An inverse regularity theorem on the domain of the infinitesimal generator of Dirichlet forms

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Abstract

On a general metric measure space X equipped with a strongly local regular Dirichlet form, we establish an inverse regularity theorem showing that, under natural geometric conditions, $\mathrm{dom}\Delta \cap C^\theta(X)$ consists solely of constant functions once θ exceeds a critical value determined by the capacity upper bound and the boundary geometry. As applications, we prove the Barlow–Perkins conjecture that on the Sierpinski gasket $\mathcal{SG}$, $\mathrm{dom}\Delta \cap C^\theta(\mathcal{SG}) = \mathbb{R}\mathbf{1}_{\mathcal{SG}}$ for every $\theta > \log(5/3)/\log 2$. We also show that on the Sierpinski carpet $\mathcal{SC}$, $\mathrm{dom}\Delta \cap \mathrm{LIP}(\mathcal{SC}) = \mathbb{R}\mathbf{1}_{\mathcal{SC}}$.

1 Introduction

Since Dirichlet forms (or diffusion processes) have been constructed on self-similar fractals, including the Sierpinski gasket $\mathcal{SG}$ by Goldstein, Kusuoka, and Barlow–Perkins [Gol, Kus, BP]; the Sierpinski carpet $\mathcal{SC}$ by Barlow–Bass and Kusuoka–Zhou [BB89, KZ]; nested fractals by Lindstrøm [Lin]; and finitely post-critical fractals by Kigami [Kig89, Kig01], many "strange" properties of these Dirichlet forms, distinct from those in the setting of smooth manifolds, have emerged. The most typical abnormal phenomenon is that the heat kernel estimates on fractals usually hold with walk dimension β strictly larger than 2, known as *sub-Gaussian heat kernel estimates*, to distinguish them from the typical case $\beta = 2$ on manifolds, known as *Gaussian heat kernel estimates*. Another abnormal phenomenon is that the energy measure on fractals is usually singular with respect to the symmetric measure of the fractal; thus, in most situations, there is no intrinsic way to express the Dirichlet form as an integral of a square power of some well-defined derivative of functions. Indeed, there is a deterministic relation between these two properties; see [KM20].

In this paper, we will prove such a "strange" property concerning the domain of the infinitesimal generator, denoted by $\mathrm{dom}\Delta$, of Dirichlet forms on fractals. Recall that

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Barlow and Perkins [BP] constructed a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the Sierpinski gasket and proved that any function in the domain of the infinitesimal generator of the form is Hölder continuous with index $\sigma = \log(5/3)/\log 2$, denoted by $C^\sigma(X)$; see [BP, Theorem 5.22]. This is mainly due to the same Hölder continuity of Green functions, which is indeed also a typical property on fractals, since the Green function on Euclidean spaces with dimension larger than 1 is not continuous. The index σ coincides with the *capacity upper bound* of their Dirichlet form

$$\mathrm{cap}(B(x,r), X\backslash B(x,2r)) \lesssim r^{-\sigma}.$$

Moreover, they proposed a conjecture concerning the sharpness of the bound σ:

$$\mathrm{dom}\Delta \cap C^\theta(X) = \mathbb{R}\mathbf{1}_X \quad \textit{for all } \theta > \sigma, \tag{1.1}$$

where $\mathbb{R}\mathbf{1}_X$ is the family of all constant functions on the Sierpinski gasket; see the first paragraph of [BP, Section 9]. This conjecture implies that, except for the trivial common part of constant functions, $\mathrm{dom}\Delta$ and $C^\theta(X)$ are actually disjoint. This is much stronger than sharpness, which only requires $\mathrm{dom}\Delta \nsubseteq C^\theta(X)$ for every $\theta > \sigma$. One of the main motivations of this paper is to prove this conjecture as an inverse regularity theorem for the domain of the generator, abbreviated as *inverse regularity theorem* in the sequel, which will be proved as a corollary of the general scheme in Theorem 1.2.

Theorem 1.1. *For the Sierpinski gasket $\mathcal{SG}$ with the self-similar Dirichlet form constructed in [BP] and $\sigma := \log(5/3)/\log 2$, we have*

$$\mathrm{dom}\Delta \cap C^\theta(\mathcal{SG}) = \mathbb{R}\mathbf{1}_{\mathcal{SG}} \quad \textit{for all } \theta > \sigma.$$

Indeed, the proof of the above theorem on the Sierpinski gasket can be easily summarized in the following three steps (readers familiar with the theory on the Sierpinski gasket in [Kig01, Str] may readily verify them):

(1) If a function u belongs to the domain of the infinitesimal generator, then its restriction to a cell also belongs to the domain of the infinitesimal generator for the cell **without** a boundary condition.

(2) If the function u is Hölder continuous with index $\theta > \sigma$, then it has a zero normal derivative on the boundary of every cell. See [Kig01, Definition 3.7.6] or [Str, Definition 2.3.1] for the definition of *normal derivative* on the Sierpinski gasket. In other words, the function u satisfies **the Neumann** boundary condition on every cell.

(3) With a constant test function, it turns out that the integral of Δu over every cell is zero, where Δ is the infinitesimal generator. Then, by the Lebesgue point theorem, we have $\Delta u = 0$ almost everywhere, which yields the desired result that u is a constant function, since $\mathcal{E}(u,u) = -\int u\Delta u = 0$.

One of the key points in the proof is that Hölder continuity with index larger than σ serves as a sufficient condition for the Neumann boundary condition, with the calculation

of the normal derivative as an intermediate step. To extend this calculation to general metric measure Dirichlet spaces, we need a mechanism to reduce the boundary behavior of a function u to a difference of the form $|u(x) - u(y)|$ for x, y near the boundary, just like the normal derivative on the Sierpinski gasket. This leads to the application of the trace of a Dirichlet form near the boundary, and to a specific requirement that the trace form be purely jumping. In this case, the trace form can be expressed as the integral of differences $|u(x) - u(y)|$ with respect to a measure, called the *jump measure*. To this end, we apply the trace theory recently established independently in [CC] and [KM24], which proves that every uniform domain in a metric measure Dirichlet space satisfying the volume doubling condition and heat kernel estimates has the desired trace form and jump measure; see Subsection 2.3. By shrinking the neighborhood of the boundary thinner and thinner, the trace form serves as a substitution for the "normal derivative" on general metric measure Dirichlet spaces.

At this stage, one may notice that the finiteness (or dimension 0) of the boundary in the Sierpinski gasket plays an important role in Theorem 1.1. If the boundary of an open set in a general metric measure Dirichlet space has upper box dimension $\gamma > 0$, then the appropriate bound on the Hölder continuity index should be changed from σ to $\sigma+\gamma$, since we need a faster decay rate of the oscillation near the boundary to cancel out the effect of the boundary volume. Roughly speaking, we will prove the following sufficient condition for the Neumann boundary condition: if the Dirichlet form on a metric measure space satisfies **the capacity upper bound condition** with index $\sigma \in \mathbb{R}$ and U is a desired open subset with **boundary upper box dimension** $\gamma > 0$, then every Hölder continuous function with index $\theta > \sigma + \gamma$ satisfies the Neumann boundary condition; see Theorem 2.20. The remaining part of the proof of the inverse regularity theorem, Theorem 1.2, remains the same as that for the Sierpinski gasket, which leads to (1.1) holding with σ replaced by $\sigma + \gamma$.

Theorem 1.2. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet (MMD) space satisfying the capacity upper bound with index $\sigma \in \mathbb{R}$. Assume there is a measure determining class $\mathcal{C}$ of X such that all $U \in \mathcal{C}$ are bounded trace admissible domains admitting an interior cutoff sequence. Let $\gamma := \sup_{U\in\mathcal{C}} \overline{\dim}_B(\partial U)$. Then for any $\theta > \sigma + \gamma$, we have*

$$\mathrm{Dom}\Delta \cap C^\theta(X) = \ker\mathcal{E}.$$

The notions in this theorem are introduced in Section 2. In the standard MMD space setting with volume doubling and heat kernel estimates, the capacity upper bound follows from [GHL] and bounded uniform domains are trace admissible by [CC, KM24]. Thus, in concrete examples, the remaining task is to construct a measure determining class of such domains with interior cutoff sequences, which is a geometric and topological verification.

A similar notion of boundary dimension, called the *topological Hausdorff dimension* and denoted by d_{tH}, is involved in the discussion of the range of bounded variation functions and the weak Bakry–Emery condition wBE(κ) in [ABCRST]. For the Sierpinski carpet, the topological Hausdorff dimension $d_{\mathrm{tH}} = \log 2/\log 3 + 1$ coincides with $\gamma + 1$, where γ is the upper box boundary dimension $\log 2/\log 3$ discussed above. They conjectured that wBE(κ) holds on the Sierpinski carpet with $\kappa = \sigma + d_{\mathrm{tH}} - 1 (= \sigma + \gamma)$, that

is,

$$|T_t f(x) - T_t f(y)| \lesssim \frac{d(x,y)^\kappa}{t^{\kappa/\beta}} \|f\|_{L^\infty}$$

for every $f \in L^\infty(X,m)$, $t > 0$ and $x, y \in X$, where T_t is the semigroup and β is the walk dimension of the Sierpinski carpet. For the Sierpinski gasket, they proved that wBE(κ) holds with $\kappa = \sigma$ and that this bound is sharp. Our present work shares a similar flavor while addressing different topics and using different approaches.

It is worth noting that, if the metric space is quasi-convex and $\sigma + \gamma \geqslant 1$, then the inverse regularity theorem is trivial, since the space $C^\theta(X)$ itself consists only of constant functions even without intersecting with domΔ. A typical situation is that of open balls in n-dimensional Euclidean space with $n \geqslant 1$, in which case we have $\sigma = 2 - n$ and $\gamma = n - 1$, and then $\sigma + \gamma = 1$. This is why such an inverse regularity theorem is not visible in Euclidean spaces. However, for the Sierpinski carpet, we can find a desired family of subsets with an upper box boundary dimension $\gamma = \log 2/\log 3 \leqslant 0.64$. By a numerical calculation, see [BB90, equation (5.5)], the capacity upper bound holds with index $\sigma \leqslant 0.23$, so $\sigma + \gamma \leqslant 0.87 < 1$. Thus the inverse regularity theorem is non-trivial on the Sierpinski carpet. In particular, we can prove that the intersection of the domain of the infinitesimal generator and the family of Lipschitz functions consists only of constant functions.

Theorem 1.3. *For the Sierpinski carpet $\mathcal{SC}$ with the self-similar Dirichlet form constructed in [BB89, KZ], we have*

$$\mathrm{dom}\Delta \cap \mathrm{LIP}(\mathcal{SC}) = \mathbb{R}\mathbf{1}_{\mathcal{SC}}.$$

A similar conjecture for the Sierpinski gasket is raised in [Mur24a, Section 6.3], with the domain of the infinitesimal generator replaced by the entire domain of the Dirichlet form. This conjecture remains open to this day.

Moreover, different from the case of the Sierpinski gasket, for general metric measure Dirichlet spaces like the Sierpinski carpet, there appears a gap between the inverse regularity result

$$\mathrm{dom}\Delta \cap C^{\sigma+\gamma+\epsilon}(X) = \mathbb{R}\mathbf{1}_X,$$

and the positive regularity result, see [Kig12, Section 4] for Dirichlet forms induced from regular resistance forms,

$$\mathrm{dom}\Delta \subset C^\sigma(X).$$

According to the conjecture in [ABCRST], we should expect that im$T_t \subset$ dom$\Delta \cap C^{\sigma+\gamma}(X)$; thus, $\sigma + \gamma$ is the optimal bound for the inverse regularity theorem on the Sierpinski carpet. Moreover, the topological dimension of the Sierpinski carpet has been proved to be $\log 2/\log 3 + 1$ [BBE, Theorem 5.4], which also shows the sharpness of the choice of γ. Nevertheless, to the best of the authors' knowledge, there has been no improvement in the positive regularity, even if the domain of the infinitesimal generator is replaced by the family of harmonic functions.

Conjecture 1.4. For the Sierpinski carpet $\mathcal{SC}$ with the self-similar Dirichlet form constructed in [BB89, KZ] and constants σ, γ as before, we have

$$\mathrm{dom}\Delta \cap C^{\sigma+\gamma}(\mathcal{SC}) \neq \mathbb{R}\mathbf{1}_{\mathcal{SC}}.$$

We should point out that even if the inverse regularity theorem, Theorem 1.2, implies a fundamental incompatibility between the Dirichlet form and Lipschitz functions on fractals, it is still possible to construct functions in the domain of the Dirichlet form with Lipschitz continuity on some "thin" continua. Such a construction plays a key role in some circumstances. For example, every function f in the domain of the Dirichlet form $\mathcal{F}$ on the Sierpinski carpet admits an approximating sequence $f_n \in \mathcal{F}$ with Lipschitz continuity (indeed, piecewise linear) on the boundary of the convex hull square; see [CQ24a] for their building-block manipulation. This idea can be extended to many fractals with flat geometric boundaries, such as the unconstrained Sierpinski carpets proposed in [CQ24a] and the golden ratio gasket proposed in [CQ24b]. As a substitution for classical arguments involving local symmetry, it leads to the existence and trace characterization of the self-similar Dirichlet form on such fractals in the absence of local symmetry.

