \varphi(x))\big)\, d\mu(x) = 0.
\end{aligned}$$

Hence there is a set $N_j \subset [0,1]^n$ with $\mathcal{L}^n(N_j) = 0$ such that, for every $t \in [0,1]^n \setminus N_j$,

$$\mu\{x \in S_j : \ \varphi(x) + t \in U\} = 0. \tag{4.1}$$

Set $N = \bigcup_{j=1}^{\infty} N_j$. Then $\mathcal{L}^n(N) = 0$, and for each $t \in [0,1]^n \setminus N$, identity (4.1) holds for every $j \in \mathbb{N}$. Since $S = \bigcup_{j=1}^{\infty} S_j$, this gives

$$\mu\{x \in S : \ \varphi(x) + t \in U\} = 0.$$

This proves the lemma. □

Given metric spaces $(X, d_X)$ and $(Y, d_Y)$, an injective map $f : X \to Y$ is a *quasisymmetric embedding* if the induced map

$$f : (X, d_X) \longrightarrow (f(X),\, d_Y|_{f(X)})$$

is quasisymmetric.

**Lemma 4.2.** *Let $n \geq 1$, let $(S, d)$ be a metric space, and let $\mu$ be a $\sigma$-finite Borel measure on $S$ with $\operatorname{spt}\mu = S$. Suppose that there is a quasisymmetric embedding $\varphi : S \to \mathbb{R}^n$. Then, for every $p > 0$, there exist a metric space $Y$ and a quasisymmetric homeomorphism $h : S \to h(S) \subset Y$ such that*

$$\dim_{\mathrm{H}}(h_{\#}\mu) \leq p.$$

*Proof.* Let $p > 0$. Romney's theorem [Rom19, Corollary 1.4] states that $\dim_{\mathrm{C}} \mathcal{L}^n = 0$. Hence there exist a metric space $Y$, a quasisymmetric homeomorphism $g : \mathbb{R}^n \to Y$, and a Borel set $E \subset \mathbb{R}^n$ such that

$$\mathcal{L}^n(\mathbb{R}^n \setminus E) = 0 \quad \text{and} \quad \dim_{\mathrm{H}} g(E) \leq p.$$

Set $U = \mathbb{R}^n \setminus E$. By Lemma 4.1, choose $t_0 \in [0,1]^n$ such that

$$\mu\{x \in S : \ \varphi(x) + t_0 \in U\} = 0.$$

Define

$$h : S \to Y, \qquad h(x) = g(\varphi(x) + t_0).$$

Since the translation $z \mapsto z + t_0$ is an isometry, $h$ is a quasisymmetric homeomorphism from $S$ onto $h(S)$. The set

$$F = \{x \in S : \ \varphi(x) + t_0 \notin U\}$$

has full $\mu$-measure in $S$, and

$$h(F) \subset g(E).$$

Thus $h(F)$ has full $h_{\#}\mu$-measure. Therefore

$$\dim_{\mathrm{H}}(h_{\#}\mu) \leq \dim_{\mathrm{H}} h(F) \leq \dim_{\mathrm{H}} g(E) \leq p.$$

This proves the lemma. □

We can now combine Assouad's embedding with Lemma 4.2.

*Alternative proof of Corollary 1.2.* Let $S = \operatorname{spt}\mu$. Since $X$ is doubling, it is separable, and hence $\mu(X \setminus S) = 0$. We therefore regard $\mu$ as a locally finite Borel measure on $S$. Since $S$ is doubling, Assouad's embedding theorem [Hei01, Theorem 12.1] gives a quasisymmetric embedding $\varphi : S \to \mathbb{R}^n$ for some $n \geq 1$.

Since $S$ has a countable base and $\mu$ is locally finite, there are Borel sets $S_j \subset S$ such that

$$S = \bigcup_{j=1}^{\infty} S_j \quad \text{and} \quad \mu(S_j) < \infty \quad \text{for all } j.$$

Thus $\mu$ is $\sigma$-finite.

By Lemma 4.2, applied to $(S,d)$, $\mu$, and $\varphi$, for every $p>0$ there exist a metric space $Y$ and a quasisymmetric homeomorphism $h:S\to h(S)\subset Y$ such that $\dim_{\mathrm{H}}(h_{\#}\mu)\le p$. Since $p>0$ is arbitrary, $\dim_{\mathrm{C}}\mu=0$. □

## 5. Singular quasisymmetric maps

This section proves Theorem 1.4. We recall the part of Romney's metric construction needed for the upper bound, use a counting argument to control the exceptional set, and finish with the sharpness statement in Proposition 5.5.

5.1. **Romney's construction.** The case $n=1$ of Theorem 1.4 is Tukia's theorem [Tuk89], so the rest of this section assumes $n\ge 2$. We first review Romney's construction [Rom19, Sections 2–4] and isolate the diameter estimate needed in the proof.

Let $Q=[0,1]^n$, equipped initially with the Euclidean metric $|\cdot|$. Fix an integer $M>2n$ and a real number $L>1$, to be chosen later. For $k\ge 1$, let $W_k=\{0,1,\dots,M^n-1\}^k$, set $W_0=\{\emptyset\}$, and write $W=\bigcup_{k\ge 0}W_k$. The standard $M$-adic grid $\mathcal{Q}_k$ consists of $M^{nk}$ closed cubes of side length $M^{-k}$ with pairwise disjoint interiors and union $Q$. We label these cubes as $Q_w$, $w\in W_k$, in such a way that $Q_{(w,i)}\subset Q_w$ for $0\le i<M^n$. Set $Q_\emptyset=Q$. We say the *level* of $Q_w$ is $k$. The vertices of the level-$k$ cubes form the set

$$\mathcal{V}_k=\left\{(i_1M^{-k},\dots,i_nM^{-k}):0\le i_1,\dots,i_n\le M^k\right\},$$

and we write $\mathcal{V}=\bigcup_{k\ge 0}\mathcal{V}_k$.

The *children* of $Q_w\in\mathcal{Q}_k$ are the cubes in $\mathcal{Q}_{k+1}$ contained in $Q_w$. Divide these children into three subcollections. For $E,F\subset Q$, write $d_{|\cdot|}(E,F)=\inf\{|x-y|:x\in E,\ y\in F\}$ for their Euclidean distance. Then

$$\begin{aligned}
\mathcal{P}_w^1&=\big\{Q\in\mathcal{Q}_{k+1}:Q\subset Q_w\text{ and }Q\cap\partial Q_w\ne\varnothing\big\},\\
\mathcal{P}_w^2&=\big\{Q\in\mathcal{Q}_{k+1}\setminus\mathcal{P}_w^1:Q\subset Q_w\text{ and }d_{|\cdot|}\big(Q,\cup\mathcal{P}_w^1\big)<(n-1)M^{-(k+1)}\big\},\\
\mathcal{P}_w^3&=\big\{Q\in\mathcal{Q}_{k+1}:Q\subset Q_w\text{ and }Q\notin\mathcal{P}_w^1\cup\mathcal{P}_w^2\big\},
\end{aligned}$$

where $\bigcup\mathcal{P}_w^1$ denotes the union of the cubes in $\mathcal{P}_w^1$. These collections form, respectively, the boundary layer, the intermediate layer, and the central region of $Q_w$. Writing $\#\mathcal{A}$ for the number of elements of a finite set $\mathcal{A}$, their cardinalities are

$$\#\mathcal{P}_w^1=M^n-(M-2)^n,\qquad \#\mathcal{P}_w^2=(M-2)^n-(M-2n)^n,\qquad \#\mathcal{P}_w^3=(M-2n)^n.$$

For $j\in\{1,2,3\}$, set $P_w^j=\bigcup\mathcal{P}_w^j$.

We next define *conformal weights* $\rho_k:Q\to(0,\infty)$. Set $\rho_0\equiv 1$. Assuming $\rho_k$ is known, for $w\in W_k$, we define $\rho_{k+1}$ on the interiors of the children of $Q_w$ as follows. Let $q_w$ denote the center of $Q_w$. Then

$$\rho_{k+1}(x)=\begin{cases}\rho_k(q_w), & \text{if } x\in P_w^1,\\ (M-2n+1)\,\rho_k(q_w), & \text{if } x\in P_w^2,\\ L^{-1}\,\rho_k(q_w), & \text{if } x\in P_w^3,\end{cases}$$

Finally, at each grid-boundary point $x$, define $\rho_{k+1}(x)$ as the minimum of the constant values assigned on the interiors of the level-$(k+1)$ cubes containing $x$. This is the lower-semicontinuous extension of the preceding definition.