This paper is organized as follows. Section 2 is devoted to proving the inverse regularity theorem, Theorem 1.2, for general metric measure Dirichlet spaces. There, we recall the setting of metric measure Dirichlet spaces, the local Dirichlet form, the domain of the Laplacian, and the trace form in Subsections 2.1, 2.2, and 2.3, respectively, and prove the main theorem in Subsection 2.4. We then apply the general scheme of Theorem 1.2 to concrete examples in Section 3, including the self-similar Dirichlet form on the Sierpinski carpet in Subsection 3.1 and resistance forms on finitely ramified fractals in Subsection 3.2.

2 Inverse regularity theorem for MMD spaces

2.1 Preliminary on Dirichlet forms

Recall the setting of metric measure Dirichlet spaces $(X, d, m, \mathcal{E}, \mathcal{F})$ as follows. (X, d, m) is said to be a *metric measure space*, if (X, d) is a complete, separable, proper metric space and m is a Radon measure on X with full support. Here a metric space is said to be *proper*, if every bounded set in the space is precompact. For each m-measurable function u on X, we write $\mathrm{spt}(u)$ for the *support* of the measure $\mathrm{spt}(u \cdot m)$. We denote by $C_0(X)$ the class of continuous functions with compact support equipped with the norm $\|u\|_{C(X)} := \sup_{x\in X} |u(x)|$. A pair $(\mathcal{E}, \mathcal{F})$ is said to be a *strongly local regular Dirichlet form* on (X, d, m), if

(1) $\mathcal{F}$ is a dense subspace of $L^2(X, m)$ and $\mathcal{E}$ is a nonnegative symmetric bilinear form on $\mathcal{F}$.

(2) *Closed.* $(\mathcal{F}, \mathcal{E}_1)$ is a Hilbert space with $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(X,m)}$ for $u, v \in \mathcal{F}$.

(3) *Markovian.* For all $u \in \mathcal{F}$ with $\overline{u} := (u \wedge 1) \vee 0$, it holds that $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\overline{u}, \overline{u}) \leqslant \mathcal{E}(u, u)$.

(4) *Regular.* $\mathcal{F} \cap C_0(X)$ is dense in both $(\mathcal{F}, \mathcal{E}_1)$ and $(C_0(X), \|\cdot\|_{C(X)})$.

(5) *Strongly local.* For any $u, v \in \mathcal{F}$ and $c \in \mathbb{R}$ with $\mathrm{spt}(u - c) \cap \mathrm{spt}(v) = \varnothing$, we have $\mathcal{E}(u, v) = 0$.

Generally, $(\mathcal{E}, \mathcal{F})$ satisfying conditions (1)–(3) is said to be a *Dirichlet form* on (X, d, m). If $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on a metric measure space (X, d, m), then $(X, d, m, \mathcal{E}, \mathcal{F})$ is said to be a *metric measure Dirichlet space*, or *MMD space* for simplicity. A detailed account of the theory of Dirichlet forms can be found in [FOT]. We denote the *kernel* of $(\mathcal{E}, \mathcal{F})$ by

$$\ker\mathcal{E} := \{u \in \mathcal{F} : \mathcal{E}(u, u) = 0\},$$

and write $\mathbb{R}\mathbf{1}_A$ for the class of all real constant functions on a set A.

One of the most typical examples of a Dirichlet form is the *first-order Hilbert space with Dirichlet boundary* $\left(H_0^1(\Omega), (\cdot,\cdot)_{H^1(\Omega)}\right)$ of an open set $\Omega \subset \mathbb{R}^n$. For each open set Ω, we define the first-order Hilbert space on Ω by

$$H^1(\Omega) := \left\{u \in L^2(\Omega, \mathcal{L}^n) : \partial_j u \in L^2(\Omega, \mathcal{L}^n), j = 1, \cdots, n\right\}$$

and bilinear form

$$(u, v)_{H^1(\Omega)} := \sum_{j=1}^{n} (\partial_j u, \partial_j v)_{L^2(\Omega, \mathcal{L}^n)} = \int_\Omega \nabla u \cdot \nabla v d\mathcal{L}^n \quad \textit{for } u, v \in H^1(\Omega),$$

where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$, $\partial_j u$ is the j-th weak partial derivative of u and $\nabla u = (\partial_1 u, \cdots, \partial_n u)$. Then the first order Hilbert space with Dirichlet boundary $H_0^1(\Omega)$ is defined as the closure of $H^1(\Omega) \cap C_0(\Omega)$ in $(H^1(\Omega), (\cdot,\cdot)_{H^1(\Omega)})$. It is known that $\left((\cdot,\cdot)_{H^1(\Omega)}, H_0^1(\Omega)\right)$ is a strongly local regular Dirichlet form on $(\Omega, d, \mathcal{L}^n)$. Clearly, $\ker H_0^1(\Omega) = \{0\}$.

For a general Dirichlet form $(\mathcal{E}, \mathcal{F})$, it is unknown whether there exists a finite integer $n \geqslant 0$ and a reasonable linear map $\nabla : \mathcal{F} \to (L^2(X, m))^n$ such that

$$\mathcal{E}(u, v) = \int_X \nabla u \cdot \nabla v dm.$$

However, there still exists a finite measure, called the *energy measure* and denoted by $\Gamma(u, v)$, on X for each $u, v \in \mathcal{F}$, which encodes the local information of $\mathcal{E}(u, v)$ on each Borel set and serves as a substitution for $(\nabla u \cdot \nabla v) \cdot \mathcal{L}^n$ in $H_0^1(\Omega)$. Hence we need to introduce the following concepts of *capacity* and *quasi continuity* to ensure that the integral of the energy measure is well-defined.

For each set $A \subset X$, we define the 1-*capacity* of A by

$$\mathrm{Cap}(A) := \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{F}, u|_U \geqslant 1 \ m\text{-a.e. for some open set } U \supset A\},$$

where $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(X,m)}$ as before. A Radon measure μ on X is said to be a *smooth measure*, if $\mu(A) = 0$ for all sets $A \subset X$ satisfying $\mathrm{Cap}(A) = 0$. Clearly the measure m is a smooth measure since $m(A) \leqslant \mathrm{Cap}(A)$ for all Borel sets A. As a counter example of smooth measure, one can take a k-dimensional subspace $S \subset \mathbb{R}^n$ with $0 \leqslant k \leqslant n-2$ and restrict the k-dimensional Hausdorff measure on S by $\mu := \mathbf{1}_S \cdot \mathcal{H}^k$. Then we have $\mathrm{Cap}(S) = 0$ and $\mu(S) > 0$, which implies that μ is not a smooth measure. The capacity in this example is defined relative to $H_0^1(\mathbb{R}^n)$.

Similar to the terminology *almost everywhere*, a statement depending on points is said to hold *quasi everywhere (q.e.)* on some $A \subset X$, if there is a set $N \subset A$ with $\mathrm{Cap}(N) = 0$ such that the statement holds for all $x \in A \backslash N$. For example, two functions $u, v : X \to \mathbb{R}$ are said to satisfy $u = v$ q.e. if $\mathrm{Cap}\big(\{x \in X : u(x) \neq v(x)\}\big) = 0$. It is worth noting that, if μ is a smooth measure on X and functions $u, v : X \to \mathbb{R}$ satisfy $u = v$ q.e., then $u = v$ μ-a.e.. See [FOT, Section 2.1] for details on capacity.

A function $u : X \to \mathbb{R}$ is said to be *quasi continuous* on a set $A \subset X$, if for any $\epsilon > 0$, there is a closed set $F \subset X$ such that $u|_{A \cap F} \in C(A \cap F)$ and $\mathrm{Cap}(X \backslash F) < \epsilon$. For each $u \in \mathcal{F}$, there is a quasi continuous function $\tilde{u} : X \to \mathbb{R}$ such that $\tilde{u} = u$ m-a.e.. The function $\tilde{u}$ is said to be a *quasi continuous version* of u, which is unique in the following sense: if both quasi continuous $\tilde{u}, \tilde{u}'$ satisfy $\tilde{u} = u, \tilde{u}' = u$ m-a.e., then $\tilde{u} = \tilde{u}'$ q.e.. We always denote a quasi continuous version of u by $\tilde{u}$. See [FOT, Section 2.2] for details of quasi continuity.

For each $u \in \mathcal{F} \cap L^\infty(X,m)$, there is a Radon measure $\Gamma(u,u)$ on X satisfying

$$\int_X \tilde{\varphi} d\Gamma(u,u) = \mathcal{E}(u\varphi, u) - \frac{1}{2}\mathcal{E}(u^2, \varphi) \quad \textit{for all } \varphi \in \mathcal{F} \cap L^\infty(X,m).$$

For general $u \in \mathcal{F}$, let $u_n := (u \wedge n) \vee (-n)$ for $n \geqslant 1$ and define $\Gamma(u,u) := \lim_{n\to\infty} \Gamma(u_n, u_n)$. For $u, v \in \mathcal{F}$, the *energy measure* of u, v is defined by polarization

$$\Gamma(u,v) := \frac{1}{4}\big(\Gamma(u+v, u+v) - \Gamma(u-v, u-v)\big).$$

Energy measure $\Gamma(u,u)$ is always a smooth measure.

For $u, \varphi \in H_0^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \big(\nabla(u\varphi) \cdot \nabla u - \frac{1}{2}\nabla(u^2) \cdot \nabla\varphi\big) d\mathcal{L}^n = \int_{\mathbb{R}^n} \varphi |\nabla u|^2 d\mathcal{L}^n$$

by Leibniz law of ∇. Thus the energy measure of $u \in H_0^1(\mathbb{R}^n)$ is $\Gamma(u,u) = |\nabla u|^2 \cdot \mathcal{L}^n$. See [FOT, Section 3.2] for details of energy measure.

Classically, a function $u \in \mathcal{F}$ is said to belong to the *domain of infinitesimal generator* Δ *related to the measure* m, if there is $g \in L^2(X,m)$ such that

$$\mathcal{E}(u,\varphi) = -\int g \cdot \varphi dm \quad \textit{for all } \varphi \in \mathcal{F},$$

and we write $\Delta u = g$, see [FOT, Corollary 1.3.1]. We write $\mathrm{dom}\Delta$ for the domain of generator relative to m.

The following definition of the Laplacian is slightly more general than the infinitesimal generator of a Dirichlet form in [FOT] (we use the term "Laplacian" to indicate the difference), since the image of the Laplacian will only be required to be a signed smooth measure. This relaxation enables us to deal with more general functions, such as Green functions on resistance form spaces, which are not included in the classical definition of the domain of the generator.

Definition 2.1. Define the *domain of Laplacian* $\mathrm{Dom}\Delta$ by $u \in \mathrm{Dom}\Delta$ if and only if $u \in \mathcal{F}$ and there is a smooth measure μ on X and a μ-measurable function $\eta : X \to \{-1, 1\}$ such that

$$\mathcal{E}(u, \varphi) = -\int_X \varphi \cdot \eta d\mu \quad \textit{for all } \varphi \in \mathcal{F} \cap C_0(X). \tag{2.1}$$

For each $u \in \mathrm{Dom}\Delta$, we write $\Delta u := \eta \cdot \mu$.

Remark 2.2. If $u \in \mathcal{F}$ satisfies (2.1) for a Radon measure μ and $\eta = 1$ μ-a.e., then for any nonnegative $\varphi \in \mathcal{F} \cap C_0(X)$ with $\varphi|_U \geqslant 1$ m-a.e. for some open set $U \supset A$, we have

$$\mu(A) \leqslant \int_X \varphi d\mu = |\mathcal{E}(u, \varphi)| \leqslant \mathcal{E}_1(u, u)^{1/2} \mathcal{E}_1(\varphi, \varphi)^{1/2},$$

which implies

$$\mu(A) \leqslant \mathcal{E}_1(u, u)^{1/2} \mathrm{Cap}(A)^{1/2}$$

for all Borel sets A. Thus μ is automatically a smooth measure. A similar argument holds if $\eta = -1$ μ-a.e.. However, it is unknown whether a Radon measure representing the total variation of the image of the Laplacian is automatically a smooth measure when both $\mu(\{\eta = 1\}) > 0$ and $\mu(\{\eta = -1\}) > 0$.

If $u \in \mathrm{dom}\Delta$ with $\Delta u = g \in L^2(X, m)$, then by defining $\eta := \mathbf{1}_{\{g \geqslant 0\}} - \mathbf{1}_{\{g<0\}}$ and $\mu := |g| \cdot m$, which is a smooth measure, we have $u \in \mathrm{Dom}\Delta$ with $\Delta u = \eta \cdot \mu$. It is worth noting that the domain of the Laplacian $\mathrm{Dom}\Delta$ is independent of the choice of the measure m, in contrast to $\mathrm{dom}\Delta$.