For $k\ge 0$, the weight $\rho_k$ induces the *length metric*

$$D_k(x,y)=\inf_\gamma\int_\gamma\rho_k\,ds,$$

where the infimum is over all curves $\gamma$ of finite Euclidean length joining $x$ to $y$, and $ds$ is Euclidean arclength. Romney proves that, for each $x, y \in \mathcal{V}$, the sequence $(D_k(x,y))$ is eventually nondecreasing and bounded. Thus

$$D(x,y) = \lim_{k\to\infty} D_k(x,y), \qquad x, y \in \mathcal{V},$$

defines a metric on $\mathcal{V}$ that extends by continuity to a metric on $Q$; see [Rom19, Section 4]. Moreover, the identity map

$$f\colon Q \to X, \qquad X = (Q, D)$$

is quasisymmetric by [Rom19, Proposition 4.1].

For $w \in W_k$, write $r(w) = \rho_k(q_w)$. For $0 \le i \le k$, let $w^{(i)} \in W_i$ be the prefix of $w$ of length $i$. Note that $Q_w \subset Q_{w^{(i)}}$ for each $i$. Define

$$c(w) = \#\big\{i \in \{1,\dots,k\} : Q_{w^{(i)}} \in \mathcal{P}^3_{w^{(i-1)}}\big\}.$$

Thus $c(w)$ is the number of visits to the central region during the first $k$ steps. Since each noncentral step multiplies the weight by at most $A := M - 2n + 1$, while each central step multiplies it by $L^{-1}$,

$$r(w) \le A^k L^{-c(w)}. \tag{5.1}$$

The diameter estimate used in Subsection 5.2 is an immediate consequence of [Rom19, Lemma 3.3].

**Lemma 5.1.** *There is a constant $C_0 = C_0(n, M, L) < \infty$ such that, for every $k \ge 0$ and every $Q_w \in \mathcal{Q}_k$,*

$$\operatorname{diam}_D f(Q_w) \le C_0 r(w) M^{-k} \le C_0 A^k L^{-c(w)} M^{-k},$$

*where $A = M - 2n + 1$.*

*Proof.* Fix a vertex $z$ of $Q_w$. Given $x \in Q_w$, choose $x_j \in \mathcal{V} \cap Q_w$ with $x_j \to x$ in the Euclidean metric. The continuous extension of $D$ and [Rom19, Lemma 3.3] give

$$\begin{aligned} D(x,z) &= \lim_{j\to\infty} D(x_j, z) = \lim_{j\to\infty} \lim_{m\to\infty} D_m(x_j, z) \\ &\le C_1 r(w) \lim_{j\to\infty} |x_j - z| = C_1 r(w)|x - z| \le C_1\sqrt{n}\, r(w) M^{-k}, \end{aligned}$$

where $C_1 = C_1(n, M, L)$. Applying this estimate to two points of $Q_w$ and using the triangle inequality yields $\operatorname{diam}_D f(Q_w) \le 2C_1\sqrt{n}\, r(w) M^{-k}$. The result now follows from (5.1), with $C_0 = 2C_1\sqrt{n}$. □

5.2. **Proof of Theorem 1.4.** The preceding lemma gives the metric estimate needed here. We combine it with a count of cubes that visit the central region too infrequently. Fix $0 < \theta < 1/2$. A cube $Q_w \in \mathcal{Q}_k$ is called *$\theta$-good* if

$$c(w) \ge \theta k.$$

Otherwise it is called *$\theta$-bad*. Let $G_k$ be the union of all $\theta$-good cubes in $\mathcal{Q}_k$, and define

$$E_\theta := \limsup_{k\to\infty} G_k = \bigcap_{N=1}^{\infty} \bigcup_{k\ge N} G_k. \tag{5.2}$$

Thus $E_\theta$ is the Borel set of points that belong to $\theta$-good cubes at infinitely many levels. We first count the bad cubes, then estimate the Euclidean dimension of the exceptional set and the $D$-dimension of $f(E_\theta)$.