Definition 2.3. A class $\mathcal{C}$ of bounded subsets in a proper metric space X is said to be a *measure determining class*, if for any two Radon measures μ, ν on X, $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}$ implies $\mu = \nu$.

For example, a typical measure determining class $\mathcal{C}$ of $\mathbb{R}^n$ is given by

$$\mathcal{C} := \left\{ \prod_{j=1}^{n} (k_j 2^m, (k_j + 1)2^m] : m \in \mathbb{Z}, k_j \in \mathbb{Z}, j = 1, \cdots, n \right\},$$

see [Kal, Lemma 1.17].

Lemma 2.4. *Let $\mathcal{C}$ be a measure determining class on X. If a Radon measure μ on X and a μ-measurable function $\eta : X \to \{-1, 1\}$ satisfy $\int_C \eta d\mu = 0$ for all $C \in \mathcal{C}$, then $\eta \cdot \mu = 0$.*

Proof. Define $\mu_1 := \mathbf{1}_{\{\eta=1\}} \cdot \mu$ and $\mu_2 := \mathbf{1}_{\{\eta=-1\}} \cdot \mu$. Then for each $C \in \mathcal{C}$, we have

$$\mu_1(C) - \mu_2(C) = \int_C \eta d\mu = 0.$$

Since both μ_1, μ_2 are Radon measures and $\mathcal{C}$ is a measure determining class, we have $\mu_1 = \mu_2$. This gives

$$\int_A \eta d\mu = \int_{A\cap\{\eta=1\}} d\mu - \int_{A\cap\{\eta=-1\}} d\mu = \mu_1(A) - \mu_2(A) = 0$$

for all Borel measurable sets A. Thus $\eta \cdot \mu = 0$. □

2.2 Local Dirichlet space and local Laplacian

Before introducing the notion of local Dirichlet spaces and local Laplacian with various boundary conditions, we first recall some geometric inequality conditions and the notion of uniform domain, see [GHL, Mur24a].

Associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$, there is a *strongly continuous semigroup* $\{T_t\}_{t>0}$ on $L^2(X, m)$, that is, a family of bounded linear operators on $L^2(X, m)$ such that $T_t T_s = T_{t+s}$, $\|T_t u\|_{L^2(X,m)} \leqslant \|u\|_{L^2(X,m)}$ and $\lim_{t\to 0} T_t u = u$ in $L^2(X, m)$ for all $s, t > 0$ and $u \in L^2(X, m)$. A non-negative Borel function $p_t : X \times X \to \mathbb{R}$ is said to be the *kernel* of T_t, if

$$T_t u(x) = \int_X u(y) p_t(x, y) dm(y) \quad m\text{-a.e.} x \in X$$

for all $u \in L^2(X, m)$ and $t > 0$.

The notion of *relative capacity* is defined as follows. For disjoint sets $A, B \subset X$, we define the relative capacity between A and B by

$$\mathrm{cap}(A, B) := \inf\{\mathcal{E}(u, u) : u \in \mathcal{F}, u|_U = 0, u|_V = 1 \ m\text{-a.e. for open sets } U \supset A, V \supset B\}. \tag{2.2}$$

We list some well-known geometric inequalities that will be involved in this paper.

Definition 2.5. Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space and $\{T_t\}_{t>0}$ be the associated semigroup.

(1) (X, d) is said to be *metric doubling*, if there is a constant $N \geqslant 1$ such that for any $x \in X$ and $r > 0$, there exists $x_1, \cdots, x_n \in X$ with $n \leqslant N$ satisfying

$$B(x, 2r) \subset \bigcup_{j=1}^{n} B(x_j, r).$$

(2) (X, d, m) is said to be *volume doubling*, if there is a constant $C \geqslant 1$ such that

$$m(B(x, 2r)) \leqslant C m(B(x, r)) \quad \text{for all } x \in X, r > 0.$$

(3) (X, d, m) is said to be *α-Ahlfors regular* for $\alpha \geqslant 0$, if there is $C \geqslant 1$ such that

$$C^{-1}r^{\alpha} \leqslant m(B(x,r)) \leqslant Cr^{\alpha} \quad \textit{for all } r \in (0, \mathrm{diam}X].$$

(4) $(X, d, m, \mathcal{E}, \mathcal{F})$ is said to satisfy *capacity upper bound with index* $\sigma \in \mathbb{R}$, if there is a constant $C \geqslant 1$ such that

$$\mathrm{cap}(B(x,r), X\backslash B(x,2r)) \leqslant Cr^{-\sigma} \quad \textit{for all } x \in X, r > 0.$$

(5) $(X, d, m, \mathcal{E}, \mathcal{F})$ is said to satisfy *heat kernel estimates with walk dimension* $\beta \geqslant 0$, HKE(β), if T_t admits a kernel $p_t(x,y)$ and there is a constant $C \geqslant 1$ such that the kernel of T_t satisfies

$$\frac{C^{-1}}{m(B(x,t^{1/\beta}))} \cdot \mathbf{1}_{\{d(x,y)\leqslant C^{-1}t^{1/\beta}\}} \leqslant p_t(x,y) \leqslant \frac{C}{m(B(x,t^{1/\beta}))} \exp\left(-\left(\frac{d(x,y)^{\beta}}{Ct}\right)^{\frac{1}{\beta-1}}\right)$$

for all $x, y \in X$ and $t > 0$.

It is well-known that if $(X, d, m, \mathcal{E}, \mathcal{F})$ is α-Ahlfors regular and satisfies heat kernel estimates with walk dimension β, then it satisfies capacity upper bound estimate with index $\sigma = \beta - \alpha$, see [GHL]. Actually, the index of Ahlfors regularity and walk dimension always satisfy the relation

$$2 \leqslant \beta \leqslant \alpha + 1,$$

see [Mur24b] for the proof and optimality of this relation. In particular, it implies $\sigma \leqslant 1$.

Definition 2.6. Let U be a connected, nonempty, open, proper subset of X. If there is a constant $A \geqslant 1$ such that for any $x, y \in U$, there is a path-connected set γ in U containing both x and y such that $\mathrm{diam}\gamma \leqslant Ad(x,y)$ and

$$\mathrm{dist}(\{z\}, \partial U) \geqslant A^{-1}(d(x,z) \wedge d(y,z)) \quad \textit{for all } z \in \gamma,$$

then U is said to be a *uniform domain* of X.

The following definition of local Dirichlet space follows from [Mur24a, Definition 2.5]. For an open set $U \subset X$, we denote the domain of *local Dirichlet space* on U by $\mathcal{F}_{\mathrm{loc}}(U)$, where $u \in \mathcal{F}_{\mathrm{loc}}(U)$ if and only if for any relatively compact open subset $V \subset U$, there is $u^{\#} \in \mathcal{F}$ such that $u = u^{\#}$ m-a.e. on V. The energy measure $\Gamma(u,u)$ of $u \in \mathcal{F}_{\mathrm{loc}}(U)$ is a Radon measure on U such that $\mathbf{1}_V \cdot \Gamma(u,u) = \mathbf{1}_V \cdot \Gamma(u^{\#}, u^{\#})$ for any relatively compact open set $V \subset U$ and any $u^{\#} \in \mathcal{F}$ such that $u = u^{\#}$ m-a.e. on V. Define a pair $(\mathcal{E}_U, \mathcal{F}(U))$ by

$$\mathcal{F}(U) := \left\{u \in \mathcal{F}_{\mathrm{loc}}(U) \cap L^2(U,m) : \int_U d\Gamma(u,u) < \infty\right\}$$

and

$$\mathcal{E}_U(u,u) := \int_U d\Gamma(u,u).$$

For $u, v \in \mathcal{F}(U)$, we write $\mathcal{E}_U(u,v) := \frac{1}{4}(\mathcal{E}_U(u+v,u+v) - \mathcal{E}_U(u-v,u-v))$.

It is worth emphasizing that the pair $(\mathcal{E}_U, \mathcal{F}(U))$ might fail to be a regular Dirichlet form on $(\overline{U}, d, m)$. For example, let $U := B(0,1)\backslash([0,1] \times \{0\}) \subset \mathbb{R}^2$. Then $H^1(U)$ as the local version of $H^1(\mathbb{R}^2)$ is not regular since $u(r\cos\theta, r\sin\theta) := \theta$ satisfies $u \in H^1(U)$ while there does not exist a sequence $u_n \in H^1(U) \cap C(\overline{U})$ such that u_n converges to u in $(\cdot,\cdot)_{H^1(U)}$. See [Mur24a, Definition 4.1, Lemma 4.2] for the following sufficient condition for $(\mathcal{E}_U, \mathcal{F}(U))$ to be a strongly local regular Dirichlet form on $(\overline{U}, d, m)$.

Definition 2.7. An open subset $U \subset X$ is said to be an *extension domain*, if there is a bounded linear operator $E : \mathcal{F}(U) \to \mathcal{F}$ such that $Eu = u$ m-a.e. on U, where the topology on $\mathcal{F}(U)$ and $\mathcal{F}$ are given by $\mathcal{E}_{U,1}$ and $\mathcal{E}_1$ respectively.

Proposition 2.8. *If $U \subset X$ is an extension domain and $m(\partial U) = 0$, then $(\mathcal{E}_U, \mathcal{F}(U))$ is a strongly local regular Dirichlet form on $(\overline{U}, d, m)$. Moreover, the 1-capacity for $A \subset \overline{U}$ with respect to $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}_U, \mathcal{F}(U))$ are comparable to each other.*

Once U is an extension domain, the notions of quasi everywhere and quasi continuity on U are equivalent for $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}_U, \mathcal{F}(U))$. We have the following sufficient condition for a set to be an extension domain; see [Mur24a, Theorem 2.7].

Proposition 2.9. *If MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies volume doubling property and heat kernel estimates, then any uniform domain in X is an extension domain.*

Next, we consider restricting the Laplacian on the whole space, defined in Definition 2.1, to an open set U. Recall that for each $u \in \mathrm{Dom}\Delta$, the Laplacian Δu is only required to be a signed smooth measure on X.

Definition 2.10. Let $U \subset X$ be an extension domain.

(1) Define the *domain of Laplacian on U without boundary condition* $\mathrm{Dom}(\Delta, U) \subset \mathcal{F}(U)$ by $u \in \mathrm{Dom}(\Delta, U)$ if and only if for any relatively compact open set $V \subset U$, there is $u^\# \in \mathrm{Dom}\Delta$ such that $u|_V = u^\#|_V$ m-a.e..

(2) If $u \in \mathrm{Dom}(\Delta, U)$, then there exists a signed smooth measure Δu on U such that $\mathbf{1}_V \cdot \Delta u = \mathbf{1}_V \cdot \Delta u^\#$ for any relatively compact open set $V \subset U$ and $u^\# \in \mathrm{Dom}\Delta$ such that $u = u^\#$ on V m-a.e..

Clearly for each $u \in \mathrm{Dom}\Delta$, we have $u|_U \in \mathrm{Dom}(\Delta, U)$ and

$$\Delta(u|_U) = \mathbf{1}_U \cdot \Delta u.$$

We need the following characterization when Δu acts on $\varphi \in \mathcal{F}(U)$ with $\tilde{\varphi}|_{\partial U} = 0$ q.e..

Lemma 2.11. *Let $U \subset X$ be an extension domain and $u \in \mathrm{Dom}(\Delta, U)$. Then for each $\varphi \in \mathcal{F}(U) \cap L^\infty(U, m)$ such that $\tilde{\varphi}|_{\partial U} = 0$ q.e., we have*

$$\mathcal{E}_U(u, \varphi) = -\int_U \tilde{\varphi} d\Delta u. \tag{2.3}$$

Proof. Write $\Delta u = \eta\cdot\mu$ as before. Since $|\varphi| \leqslant \|\varphi\|_{L^\infty(X,m)}$ m-a.e., we have $|\tilde{\varphi}| \leqslant \|\varphi\|_{L^\infty(X,m)}$ q.e., which implies $\|\tilde{\varphi}\|_{L^\infty(X,\mu)} \leqslant \|\varphi\|_{L^\infty(X,m)}$. By [FOT, Corollary 2.3.1], there is a sequence $\varphi_n \in \mathcal{F}(U)\cap C_0(U)$ such that $\lim_{n\to\infty}\mathcal{E}_{U,1}(\varphi-\varphi_n,\varphi-\varphi_n)=0$. Moreover, by [FOT, Theorem 1.4.2 (v)], we further assume $\|\tilde{\varphi}_n\|_{L^\infty(U,\mu)} \leqslant \|\tilde{\varphi}\|_{L^\infty(U,\mu)}$ for all $n\geqslant 1$. By the definition of Δu, we have

$$\mathcal{E}_U(u,\varphi_n)=\mathcal{E}(u^\#,\varphi_n^0)=-\int_X \varphi_n^0 d\Delta u^\# = -\int_U \varphi_n\cdot\eta d\mu,$$

where φ_n^0 is defined on X as the zero extension of φ_n, and $u^\#\in\mathrm{Dom}(\Delta)$ such that $u=u^\#$ on the support of φ_n. Clearly we have

$$\lim_{n\to\infty}\mathcal{E}_U(u,\varphi_n)=\mathcal{E}_U(u,\varphi),$$

so it remains to show that

$$\lim_{n\to\infty}\int_U \varphi_n\cdot\eta d\mu = \int_U\tilde{\varphi}\cdot\eta d\mu.$$

Since μ is a smooth measure, by [FOT, Theorem 2.2.2, Theorem 2.2.4], there is an increasing sequence of relatively closed sets $F_j\subset U$ such that $\mathrm{Cap}(U\backslash F_j)\to 0, \mu(U\backslash F_j)\to 0$ and

$$\left|\int_{U\cap F_j}\tilde{v}d\mu\right| \leqslant C_j\sqrt{\mathcal{E}_{U,1}(v,v)} \quad \textit{for all } v\in\mathcal{F}(U)$$

for some constant $C_j\geqslant 1$ depending on j.