**Lemma 5.2.** *There is a constant $C_n > 1$, depending only on $n$, such that, for all sufficiently large $M$, the number $B_k$ of $\theta$-bad cubes in $\mathcal{Q}_k$ satisfies*

$$B_k \le (k+1)\exp(kH(\theta))(C_nM^{n-1})^k(C_nM)^{\theta k},$$

*where*

$$H(\theta) = -\theta\log\theta - (1-\theta)\log(1-\theta)$$

*is the* entropy function.

*Proof.* Set

$$a = M^n - (M-2n)^n \quad \text{and} \quad b = (M-2n)^n.$$

The numbers of noncentral and central choices at each step are $a$ and $b$, respectively. Consequently,

$$B_k = \sum_{j<\theta k}\binom{k}{j}b^j a^{k-j} = \sum_{j<\theta k}\binom{k}{j}\left(\frac{b}{a}\right)^j a^k.$$

For all sufficiently large $M$, a constant $C_n > 1$ depending only on $n$ can be chosen so that

$$C_n^{-1}M^{n-1} \le a \le C_nM^{n-1}, \qquad \frac{b}{a} \le C_nM.$$

The standard entropy estimate

$$\binom{k}{j} \le \exp\big(kH(j/k)\big), \qquad 0 \le j \le k,$$

holds with the convention $0\log 0 = 0$. Since $H$ is increasing on $[0, 1/2]$ and $0 < \theta < 1/2$,

$$\sum_{j<\theta k}\binom{k}{j} \le (k+1)\exp(kH(\theta)).$$

For $j < \theta k$, we also have $(b/a)^j \le (C_nM)^{\theta k}$. Combining these estimates proves the lemma. □

**Lemma 5.3.** *For every $p > 0$, there is a $\theta \in (0, 1/2)$ such that, for all sufficiently large $M$,*

$$\dim_{\mathrm{H}}(Q \setminus E_\theta) \le n - 1 + p.$$

*Proof.* For $N \ge 1$, set

$$F_N := \bigcap_{k\ge N}(Q \setminus G_k).$$

Then $Q \setminus E_\theta = \bigcup_{N\ge1} F_N$. For every $k \ge N$, the set $F_N$ is covered by the $\theta$-bad cubes in $\mathcal{Q}_k$, each of which has Euclidean diameter at most $\sqrt{n}\,M^{-k}$.

Let

$$s > n - 1 + \theta + \frac{H(\theta) + (\theta+1)\log C_n}{\log M}.$$

By Lemma 5.2, the covering sum in the definition of $\mathcal{H}^s(F_N)$, using cubes of diameter at most $\sqrt{n}M^{-k}$, is bounded by

$$B_k(\sqrt{n}\,M^{-k})^s \le (k+1)n^{s/2}\left[\exp(H(\theta))C_n^{\theta+1}M^{n-1+\theta-s}\right]^k.$$

The bracket is strictly less than 1. Letting $k \to \infty$ gives $\mathcal{H}^s(F_N) = 0$. Since $Q \setminus E_\theta$ is the countable union of the sets $F_N$,

$$\dim_{\mathrm{H}}(Q \setminus E_\theta) \le n - 1 + \theta + \frac{H(\theta) + (\theta+1)\log C_n}{\log M}.$$

First choose $\theta > 0$ sufficiently small and then choose $M$ sufficiently large so that the right-hand side is at most $n - 1 + p$. □

**Lemma 5.4.** *Fix $p > 0$, $\theta \in (0, 1/2)$, and $M > 2n$. If $L$ is sufficiently large, then*

$$\dim_{\mathrm{H}} f(E_\theta) \le p.$$

*Proof.* The set $G_k$ is the union of at most $M^{nk}$ cubes $Q_w$ satisfying $c(w) \ge \theta k$. By Lemma 5.1, each such cube satisfies

$$\operatorname{diam}_D f(Q_w) \le C_0 A^k L^{-\theta k} M^{-k} = C_0 \left(\frac{A}{M}\right)^k L^{-\theta k},$$

where $A = M - 2n + 1$. For every $N \ge 1$,

$$E_\theta \subset \bigcup_{k \ge N} G_k,$$

so these images form a cover of $f(E_\theta)$. The sum of the $p$-powers of their diameters is at most

$$\sum_{k \ge N} M^{nk} \left[ C_0 \left(\frac{A}{M}\right)^k L^{-\theta k} \right]^p.$$