Then we have

$$\begin{aligned}\left|\int_U(\varphi_n-\tilde{\varphi})\cdot\eta d\mu\right| &\leqslant \int_{U\cap F_j}|\varphi_n-\tilde{\varphi}|d\mu+\int_{U\backslash F_j}|\varphi_n-\tilde{\varphi}|d\mu\\ &\leqslant C_j\sqrt{\mathcal{E}_{U,1}(\varphi_n-\varphi,\varphi_n-\varphi)}+2\mu(U\backslash F_j)\|\tilde{\varphi}\|_{L^\infty(U,\mu)},\end{aligned}$$

which implies

$$\limsup_{n\to\infty}\left|\int_U(\varphi_n-\tilde{\varphi})\cdot\eta d\mu\right| \leqslant 2\mu(U\backslash F_j)\|\tilde{\varphi}\|_{L^\infty(U,\mu)}$$

for all $j\geqslant 1$. Then the desired result follows by taking $j\to\infty$. □

Recall the Gauss-Green formula for open sets in Euclidean spaces. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with smooth boundary. Then for each $u,\varphi\in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega \nabla u\cdot\nabla\varphi d\mathcal{L}^n = -\int_\Omega\Delta u\cdot\varphi d\mathcal{L}^n+\int_{\partial\Omega}\partial_\nu u\cdot\varphi d\mathcal{H}^{n-1},$$

where $\partial_\nu u$ is the normal derivative of u on $\partial\Omega$ oriented outside of Ω. If $u\in C^\infty(\overline{\Omega})$ satisfies

$$\int_\Omega\nabla u\cdot\nabla\varphi d\mathcal{L}^n=-\int_\Omega\Delta u\cdot\varphi d\mathcal{L}^n \quad \textit{for all } \varphi\in C^\infty(\overline{\Omega}),$$

then $\partial_\nu u|_{\partial\Omega}=0$ $\mathcal{H}^{n-1}$-a.e., and in this case we say u has Neumann boundary on $\partial\Omega$.

Similarly, we define the *domain of Laplacian on U with Neumann boundary* for Dirichlet forms as follows.

Definition 2.12. Let $U \subset X$ be an extension domain. Define the *domain of Laplacian on U with Neumann boundary* by

$$\mathrm{Dom}_N(\Delta, U) := \left\{ u \in \mathrm{Dom}(\Delta, U) : \mathcal{E}_U(u, \varphi) = -\int_U \tilde{\varphi} d\Delta u \text{ for all } \varphi \in \mathcal{F}(U) \cap L^\infty(U, m) \right\}.$$

Compared with Lemma 2.11, if u satisfies the Neumann boundary condition, then (2.3) holds not only for a function φ with zero boundary values on ∂U, but for all essentially bounded functions in the domain of the local Dirichlet form.

In the case of a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and $u \in C^\infty(\overline{\Omega})$, if there are $\theta > 1$ and $C \geqslant 1$ such that

$$|u(x) - u(y)| \leqslant C d(x, y)^\theta \quad \text{for all } x \in \partial\Omega, y \in \Omega,$$

then $\partial_\nu u|_{\partial\Omega} = 0$, and thus u satisfies the Neumann boundary condition.

2.3 Trace of Dirichlet forms and estimates of the jump measure

The following definition of the trace form mainly follows [KM24], using different notation. Let $U \subset X$ be an extension domain. Then $(\mathcal{E}_U, \mathcal{F}(U))$ is a strongly local regular Dirichlet form on $(\overline{U}, d, m)$. Define a space $\check{\mathcal{F}}(\partial U)$ by $\tilde{u} \in \check{\mathcal{F}}(\partial U)$ if and only if $\tilde{u} : \partial U \to \mathbb{R}$ is quasi continuous on ∂U and there is $\tilde{u}^\# \in \mathcal{F}(U)$ such that $\tilde{u}^\#|_{\partial U} = \tilde{u}$ q.e.. For each $\tilde{u} \in \check{\mathcal{F}}(\partial U)$, define

$$\check{\mathcal{E}}_{\partial U}(\tilde{u}, \tilde{u}) := \inf\{\mathcal{E}_U(\tilde{u}^\#, \tilde{u}^\#) : \tilde{u}^\# \in \mathcal{F}(U), \tilde{u}^\# = \tilde{u} \text{ q.e. on } \partial U\}.$$

For $\tilde{u}, \tilde{v} \in \check{\mathcal{F}}(\partial U)$, we define $\check{\mathcal{E}}(\tilde{u}, \tilde{v})$ by polarization. The pair $(\check{\mathcal{E}}_{\partial U}, \check{\mathcal{F}}(\partial U))$ is called the *trace* of $(\mathcal{E}_U, \mathcal{F}(U))$ on ∂U.

Assume $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the volume doubling property and heat kernel estimates. If U is a uniform domain of X, then there is a smooth measure μ on ∂U, called the *$\mathcal{E}$-elliptic measure*; see [KM24, Proposition 4.1]. Moreover, $(\check{\mathcal{E}}_U, \check{\mathcal{F}}(U))$ is a Dirichlet form on $(\partial U, d, \mu)$. The following characterization of the trace mainly follows from [KM24, Theorem 1.5, Definition 2.33].

Proposition 2.13. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying the volume doubling property and heat kernel estimates, and let U be a uniform domain of X.*

(1) There is a linear operator $H_{\partial U} : \check{\mathcal{F}}(\partial U) \to \mathcal{F}(U)$ satisfying the following properties.

- *(a) $H_{\partial U}\tilde{u}|_{\partial U} = \tilde{u}$ q.e. for all $\tilde{u} \in \check{\mathcal{F}}(\partial U)$.*
- *(b) $\mathcal{E}_U(H_{\partial U}\tilde{u}, \varphi) = \check{\mathcal{E}}_{\partial U}(\tilde{u}, \tilde{\varphi}|_{\partial U})$ for all $\tilde{u} \in \check{\mathcal{F}}(\partial U)$ and $\varphi \in \mathcal{F}(U)$.*
- *(c) $\mathcal{E}_U(H_{\partial U}\tilde{u}, H_{\partial U}\tilde{u}) = \check{\mathcal{E}}_{\partial U}(\tilde{u}, \tilde{u})$ for all $\tilde{u} \in \check{\mathcal{F}}(\partial U)$.*
- *(d) $\|H_{\partial U}\tilde{u}\|_{L^\infty(U,m)} \leqslant \|\tilde{u}\|_{L^\infty(\partial U, \mathcal{E})}$, where we define*

$$\|\tilde{u}\|_{L^\infty(\partial U, \mathcal{E})} := \inf\{c \in [0, \infty] : \mathrm{Cap}(\{|\tilde{u}| > c\}) = 0\}.$$

(2) There is a Radon measure $J_{\partial U}$ on $\partial U \times_{\mathrm{od}} \partial U$ such that

$$\check{\mathcal{E}}_{\partial U}(u,v) = \int_{\partial U \times_{\mathrm{od}} \partial U} (u(x)-u(y))(v(x)-v(y)) dJ_{\partial U}(x,y) \quad \text{for all } u,v \in \check{\mathcal{F}}(\partial U).$$

Here for each $A, B \subset X$, we write $A \times_{\mathrm{od}} B := \{(x,y) \in A \times B : x \neq y\}$, where 'od' means off-diagonal.

The linear operator $H_{\partial U}$ is said to be the harmonic extension operator*, and the measure $J_{\partial U}$ is said to be the* jump measure*.*

Since the open sets considered below might not even be connected, and thus may not be uniform domains, we need the following generalization.

Definition 2.14. Let $U \subset X$ be an extension domain. If there exist a linear operator $H_{\partial U}$ and a Radon measure $J_{\partial U}$ that satisfy the properties in Proposition 2.13, respectively, then U is said to be a *trace admissible domain* in X.

Lemma 2.15. *Let U_i be trace admissible domains in X for $1 \leqslant i \leqslant n$ such that $\mathrm{dist}(U_i, U_j) > 0$ for all $1 \leqslant i < j \leqslant n$. Then $U := \bigcup_{j=1}^n U_j$ is still a trace admissible domain.*

Proof. Note that by definition, it is easy to see that $u \in \mathcal{F}(U)$ if and only if $u|_{U_i} \in \mathcal{F}(U_i)$ for all $1 \leqslant i \leqslant n$, and

$$\mathcal{E}_U(u,u) = \sum_{i=1}^n \mathcal{E}_{U_i}(u|_{U_i}, u|_{U_i}).$$

In particular, if $\mathrm{spt}(u) \subset \overline{U}_i$ and $\mathrm{spt}(v) \subset \overline{U}_j$ for $i \neq j$, then $\mathcal{E}_U(u,v) = 0$.

Moreover, we have $u \in \check{\mathcal{F}}(\partial U)$ if and only if $u|_{\partial U_i} \in \check{\mathcal{F}}(\partial U_i)$ for all $1 \leqslant i \leqslant n$. Then we can define $H_{\partial U} : \check{\mathcal{F}}(\partial U) \to \mathcal{F}(U)$ by

$$H_{\partial U} u := \sum_{i=1}^n H_{\partial U_i}(u|_{\partial U_i}) \cdot \mathbf{1}_{\overline{U}_i}$$

and define $J_{\partial U}$ on $\partial U \times_{\mathrm{od}} \partial U$ by

$$J_{\partial U} := \sum_{i=1}^n J_{\partial U_i}.$$

Then $H_{\partial U}$ and $J_{\partial U}$ give the desired harmonic extension operator and jump measure. □

The following upper bound estimate of the jump measure will be useful in the proof of our main theorem.

Lemma 2.16. *Let U be a trace admissible domain of X.*

(1) For any disjoint open sets $A, B \subset X$, we have

$$J_{\partial U}((A \cap \partial U) \times (B \cap \partial U)) \leqslant \mathrm{cap}(A, B),$$

where the relative capacity is defined in (2.2).

(2) Assume $(\mathcal{E}, \mathcal{F})$ satisfies capacity upper bound in Definition 2.5 (4) with index $\sigma \in \mathbb{R}$. Let $K \subset \partial U$ and $N \subset K$ such that $K \subset \bigcup_{x \in N} B(x, \delta/3)$ for $\delta > 0$. Then for $F := \{(x, y) \in K \times \partial U : d(x, y) \geqslant \delta\}$, we have

$$J_{\partial U}(F) \leqslant \#N \cdot C\delta^{-\sigma},$$

where $C \geqslant 1$ is a constant only depending on the capacity upper bound condition.

Proof. (1) It is enough to consider the case $\mathrm{cap}(A, B) < \infty$. Take $\psi \in \mathcal{F}$ such that $\psi|_A = 1, \psi|_B = 0$ m-a.e. and $\mathcal{E}(\psi, \psi) \leqslant \mathrm{cap}(A, B) + \epsilon$ for some $\epsilon > 0$. Then we have

$$\begin{aligned}
\mathcal{E}(\psi, \psi) &\geqslant \mathcal{E}_U(\psi|_U, \psi|_U) \geqslant \check{\mathcal{E}}_{\partial U}(\tilde{\psi}|_{\partial U}, \tilde{\psi}|_{\partial U}) \\
&= \int_{\partial U \times_{\mathrm{od}} \partial U} (\tilde{\psi}(x) - \tilde{\psi}(y))^2 dJ_{\partial U}(x, y) \\
&\geqslant \int_{(A \cap \partial U) \times (B \cap \partial U)} (\tilde{\psi}(x) - \tilde{\psi}(y))^2 dJ_{\partial U}(x, y) \\
&\geqslant J_{\partial U}((A \cap \partial U) \times (B \cap \partial U)).
\end{aligned}$$

Then the desired result follows by taking $\epsilon \to 0$.

(2) For each $x \in N$, write $B_x := B(x, \delta/3)$ and $A_x := X \backslash B(x, 2\delta/3)$. For any $(x, y) \in K \times \partial U$ with $d(x, y) > \delta$, there exists $x' \in N$ such that $x \in B_{x'}$ and $y \in A_{x'}$. Then we have

$$J_{\partial U}(F) \leqslant \sum_{x \in N} J_{\partial U}((B_x \cap K) \times (A_x \cap \partial U)) \leqslant \sum_{x \in N} \mathrm{cap}(B_x, A_x) \leqslant \#N \cdot C\delta^{-\sigma}.$$

This gives the desired estimate. □

Lemma 2.17. *Let $U_j \subset X, j \geqslant 0$ be a sequence of bounded trace admissible domains such that $U_{j+1} \subset U_j$ for all $j \geqslant 0$ and $\bigcap_{j=0}^{\infty} U_j = \varnothing$.*

(1) Let $A, B \subset X$ be disjoint open sets such that $\mathrm{cap}(A, B) < \infty$. Then

$$\lim_{j \to \infty} J_{\partial U_j}((A \cap \partial U_j) \times (B \cap \partial U_j)) = 0.$$

(2) Let $\delta > 0$ and define $E := \{(x, y) \in \overline{U}_0 \times \overline{U}_0 : d(x, y) > \delta\}$. Then we have

$$\lim_{j \to \infty} J_{\partial U_j}(E \cap (\partial U_j \times \partial U_j)) = 0.$$

Proof. (1) Take $\psi \in \mathcal{F}$ such that $\psi|_A = 1, \psi|_B = 0$ m-a.e.. Then we have

$$\begin{aligned}\int_{U_j} d\Gamma(\psi,\psi) &= \mathcal{E}_{U_j}(\psi|_{U_j}, \psi|_{U_j}) \geqslant \check{\mathcal{E}}_{\partial U_j}(\tilde{\psi}|_{\partial U_j}, \tilde{\psi}|_{\partial U_j}) \\ &= \int_{\partial U_j \times_{\mathrm{od}} \partial U_j} (\tilde{\psi}(x) - \tilde{\psi}(y))^2 dJ_{\partial U_j}(x,y) \\ &\geqslant \int_{(A\cap\partial U_j)\times(B\cap\partial U_j)} (\tilde{\psi}(x) - \tilde{\psi}(y))^2 dJ_{\partial U_j}(x,y) \\ &\geqslant J_{\partial U_j}((A\cap\partial U_j)\times(B\cap\partial U_j)).\end{aligned}$$

Since U_j decreases to empty set, we have

$$\lim_{j\to\infty} J_{\partial U_j}((A\cap\partial U_j)\times(B\cap\partial U_j)) \leqslant \lim_{j\to\infty}\int_{U_j} d\Gamma(\psi,\psi) = 0.$$

This gives the desired result.

(2) Since U_0 is bounded, there is a finite set $N \subset U_0$ such that $U_0 \subset \bigcup_{x\in N} B_x$, where $B_x := B(x, \delta/5)$. Then for any $x, y \in U_0$ such that $d(x,y) > \delta$, there exists $x', y' \in N$ such that $x \in B_{x'}, y \in B_{y'}$ and $\mathrm{dist}(B_{x'}, B_{y'}) > \delta/5$. Note that in this case, we have $\mathrm{cap}(B_{x'}, B_{y'}) < \infty$, which gives

$$\lim_{j\to\infty} J_{\partial U_j}((B_{x'}\cap\partial U_j)\times(B_{y'}\cap\partial U_j)) = 0.$$

Thus we have

$$\lim_{j\to\infty} J_{\partial U_j}(E\cap(\partial U_j\times\partial U_j)) \leqslant \sum_{x',y'}\lim_{j\to\infty} J_{\partial U_j}((B_{x'}\cap\partial U_j)\times(B_{y'}\cap\partial U_j)) = 0,$$

where the sum is taken for all $x', y' \in N$ such that $\mathrm{dist}(B_{x'}, B_{y'}) > \delta/5$. □

2.4 Inverse regularity of the domain of Laplacian

In this subsection, we prove the two main theorems in this paper. Theorem 2.20 provides a sufficient condition of Neumann boundary condition in terms of Hölder continuity, and Theorem 1.2 proves the inverse regularity of the domain of Laplacian.

Definition 2.18. (1) For each bounded set $K \subset X$ and $\delta > 0$, we denote by $s_\delta(K)$ the smallest integer such that there is $N \subset X$ with $\#N = s_\delta(K)$ such that

$$K \subset \bigcup_{x\in N} B(x,\delta).$$

Moreover, we write the *upper box dimension* of K by

$$\overline{\dim}_B K := \limsup_{\delta\to 0} \frac{\log s_\delta(K)}{-\log\delta}.$$

(2) For each $\theta \geqslant 0$ and $A \subset X$, we write $C^\theta(A)$ for the class of all functions on A satisfying

$$|f(x) - f(y)| \leqslant Cd(x,y)^\theta \quad \textit{for all } x, y \in A$$

for some constant $C \geqslant 1$. In particular, we write $\mathrm{LIP}(X) := C^1(X)$. Note that if $X = \mathbb{R}^n$ and $\theta > 1$, the notion $C^\theta(A)$ is different from its usual meaning.

For two trace admissible open sets $V \subset U \subset X$ with $\partial U \subset \partial V$ and $k \in \mathbb{Z}$, we write

$$F_k(U,V) := \{(x,y) \in \partial U \times \partial V : 2^{-k} \leqslant d(x,y) < 2^{-k+1}\},$$

$$E_k(U) := \{(x,y) \in \overline{U} \times \overline{U} : d(x,y) \geqslant 2^{-k}\}.$$

Definition 2.19. Let U be a bounded trace admissible domain. A decreasing sequence of trace admissible domains $V_j \subset U$ for $j \geqslant 0$ is said to be an *interior cutoff sequence* of U, if $V_0 = U$, $\bigcap_{j=0}^\infty V_j = \varnothing$, $\partial U \subset \partial V_j$, and $\mathrm{dist}(\partial U, \partial V_j \backslash \partial U) > 0$ for all $j \geqslant 1$.

Theorem 2.20. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space satisfying the capacity upper bound condition with index $\sigma \in \mathbb{R}$. Let $U \subset X$ be a bounded trace admissible domain admitting an interior cutoff sequence $\{V_j\}_{j \geqslant 0}$. Then for any $\theta > \sigma + \overline{\dim}_B(\partial U)$, we have*

$$\mathrm{Dom}(\Delta, U) \cap C^\theta(\overline{U}) \subset \mathrm{Dom}_N(\Delta, U).$$

Proof. Write $\gamma := \overline{\dim}_B(\partial U)$ and take $\epsilon > 0$ such that $\sigma + \gamma + \epsilon < \theta$. Let $u \in \mathrm{Dom}(\Delta, U) \cap C^\theta(\overline{U})$ and write $\Delta u = \eta \cdot \mu$. Then there is a constant $C \geqslant 1$ such that

$$|u(x) - u(y)| \leqslant Cd(x,y)^\theta \quad \textit{for all } x, y \in \overline{U}. \tag{2.4}$$

Let $\varphi \in \mathcal{F}(U) \cap L^\infty(U, m)$. Since $\mathrm{dist}(\partial U, \partial V_j \backslash \partial U) > 0$, there is $\psi \in \mathcal{F} \cap L^\infty(X, m)$ such that $\tilde{\psi}|_{\partial U} = 1$ and $\tilde{\psi}|_{\partial V_j \backslash \partial U} = 0$ q.e., which implies $(\mathbf{1}_{\partial U} \cdot \tilde{\varphi})|_{\partial V_j} = (\tilde{\psi} \cdot \tilde{\varphi})|_{\partial V_j} \in \check{\mathcal{F}}(\partial V_j)$. Define $\varphi_j : U \to \mathbb{R}$ first on V_j by $\varphi_j := H_{\partial V_j}((\mathbf{1}_{\partial U} \cdot \tilde{\varphi})|_{\partial V_j})$ and extending to $U \backslash V_j$ by assigning value zero. Since $\tilde{\varphi}_j|_{\partial V_j \backslash \partial U} = 0$ q.e., we have $\varphi_j \in \mathcal{F}(U)$ and $\tilde{\varphi}_j|_{\partial U} = \tilde{\varphi}|_{\partial U}$ q.e.. Then by Lemma 2.11, we have

$$|\mathcal{E}_U(u, \varphi - \varphi_j)| = \int_U (\tilde{\varphi} - \tilde{\varphi}_j) d\Delta u. \tag{2.5}$$

It remains to prove

$$\lim_{j \to \infty} \mathcal{E}_U(u, \varphi_j) = 0 \quad \textit{and} \quad \lim_{j \to \infty} \int_U \tilde{\varphi}_j d\Delta u = 0.$$

On one hand, we have

$$\|\tilde{\varphi}_j\|_{L^\infty(U,\mu)} \leqslant \|\varphi_j\|_{L^\infty(U,m)} = \|\mathbf{1}_{\partial U} \cdot \tilde{\varphi}\|_{L^\infty(\partial U, \mathcal{E})} \leqslant \|\varphi\|_{L^\infty(U,m)},$$

which implies

$$\lim_{j \to \infty} \left| \int_U \tilde{\varphi}_j d\Delta u \right| \leqslant \lim_{j \to \infty} \|\varphi_j\|_{L^\infty(X,\mu)} \cdot \mu(V_j) \leqslant \|\varphi\|_{L^\infty(U,m)} \cdot \lim_{j \to \infty} \mu(V_j) = 0. \tag{2.6}$$

On the other hand, fix $\delta_0 > 0$ such that $s_\delta(\partial U) \lesssim \delta^{-\gamma-\epsilon}$ for all $\delta < \delta_0$ and take $k_0 \in \mathbb{Z}$ such that $\delta_0 > 2^{-k_0}$. Then we have

$$\begin{aligned}
|\mathcal{E}_U(u,\varphi_j)| &= |\mathcal{E}_{V_j}(u|_{V_j},\varphi_j|_{V_j})| = |\check{\mathcal{E}}_{\partial V_j}(u|_{\partial V_j},\tilde{\varphi}_j|_{\partial V_j})| \\
&= \left| \int_{\partial V_j \times_{\mathrm{od}} \partial V_j} (u(x)-u(y))(\tilde{\varphi}_j(x)-\tilde{\varphi}_j(y)) dJ_{\partial V_j}(x,y) \right| \\
&= 2\left| \int_{\partial V_j \times_{\mathrm{od}} \partial V_j} (u(x)-u(y))\tilde{\varphi}_j(x) dJ_{\partial V_j}(x,y) \right| \\
&\leqslant 2\int_{\partial U \times_{\mathrm{od}} \partial V_j} |u(x)-u(y)||\tilde{\varphi}(x)| dJ_{\partial V_j}(x,y) \\
&\leqslant 2\|\varphi\|_{L^\infty(U,m)} \int_{\partial U \times_{\mathrm{od}} \partial V_j} |u(x)-u(y)| dJ_{\partial V_j}(x,y) \\
&\leqslant 2\|\varphi\|_{L^\infty(U,m)} \Bigg(\int_{E_{k_0}(U)\cap(\partial V_j \times \partial V_j)} |u(x)-u(y)| dJ_{\partial V_j}(x,y) \\
&+ \sum_{k=k_0+1}^{\infty} \int_{F_k(U,V_j)} |u(x)-u(y)| dJ_{\partial V_j}(x,y) \Bigg) \\
&\leqslant 2\|\varphi\|_{L^\infty(U,m)} \left(2\|u\|_{C(\overline{U})} J_{\partial V_j}(E_{k_0}(U)\cap(\partial V_j \times \partial V_j)) + C \sum_{k=k_0+1}^{\infty} 2^{-k\theta} J_{\partial V_j}(F_k(U,V_j)) \right) \\
&\leqslant 2\|\varphi\|_{L^\infty(U,m)} \left(2\|u\|_{C(\overline{U})} J_{\partial V_j}(E_{k_0}(U)\cap(\partial V_j \times \partial V_j)) + C \sum_{k=k_0+1}^{\infty} 2^{-k(\theta-\sigma-\gamma-\epsilon)} \right) \\
&\leqslant 2\|\varphi\|_{L^\infty(U,m)} \left(2\|u\|_{C(\overline{U})} J_{\partial V_j}(E_{k_0}(U)\cap(\partial V_j \times \partial V_j)) + C 2^{-k_0(\theta-\sigma-\gamma-\epsilon)} \right),
\end{aligned}$$

where the last second line follows from Lemma 2.16 (2) and the assumption $s_\delta(\partial U) \lesssim \delta^{-\gamma-\epsilon}$, and the last line follows from that $\theta > \sigma+\gamma+\epsilon$. Then by Lemma 2.17 (2), we have

$$\limsup_{j\to\infty} |\mathcal{E}_U(u,\varphi_j)| \leqslant 2\|\varphi\|_{L^\infty(U,m)} \cdot C 2^{-k_0(\theta-\sigma-\gamma-\epsilon)}$$

for all $k_0 \in \mathbb{Z}$ such that $\delta_0 > 2^{-k_0}$. Let $k_0 \to \infty$, we derive

$$\lim_{j\to\infty} |\mathcal{E}_U(u,\varphi_j)| = 0. \tag{2.7}$$

Then combining (2.5), (2.6) and (2.7), we have

$$\mathcal{E}_U(u,\varphi) = \int_U \tilde{\varphi} d\Delta u$$

for all $\varphi \in \mathcal{F}(U) \cap L^\infty(U,m)$, thus $u \in \mathrm{dom}_N(\Delta, U)$. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let $u \in \mathrm{Dom}\Delta \cap C^\theta(X)$. Then for each $U \in \mathcal{C}$, we have $u|_U \in \mathrm{Dom}(\Delta, U)$ with $\Delta(u|_U) = \mathbf{1}_U \cdot \Delta u$. Then by Theorem 2.20, we have $u|_U \in \mathrm{Dom}_N(\Delta, U)$, that is,

$$\mathcal{E}_U(u|_U, \varphi) = -\int_U \tilde{\varphi} d\Delta u \quad \textit{for all } \varphi \in \mathcal{F}(U) \cap L^\infty(U, m).$$

Since U is bounded, we have $\mathbf{1}_U \in \mathcal{F}(U) \cap L^\infty(U, m)$, which gives

$$\int_U d\Delta u = -\mathcal{E}_U(u|_U, \mathbf{1}_U) = 0.$$

Then by Lemma 2.4, we have $\Delta u = 0$, which gives

$$\mathcal{E}(u, u) = \int_U \tilde{u} d\Delta u = 0,$$

that is, $u \in \ker\mathcal{E}$. □

3 Examples

We are mainly concerned with two examples: the self-similar Dirichlet form on the Sierpinski carpet, and the Dirichlet forms induced by resistance forms on finitely ramified fractals, including the Sierpinski gasket.