This equals

$$C_0^p \sum_{k \ge N} \left[ M^n \left(\frac{A}{M}\right)^p L^{-p\theta} \right]^k.$$

Choose $L$ so large that

$$\lambda := M^n \left(\frac{A}{M}\right)^p L^{-p\theta} < 1.$$

Although $C_0$ may depend on $L$, it is independent of $k$ and appears only as the fixed prefactor $C_0^p$ in the preceding series. Since $\lambda < 1$, the diameters in this cover tend uniformly to zero as $k \to \infty$, and the displayed sum tends to zero as $N \to \infty$. Hence

$$\mathcal{H}^p(f(E_\theta)) = 0,$$

and therefore

$$\dim_{\mathrm{H}} f(E_\theta) \le p.$$

□

*Proof of Theorem 1.4.* The case $n = 1$ follows from Tukia's theorem [Tuk89], so assume $n \ge 2$. Romney's construction with parameters $M, L$ gives a metric space

$$X = (Q, D)$$

and a quasisymmetric homeomorphism

$$f : (Q, |\cdot|) \to X.$$

For $p > 0$, choose $\theta \in (0, 1/2)$ and then $M$ so that Lemma 5.3 gives

$$\dim_{\mathrm{H}}(Q \setminus E_\theta) \le n - 1 + p.$$

Next choose $L$ so that Lemma 5.4 gives

$$\dim_{\mathrm{H}} f(E_\theta) \le p.$$

The set $E = E_\theta$ is Borel by (5.2), and it has the required properties. □

We finish by showing that, in higher dimensions, one cannot obtain the precise analogue of Tukia's conclusion, with both $[0,1]^n \setminus E$ and $f(E)$ of arbitrarily small Hausdorff dimension.

**Proposition 5.5.** *Let $n \geq 2$, let $f : [0,1]^n \to X$ be a homeomorphism onto a metric space $X$, and let $E \subset [0,1]^n$. If*

$$\dim_{\mathrm{H}} f(E) < 1,$$

*then*

$$\dim_{\mathrm{H}}([0,1]^n \setminus E) \geq n-1.$$

*Proof.* Still write $Q = [0,1]^n$. Suppose, toward a contradiction, that

$$\dim_{\mathrm{H}}(Q \setminus E) < n-1.$$

Since $f$ is a homeomorphism, $X$ is a separable metric space satisfying

$$X = f(E) \cup f(Q \setminus E).$$

For a separable metric space $Z$, its *small inductive dimension* $\operatorname{ind} Z$ agrees with its *covering dimension* and satisfies $\operatorname{ind} Z \leq \dim_{\mathrm{H}} Z$; see [Edg08, Theorem 6.3.10] and [Eng78, Theorem 4.1.5]. Hence

$$\operatorname{ind} f(E) < 1 \quad \text{and} \quad \operatorname{ind} f(Q \setminus E) = \operatorname{ind}(Q \setminus E) \leq \dim_{\mathrm{H}}(Q \setminus E) < n-1.$$

Here the equality uses the topological invariance of ind [Edg08, Theorem 3.4.2]. Since ind is integer-valued on separable metric spaces, and the contradiction assumption forces $E$ to be nonempty, we obtain

$$\operatorname{ind} f(E) = 0, \qquad \operatorname{ind} f(Q \setminus E) \leq n-2.$$

The union theorem [Edg08, Theorem 3.4.11] now gives

$$\operatorname{ind} X \leq \operatorname{ind} f(E) + \operatorname{ind} f(Q \setminus E) + 1 \leq 0 + (n-2) + 1 = n-1.$$

But $X$ is homeomorphic to $Q = [0,1]^n$, so

$$\operatorname{ind} X = n,$$

a contradiction. Hence $\dim_{\mathrm{H}}(Q \setminus E) \geq n-1$. □

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