3.1 Sierpinski carpet

The Sierpinski carpet is a compact subset of $\mathbb{R}^2$. Let $Q_0 = [0,1]^2 \subset \mathbb{R}^2$ and define $F_i : Q_0 \to Q_0$ by $F_i(x) = p_i + (x - p_i)/3$ for $1 \leqslant i \leqslant 8$, where q_1–q_4 are the vertices of Q_0 and q_5–q_8 are the midpoints of the edges of Q_0. Then there is a unique nonempty compact set $K \subset Q_0$ satisfying $K = \bigcup_{i=1}^8 F_i(K)$, which is the *Sierpinski carpet*. Let d be the Euclidean distance and m be the $(\log 8/\log 3)$-dimensional Hausdorff measure on K, which is α-Ahlfors regular with $\alpha = \log 8/\log 3$. In [BB89, KZ], a Dirichlet form $(\mathcal{E}, \mathcal{F})$ was constructed on K, satisfying heat kernel estimates HKE(β) for some $\beta > 2$. See [BB99] for the heat kernel estimates. Moreover, it has the following self-similarity; see [KZ, Theorem 6.9, Theorem 7.19]:

(1) If $f \in \mathcal{F}$, then $f \circ F_i \in \mathcal{F}$ for all $1 \leqslant i \leqslant 8$, and

$$\mathcal{E}(f, f) = 3^\sigma \sum_{i=1}^8 \mathcal{E}(f \circ F_i, f \circ F_i). \tag{3.1}$$

(2) If $f \in C(K)$ and $f \circ F_i \in \mathcal{F}$ for all $1 \leqslant i \leqslant 8$, then $f \in \mathcal{F}$.

Here $\sigma = \beta - \alpha$ is the index of capacity upper bound. Every function in $\mathcal{F}$ admits a continuous version, thus we can directly assume $\mathcal{F}$ to be a subfamily of $C(K)$. In [BB90, equation (5.5)], it has been proved that $\rho = 3^\sigma \leqslant 1.27656$, which implies $\sigma \leqslant 0.23$.

First we give a measure determining class $\mathcal{C}$ of $\mathbb{R}^2$. Then for any similitude $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(\mathcal{C}):=\{f(C):C\in\mathcal{C}\}$ is still a measure determining class of $\mathbb{R}^2$.

For each $n\in\mathbb{Z}$, define *n-level triadic cubes* in $\mathbb{R}^2$ by

$$\mathcal{Q}_n:=\big\{[k_1\cdot 3^{-n},(k_1+1)\cdot 3^{-n}]\times[k_2\cdot 3^{-n},(k_2+1)\cdot 3^{-n}]:k_1,k_2\in\mathbb{Z}\big\}.$$

Two cubes Q_1,Q_2 in $\mathcal{Q}_n$ are said to be *adjacent*, if $Q_1\cap Q_2$ is a one-dimensional interval. Then define the collection of rectangles formed by two n-level triadic cubes

$$\mathcal{Q}_n':=\{Q_1\cup Q_2:Q_1,Q_2\in\mathcal{Q}\ \textit{are adjacent}\}.$$

Moreover, define the collection of cubes formed by four n-level triadic cubes

$$\mathcal{Q}_n'':=\big\{[k_1\cdot 3^{-n},(k_1+2)\cdot 3^{-n}]\times[k_2\cdot 3^{-n},(k_2+2)\cdot 3^{-n}]:k_1,k_2\in\mathbb{Z}\big\}.$$

Recall that a class of subsets $\mathcal{C}$ of X is said to be a *π-system*, if $A,B\in\mathcal{C}$ imply $A\cap B\in\mathcal{C}$. A class of subsets $\mathcal{G}$ of X is said to be a *λ-system*, if $X\in\mathcal{G}$, $A,B\in\mathcal{G}$ with $A\subset B$ imply $B\backslash A\in\mathcal{G}$, and an increasing sequence $A_n\in\mathcal{G}$ implies $\bigcup_{n\geqslant 1}A_n\in\mathcal{G}$. Clearly, the intersection of a collection of λ-systems is still a λ-system; thus, every class of subsets $\mathcal{C}$ of X admits a smallest λ-system $\mathcal{G}$ such that $\mathcal{C}\subset\mathcal{G}$, which is called the λ-system generated by $\mathcal{C}$. The well-known monotone class theorem asserts that if $\mathcal{C}$ is a π-system, then the λ-system generated by $\mathcal{C}$ coincides with the σ-field generated by $\mathcal{C}$.

Moreover, for each $E\subset X$ and class of subsets $\mathcal{G}$ of X, we write $\mathcal{G}\cap E:=\{A\cap E:A\in\mathcal{G}\}$ as a collection of subsets of E.

Lemma 3.1. *Define $\mathcal{C}$ as the family of the interior of the closed sets in $\bigcup_{n\in\mathbb{Z}}(\mathcal{Q}_n\cup\mathcal{Q}_n'\cup\mathcal{Q}_n'')$. Then $\mathcal{C}$ is a measure determining class of $\mathbb{R}^2$.*

Proof. Fix two Radon measures μ and ν on $\mathbb{R}^2$ such that $\mu(C)=\nu(C)$ for all $C\in\mathcal{C}$.

Define the collection of 2-dimensional n-level triadic open squares $\mathcal{C}_n^{(2)}$ in $\mathbb{R}^2$ as the interior of all the closed sets in $\mathcal{Q}_n$. Then $\mathcal{C}_n^{(2)}$ is a subfamily of $\mathcal{C}$, hence $\mu(C)=\nu(C)$ for all $C\in\mathcal{C}_n^{(2)}$. Similarly, define the lower dimensional n-level triadic open intervals on $\mathbb{R}^2$ as $\mathcal{C}_n^{(1)}$, $\mathcal{C}_n^{(0)}$ by

$$\mathcal{C}_n^{(1)}:=\big\{(k_1\cdot 3^{-n},(k_1+1)\cdot 3^{-n})\times\{k_2\cdot 3^{-n}\},\{k_1\cdot 3^{-n}\}\times(k_2\cdot 3^{-n},(k_2+1)\cdot 3^{-n}):k_1,k_2\in\mathbb{Z}\big\}$$

$$\mathcal{C}_n^{(0)}:=\big\{\{k_1\cdot 3^{-n}\}\times\{k_2\cdot 3^{-n}\}:k_1,k_2\in\mathbb{Z}\big\}.$$

We claim that $\mu(C)=\nu(C)$ for all $C\in\mathcal{C}_n^{(1)}\cup\mathcal{C}_n^{(0)}$.

For $C\in\mathcal{C}_n^{(1)}$, without loss of generality, we assume $C=(0,3^{-n})\times\{0\}$. Let $C_1:=(0,3^{-n})\times(-3^{-n},3^{-n})$, $C_2:=(0,3^{-n})\times(-3^{-n},0)$ and $C_3:=(0,3^{-n})\times(0,3^{-n})$, which satisfy that $C_1,C_2,C_3\in\mathcal{C}$, $C_2\cap C_3=\varnothing$, $C_2\cup C_3\subset C_1$. Moreover, we have $C=C_1\backslash(C_2\cup C_3)$. Then we have

$$\mu(C)=\mu(C_1)-\mu(C_2)-\mu(C_3)=\nu(C_1)-\nu(C_2)-\nu(C_3)=\nu(C).$$

Thus $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}_n^{(1)}$.

For $C \in \mathcal{C}_n^{(0)}$, without loss of generality, we assume $C = \{(0,0)\}$. Let $C' := (-3^{-n}, 3^{-n}) \times \{0\}$ and we first prove $\mu(C') = \nu(C')$. Indeed, by taking $C_1' := (-3^{-n}, 3^{-n}) \times (-3^{-n}, 3^{-n})$, $C_2' := (-3^{-n}, 3^{-n}) \times (-3^{-n}, 0)$ and $C_3' := (-3^{-n}, 3^{-n}) \times (0, 3^{-n})$, a similar argument as above shows that $\mu(C') = \nu(C')$. Then we define $C_1'' := (-3^{-n}, 0) \times \{0\}$, $C_2'' := (0, 3^{-n}) \times \{0\}$, which satisfies $C_1'', C_2'' \in \mathcal{C}_n^{(1)}$, and then

$$\mu(C) = \mu(C') - \mu(C_1'') - \mu(C_2'') = \nu(C') - \nu(C_1'') - \nu(C_2'') = \nu(C).$$

Thus $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}_n^{(0)}$.

Now we define $\mathcal{G} := \{B \in \mathcal{B}(\mathbb{R}^2) : \mu(B) = \nu(B)\}$ and let $\mathcal{C}_n^{(*)} := \bigcup_{j=0}^{2} \mathcal{C}_n^{(j)}$, $\mathcal{C}_*^{(*)} := \bigcup_{n\in\mathbb{Z}} \mathcal{C}_n^{(*)}$. Then we have $\mathcal{C}_*^{(*)} \subset \mathcal{G}$ as proved before. Moreover, each $\mathcal{C}_n^{(*)}$ gives a disjoint partition of $\mathbb{R}^2$ and $\mathcal{C}_{n+1}^{(*)}$ is finer than $\mathcal{C}_n^{(*)}$ for all $n \in \mathbb{Z}$. Then for each $C_1, C_2 \in \mathcal{C}_*^{(*)}$, we have $C_1 \cap C_2 \in \{C_1, C_2, \varnothing\}$. Thus $\mathcal{C}_*^{(*)} \cup \{\varnothing\}$ is a π-system.

Take $Q_n := (-3^{-n}, 3^{-n})^2 \in \mathcal{C}$ for $n \in \mathbb{Z}$ and let $\mathcal{G}_n := \{B \in \mathcal{B}(Q_n) : \mu(B) = \nu(B)\}$. Since μ, ν are Radon measures, their restriction on Q_n are finite measures, thus $A, B \in \mathcal{G}_n$ and $A \subset B$ implies $B \backslash A \in \mathcal{G}_n$. Furthermore, $Q_n \in \mathcal{C} \cap Q_n \subset \mathcal{G}_n$ and $\mathcal{G}_n$ is closed under increasing limit, thus $\mathcal{G}_n$ is a λ-system. Note that $\mathcal{C}_*^{(*)} \cap Q_n$ is a π-system generating $\mathcal{B}(Q_n)$ and $\mathcal{C}_*^{(*)} \cap Q_n \subset \mathcal{G}_n$. By monotone class theorem, we conclude $\mathcal{B}(Q_n) = \mathcal{G}_n$.

Then for any Borel set $B \in \mathcal{B}(\mathbb{R}^2)$, we have

$$\mu(B) = \lim_{n\to-\infty} \mu(B \cap Q_n) = \lim_{n\to-\infty} \nu(B \cap Q_n) = \nu(B).$$

Thus $\mathcal{G} = \mathcal{B}(\mathbb{R}^2)$ and then $\mu = \nu$. □

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) := (x + \frac{1}{2}, y + \frac{1}{2})$. Then the class $f(\mathcal{C})$ is also a measure determining class of $\mathbb{R}^2$, where $\mathcal{C}$ is the class defined in the above lemma. Each $Q \in f(\mathcal{Q}_n)$ can be expressed as

$$Q = \left[\frac{2k_1 - 1}{2 \cdot 3^n}, \frac{2k_1 + 1}{2 \cdot 3^n}\right] \times \left[\frac{2k_2 - 1}{2 \cdot 3^n}, \frac{2k_2 + 1}{2 \cdot 3^n}\right]$$

for some $k_1, k_2 \in \mathbb{Z}$. For the Sierpinski carpet K, we define $\mathcal{C}_K := f(\mathcal{C}) \cap K$. Then $\mathcal{C}_K$ is a measure determining class of K. For each $U \in \mathcal{C}_K$, we write $\partial_K U$ for the boundary of U relative to the topology in K.

Lemma 3.2. *Let $\mathcal{C}_K$ be defined as above. Then*

(1) For each $U \in \mathcal{C}_K$, we have $\overline{\dim}_B U \leqslant \log 2/\log 3$.

(2) Each $U \in \mathcal{C}_K$ is a uniform domain in K.

(3) Each $U \in \mathcal{C}_K$ admits an interior cutoff sequence in K.

Proof. (1) Note that for every $n, k \in \mathbb{Z}$, $(\{\frac{2k+1}{2\cdot 3^n}\} \times [0,1]) \cap K$ and $([0,1] \times \{\frac{2k+1}{2\cdot 3^n}\}) \cap K$ are finite union of Cantor sets with upper box dimension $\log 2/\log 3$. Since the boundary of each $U \in \mathcal{C}_K$ is contained in a finite union of these sets, $\overline{\dim}_B(\partial_K U) \leqslant \log 2/\log 3$.

(2) We only prove the case $U = ((1/2,1] \times [0,1]) \cap K$ (the right half of K), since the other cases follow in a similar manner. Write $S := \{1/2\} \times [0,1]$. Then $\partial_K U = S \cap K$.

First we collect all triadic cubes removed in the construction of K, that is the collection of all connected components of $[0,1]^2 \backslash K$. We denote this collection of open sets by $\hat{\mathcal{Q}}$. Moreover, collect all cubes in $\hat{\mathcal{Q}}$ intersecting with the symmetry axis S, that is

$$\hat{\mathcal{Q}}_1 := \{Q \in \hat{\mathcal{Q}} : Q \cap S \neq \varnothing\}.$$

Then remove all $Q \in \hat{\mathcal{Q}}_1$ from the unit cube $[0,1]^2$ and the right-half cube $(1/2,1] \times [0,1]$, that is

$$E := [0,1]^2 \backslash \bigcup_{Q \in \hat{\mathcal{Q}}_1} Q \quad \textit{and} \quad F := ((1/2,1] \times [0,1]) \backslash \bigcup_{Q \in \hat{\mathcal{Q}}_1} Q.$$

Then F is a uniform domain in E. Indeed, one can connect any two points $x, y \in F$ by a curve moving first horizontally, then vertically and finally horizontally such that this curve satisfies the conditions of uniform domain.

For each curve γ in F, we define the collection of all cubes in $\hat{\mathcal{Q}}$ intersecting with γ by $\hat{\mathcal{Q}}(\gamma) := \{Q \in \hat{\mathcal{Q}} : Q \cap \gamma \neq \varnothing\}$. Note that for each $Q \in \hat{\mathcal{Q}}(\gamma)$, there is a path-connected subset $\tilde{\gamma}(Q) \subset \partial Q$ such that $(\gamma \backslash Q) \cup \tilde{\gamma}(Q)$ is path-connected and

$$\mathrm{diam}\tilde{\gamma}(Q) \leqslant 2 \cdot \mathrm{diam}(\gamma \cap Q) \leqslant 2 \cdot \mathrm{diam}\gamma. \tag{3.2}$$

Since $\gamma \subset F$, we have $\hat{\mathcal{Q}}(\gamma) \cap \hat{\mathcal{Q}}_1 = \varnothing$. For each γ in F, we define its modification in K by

$$\tilde{\gamma} := (\gamma \cap K) \cup \bigcup_{Q \in \hat{\mathcal{Q}}(\gamma)} \tilde{\gamma}(Q).$$

In other words, we substitute $\gamma \cap Q$ by $\tilde{\gamma}(Q) \subset \partial Q$ for every $Q \in \hat{\mathcal{Q}}(\gamma)$. By substituting each $Q \in \hat{\mathcal{Q}}(\gamma)$ one by one, we can prove that $\tilde{\gamma}$ is a path-connected set in $F \cap K = ((1/2,1] \times [0,1]) \cap K$ since it is a limit of path-connected closed sets in Gromov-Hausdorff distance.

Note that there is a constant $C \geqslant 1$ such that for each $Q \in \hat{\mathcal{Q}} \backslash \hat{\mathcal{Q}}_1$, we have

$$\mathrm{diam}(Q) \leqslant C \cdot \mathrm{dist}(Q, S).$$

For each $x, y \in U$, take $\gamma \subset F$ such that $x, y \in \gamma$, $\mathrm{diam}\gamma \leqslant A \cdot d(x,y)$ and

$$d(x,z) \wedge d(y,z) \leqslant A \cdot \mathrm{dist}(\{z\}, S) \quad \textit{for all } z \in \gamma,$$

where $A \geqslant 1$ is the constant for uniform domain F. We will prove that the modified curve $\tilde{\gamma}$ in U satisfies the conditions of uniform domain. Since $x, y \in U \subset K$ and $\gamma \cap K \subset \tilde{\gamma}$, we have $x, y \in \tilde{\gamma}$.

For each $Q_j \in \hat{\mathcal{Q}}(\gamma)$ and $z_j' \in \tilde{\gamma}(Q_j)$ with $j = 1, 2$, take $z_j \in \gamma \cap Q_j$, which gives

$$\begin{aligned} d(z_1', z_2') &\leqslant d(z_1, z_2) + d(z_1, z_1') + d(z_2, z_2') \leqslant \mathrm{diam}\gamma + \mathrm{diam}\tilde{\gamma}(Q_1) + \mathrm{diam}\tilde{\gamma}(Q_2) \\ &\leqslant 5\mathrm{diam}\gamma \leqslant 5A \cdot d(x, y), \end{aligned}$$

where the third inequality follows by (3.2). Similarly, we have $d(z_1', z_2') \leqslant A' \cdot d(x, y)$ for all $z_1', z_2' \in \tilde{\gamma}$, where $A' \geqslant 1$ is a constant depending only on A. Hence

$$\mathrm{diam}\tilde{\gamma} \leqslant A' \cdot d(x, y).$$

For each $Q \in \hat{\mathcal{Q}}(\gamma)$ and $z' \in \tilde{\gamma}(Q)$ take $z \in \gamma \cap Q$. Without loss of generality we assume $d(x, z) \leqslant A \cdot \mathrm{dist}(\{z\}, S)$. Then

$$\begin{aligned} d(x, z') \leqslant d(x, z) + d(z, z') &\leqslant A \cdot \mathrm{dist}(\{z\}, S) + \mathrm{diam}(Q) \\ &\leqslant A \cdot \mathrm{dist}(\{z'\}, S) + (A + 1) \cdot \mathrm{diam}(Q) \\ &\leqslant A \cdot \mathrm{dist}(\{z'\}, S) + C(A + 1) \cdot \mathrm{dist}(Q, S) \\ &\leqslant (CA + C + A)\mathrm{dist}(\{z'\}, S). \end{aligned}$$

Thus $\tilde{\gamma}$ satisfies that

$$d(x, z') \wedge d(y, z') \leqslant A'' \cdot \mathrm{dist}(\{z'\}, S) \quad \textit{for all } z' \in \tilde{\gamma}$$

for some constant $A'' \geqslant 1$. Thus U is a uniform domain.

(3) We still only consider $U = ((1/2, 1] \times [0, 1]) \cap K$. Note that $(K, d, m, \mathcal{E}, \mathcal{F})$ is volume doubling and satisfies heat kernel estimates. We define $V_0 := U$ and

$$V_j := K \cap \left((\frac{1}{2}, \frac{1}{2} + \frac{1}{3^j}) \times [0, 1] \right).$$

Clearly it satisfies $\bigcap_{j=0}^{\infty} V_j = \varnothing$, $\partial_K U \subset \partial_K V_j$ and $\mathrm{dist}(\partial_K U, \partial_K V_j \backslash \partial_K U) > 0$ for all $j \geqslant 1$. It remains to prove that each V_j is trace admissible.

Since $V_0 \in \mathcal{C}$, it is trace admissible. For each $j \geqslant 1$, the same discussion as in the proof of (2) yields that every connected component of V_j is a uniform domain, and is thus trace admissible by Propositions 2.9 and 2.13. Since each V_j has only finitely many components and each pair of components has positive distance, by Lemma 2.15, each V_j is trace admissible. Thus V_j, $j \geqslant 0$, is an interior cutoff sequence of U. □

Theorem 3.3. *Let $\sigma \in \mathbb{R}$ be the same constant in* (3.1) *and $\gamma := \log 2/\log 3$. Then for any $\theta > \sigma + \gamma$, we have*

$$\mathrm{Dom}\Delta \cap C^{\theta}(K) = \mathbb{R}\mathbf{1}_K.$$

Proof. This follows by combining Theorem 1.2 and Lemmas 3.1 and 3.2. □

Proof of Theorem 1.3. This follows by Theorem 3.3, and the estimate that $\sigma + \gamma < 1$. □

3.2 Resistance form on finitely ramified fractals

3.2.1 Basic setting

Before introducing the geometric setting, we first recall the notion of the resistance form and the resistance distance in [Kig12].

Resistance form theory is parallel to Dirichlet form theory, with the distinctive feature that a resistance form requires every single point to have positive capacity. A detailed study of resistance form theory on p.c.f. fractals, as well as its relation to Dirichlet form theory, can be found in [Kig01, Kig12].

For each set S, we write $l(S)$ as the family of all real functions defined on S. Let X be a set. A pair $(\mathcal{E}_{\rm res}, \mathcal{F}_{\rm res})$ is said to be a *resistance form* on X, if

(1) $\mathcal{F}_{\rm res}$ is a subspace of $l(X)$ containing all constant functions, and $\mathcal{E}_{\rm res}$ is a nonnegative symmetric bilinear form on $\mathcal{F}_{\rm res}$ such that $\mathcal{E}_{\rm res}(u,u) = 0$ if and only if u is a constant function.

(2) $(\mathcal{E}_{\rm res}, \mathcal{F}_{\rm res}/\mathcal{R})$ is a Hilbert space, where $\mathcal{R}$ is an equivalence relation defined on $\mathcal{F}_{\rm res}$ by $u\mathcal{R}v$ if and only if $u - v$ is a constant function.

(3) For any nonempty finite subset $F \subset X$ and $u \in l(F)$, there exists $\tilde{u} \in \mathcal{F}_{\rm res}$ such that $\tilde{u}|_F = u$.

(4) For any $x, y \in X$,

$$R(x,y) := \sup\{|u(x) - u(y)|^2 : u \in \mathcal{F}_{\rm res}, \mathcal{E}_{\rm res}(u,u) \leqslant 1\} < \infty. \tag{3.3}$$

(5) For any $u \in l(X)$ define $\overline{u} := (u \wedge 1)_+$. Then $u \in \mathcal{F}_{\rm res}$ implies $\overline{u} \in \mathcal{F}_{\rm res}$ and $\mathcal{E}_{\rm res}(\overline{u}, \overline{u}) \leqslant \mathcal{E}_{\rm res}(u,u)$.

Moreover, $(\mathcal{E}_{\rm res}, \mathcal{F}_{\rm res})$ is said to be *regular*, if $\mathcal{F}_{\rm res} \cap C_0(X,R)$ is dense in $C_0(X,R)$. $(\mathcal{E}_{\rm res}, \mathcal{F}_{\rm res})$ is said to be *local*, if $\mathcal{E}_{\rm res}(u,v) = 0$ for all $u, v \in \mathcal{F}_{\rm res}$ satisfying $\mathrm{dist}_R(\{u \neq 0\}, \{v \neq 0\}) > 0$. See [Kig12, Definition 3.1, Definition 6.2, Definition 7.5] for details.

The function R defined by (3.3) is a distance on X by [Kig12, Proposition 3.3], called the *resistance distance* induced by $(\mathcal{E}_{\rm res}, \mathcal{F}_{\rm res})$. It is worth noting that in a typical situation, (X,R) is not a quasi-convex metric space, which implies that the family of Hölder continuous functions $C^\theta(X,R)$ might contain non-constant functions even if $\theta > 1$. Here $C^\theta(X,d)$ is defined as in Definition 2.18.

For example, the resistance distance R of the Sierpinski gasket $\mathcal{SG}$ satisfies $R(x,y) \asymp d(x,y)^\sigma$ with $\sigma = \frac{\log(5/3)}{\log 2} < 1$, where $d(x,y)$ is the Euclidean distance between x and y. Therefore $C^\theta(X,R) = C^{\sigma\theta}(X,d)$ contains non-constant functions when $0 < \theta \leqslant 1/\sigma$.

The following relation between resistance forms and Dirichlet forms can be found in [Kig12, Theorem 9.4].

Proposition 3.4. *Let $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ be a local regular resistance form on X, let R be the corresponding resistance distance such that (X, R) is separable, complete, and locally compact, and let m be a Radon measure on X such that $0 < m(B(x, r)) < \infty$ for all $x \in X, r > 0$. Define $\mathcal{E} = \mathcal{E}_{\mathrm{res}}$ and $\mathcal{F}$ as the $\mathcal{E}_1$-closure of $\mathcal{F} \cap C_0(X, R)$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on (X, R, m) satisfying $\ker\mathcal{E} \subset \mathbb{R}\mathbf{1}_X$.*

Furthermore, if $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ is local, then $(\mathcal{E}, \mathcal{F})$ is strongly local. If (X, R) is compact, then $\mathcal{F}_{\mathrm{res}} = \mathcal{F}$.

The following capacity upper bound estimate for resistance form is adapted from [Kig12, Corollary 7.13]. Recall that a metric space (X, d) is said to be *uniformly perfect*, if there is a constant $A \geqslant 1$ such that for any $x \in X$ and $r > 0$, if there is $y \in X$ such that $d(x, y) \geqslant r$, then there is $z \in X$ such that $r \leqslant d(x, z) \leqslant Ar$. Every connected metric space is uniformly perfect.

Proposition 3.5. *Let $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ be a local resistance form on X and R be the resistance distance. Assume (X, R) is metric doubling and uniformly perfect. Let m be a Radon measure on X and write $(\mathcal{E}, \mathcal{F})$ for the Dirichlet form on (X, R, m) induced by $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$. Then $(X, R, m, \mathcal{E}, \mathcal{F})$ satisfies capacity upper bound with index $\sigma = 1$.*

For each closed set $F \subset X$, we define the *trace* of $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ on F by

$$\check{\mathcal{F}}_{\mathrm{res}}(F) := \{u \in l(F) : \textit{there is } u^{\#} \in \mathcal{F}_{\mathrm{res}} \textit{ such that } u^{\#}|_F = u\},$$

and

$$\check{\mathcal{E}}_{\mathrm{res},F}(u, u) := \inf\{\mathcal{E}_{\mathrm{res}}(u^{\#}, u^{\#}) : u^{\#} \in \mathcal{F}_{\mathrm{res}}, u^{\#}|_F = u\}$$

for $u \in \check{\mathcal{F}}_{\mathrm{res}}(F)$. By [Kig12, Theorem 8.4], if $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ is regular, then $(\check{\mathcal{E}}_{\mathrm{res},F}, \check{\mathcal{F}}_{\mathrm{res}}(F))$ is a regular resistance form. This coincides with the trace in the setting of a Dirichlet form. Moreover, there exists a linear map $H_F : \check{\mathcal{F}}_{\mathrm{res}}(F) \to \mathcal{F}_{\mathrm{res}}$ that satisfies all the conditions in Proposition 2.14.

When the base space F is finite, we can write the trace form as a matrix form. In particular, it provides us a desired jump measure.

Lemma 3.6. *If F is a nonempty finite set, then there is a matrix $q_F(x, y) \geqslant 0$ for $(x, y) \in F \times_{\mathrm{od}} F$ such that $\check{\mathcal{F}}_{\mathrm{res}}(F) = l(F)$ and*

$$\check{\mathcal{E}}_{\mathrm{res},F}(u, u) = \sum_{(x,y)\in F\times_{\mathrm{od}}F} q_F(x, y)(u(x) - u(y))^2 \quad \textit{for all } u \in l(F).$$

This lemma together with the above discussion provides a sufficient condition of open sets to be trace admissible in the setting of resistance forms.

Corollary 3.7. *If $U \subset X$ is an extension domain of $(X, R, m, \mathcal{E}, \mathcal{F})$ and ∂U is finite, then U is trace admissible.*

Now we introduce the geometric setting of finitely ramified fractals. The following definition of finitely ramified fractal is a slight generalization of [Typ, Definition 2.1].

Definition 3.8. A locally compact, complete, unbounded metric space X is said to be a *unbounded finitely ramified fractal*, if there are two collections $\{F_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of X satisfying the following conditions.

(FR1) $\mathcal{A}$ is a countable index set.

(FR2) F_α are distinct compact connected subsets of X.

(FR3) Each V_α is a finite subset of F_α containing at least two elements.

(FR4) There is a filtration $\{\mathcal{A}_n\}_{n\in\mathbb{Z}}$ such that

- (a) Each $\mathcal{A}_n$ is a subset of $\mathcal{A}$ such that $X = \bigcup_{\alpha\in\mathcal{A}_n} F_\alpha$, and for each $x \in X$, there is $n \in \mathbb{Z}$ such that there is a unique $\alpha \in \mathcal{A}_n$ such that $x \in F_\alpha$.
- (b) $\mathcal{A}_n \cap \mathcal{A}_m = \varnothing$ if $n \neq m$ and $\mathcal{A} = \bigcup_{n\in\mathbb{Z}} \mathcal{A}_n$.
- (c) For any $\alpha \in \mathcal{A}_n$, there is a finite subset $\mathcal{A}_\alpha \subset \mathcal{A}_{n+1}$ such that $F_\alpha = \bigcup_{\beta\in\mathcal{A}_\alpha} F_\beta$.

(FR5) If $F_\alpha = \bigcup_{\beta\in\mathcal{A}'} F_\beta$ for some finite subset $\mathcal{A}' \subset \mathcal{A}$, then $V_\alpha \subset \bigcup_{\beta\in\mathcal{A}'} V_\beta$.

(FR6) For any distinct $\alpha \neq \beta \in \mathcal{A}_n$, $F_\alpha \cap F_\beta \subset V_\alpha \cap V_\beta$.

(FR7) For any strictly decreasing sequence $F_{\alpha_1} \supsetneq F_{\alpha_2} \supsetneq \cdots$, there is $x \in X$ such that $\bigcap_{j=1}^{\infty} F_{\alpha_j} = \{x\}$.

Each F_α with $\alpha \in \mathcal{A}_n$ is said to be a *n-cell* of X and V_α is said to be the *cell boundary* of F_α. The tuple $(X, \{F_\alpha\}_{\alpha\in\mathcal{A}}, \{V_\alpha\}_{\alpha\in\mathcal{A}})$ is said to be a *finitely ramified cell structure* of X.

Alternatively, if X is a compact metric space, then the filtration is taken to be $\{\mathcal{A}_n\}_{n\geqslant 0}$, and the condition (FR4.a) is strengthened by that each $\mathcal{A}_n$ is a finite subset of $\mathcal{A}$ with $\mathcal{A}_0 = \{0\}$, $F_0 = X$. In this case, X is said to be a *bounded finitely ramified fractal*.

Roughly speaking, a locally compact metric space X is a finitely ramified fractal if it can be broken into small pieces at every scale such that the boundary of each piece consists only of finitely many points; see also [Typ, Proposition 2.7]. A typical example of a finitely ramified fractal is the one-dimensional ray $\mathbb{R}_+$ with a finitely ramified cell structure given by the collection of all dyadic intervals.

3.2.2 Local finiteness of finitely ramified fractals

The definition of a bounded finitely ramified fractal exactly coincides with the definition in [Typ, Definition 2.1]. The generalization to the unbounded situation causes no difficulty, since each cell F_α is compact and $(F_\alpha, \{F_\beta\}_{\beta\in\mathcal{A}'}, \{V_\beta\}_{\beta\in\mathcal{A}'})$ with $\mathcal{A}' := \{\beta \in \mathcal{A} : \mathcal{A}_\beta \subset \mathcal{A}_\alpha\}$ gives a finitely ramified cell structure on F_α. All the propositions in [Typ, Section 2] can be proved for locally compact finitely ramified fractals with the same arguments. Nevertheless, because of the relaxation of condition (FR4.a), we provide some properties concerning the locally finite property of X.

Lemma 3.9. *Assume X is an unbounded finitely ramified fractal. For each $\alpha \in \mathcal{A}_n, n \in \mathbb{Z}$, take a finite class $\mathcal{A}_\alpha \subset \mathcal{A}_{n+1}$ such that $F_\alpha = \bigcup_{\beta\in\mathcal{A}_\alpha} F_\beta$ as ensured by (FR4.c). Moreover, for each $\alpha \in \mathcal{A}_n$ and $k \geqslant 1$, we define*

$$\mathcal{A}_{\alpha,k} := \{\beta \in \mathcal{A}_{n+k} : F_\beta \subset F_\alpha\}.$$

(1) $\mathcal{A}_{\alpha,k} \cap \mathcal{A}_{\alpha',k} = \varnothing$ for distinct $\alpha, \alpha' \in \mathcal{A}_n$ and $k \geqslant 1$.

(2) $\mathcal{A}_{n+1} = \bigcup_{\alpha\in\mathcal{A}_n} \mathcal{A}_\alpha$.

(3) $\mathcal{A}_{\alpha,1} = \mathcal{A}_\alpha$ for all $\alpha \in \mathcal{A}_n$.

(4) $\mathcal{A}_{n+k} = \bigcup_{\alpha\in\mathcal{A}_n} \mathcal{A}_{\alpha,k}$.

(5) $\mathcal{A}_{\alpha,k+1} = \bigcup_{\beta\in\mathcal{A}_{\alpha,k}} \mathcal{A}_\beta$ for $k \geqslant 1$.

(6) For each $\alpha \in \mathcal{A}_n$ and $k \geqslant 1$, the class $\mathcal{A}_{\alpha,k}$ is finite.

Proof. (1) If $\beta \in \mathcal{A}_{\alpha,k} \cap \mathcal{A}_{\alpha',k}$, $F_\beta \in F_\alpha \cap F_{\alpha'}$. Then a contradiction follows from that F_β is uncountable by [Typ, Proposition 2.8] and $F_\alpha \cap F_{\alpha'} = V_\alpha \cap V_{\alpha'}$ is a finite set.

(2) It is enough to prove that the left hand side is contained in the right hand side. Take $\gamma \in \mathcal{A}_{n+1}$. Then F_γ is an uncountable set by [Typ, Proposition 2.8]. Then there is $\alpha \in \mathcal{A}_n$ such that $F_\gamma \cap F_\alpha$ is uncountable. Moreover, there is $\beta \in \mathcal{A}_\alpha$ such that $F_\gamma \cap F_\beta$ is uncountable, which implies $\gamma = \beta \in \mathcal{A}_\alpha$ by (FR6).

(3) It is easy to see $\mathcal{A}_\alpha \subset \mathcal{A}_{\alpha,1}$, and then the other side follows from

$$\mathcal{A}_{\alpha,1} \subset \mathcal{A}_{n+1}\setminus \bigcup_{\alpha'\in\mathcal{A}_n\setminus\{\alpha\}} \mathcal{A}_{\alpha',1} \subset \mathcal{A}_{n+1}\setminus \bigcup_{\alpha'\in\mathcal{A}_n\setminus\{\alpha\}} \mathcal{A}_{\alpha'} \subset \mathcal{A}_\alpha,$$

where the first containing relation follows from (1) and the third containing relation follows from (2).

(4) It is enough to prove the left hand side is contained in the right hand side. The case $k = 1$ follows by combining (2) and (3). Assume the claim in (4) holds for the case k and we consider the case $k + 1$. For $\gamma \in \mathcal{A}_{n+k+1}$, by the assumption, there is $\beta \in \mathcal{A}_{n+1}$ such that $\gamma \in \mathcal{A}_{\beta,k}$, that is $F_\gamma \subset F_\beta$. Then by (3), there is $\alpha \in \mathcal{A}_n$ such that $\beta \in \mathcal{A}_{\alpha,1}$, that is $F_\beta \subset F_\alpha$. Then we have $F_\gamma \subset F_\alpha$, that is $\gamma \in \mathcal{A}_{\alpha,k+1}$. Thus the desired result follows by induction.

(5) It is enough to prove the left hand side is contained in the right hand side. Take $\gamma \in \mathcal{A}_{\alpha,k+1}$ and then by (2) and (3), there is $\beta \in \mathcal{A}_{n+k}$ such that $\gamma \in \mathcal{A}_\beta$. Then by (1) and (4), $\beta \in \mathcal{A}_{\alpha,k}$. Thus the desired result follows.

(6) This follows by the finiteness of $\mathcal{A}_\alpha$ and induction by (5). □

Proposition 3.10. *For each $x \in X$ and $n \in \mathbb{Z}$, define the n-neighborhood index of x by*

$$\mathcal{N}_{n,x} := \{\alpha \in \mathcal{A}_n : x \in F_\alpha\}. \tag{3.4}$$

Then $\mathcal{N}_{n,x}$ is finite for all $x \in X$ and $n \in \mathbb{Z}$.

Proof. It is enough to consider the unbounded case, since in the bounded case, each $\mathcal{A}_n$ is finite. Fix $x \in X$ and write $N_n := \#\mathcal{N}_{n,x}$. Then N_n is increasing by (FR4.c) and Lemma 3.9 (1). Note that by (FR4.a), there are $n_0 \in \mathbb{Z}$ and $\alpha_0 \in \mathcal{A}_{n_0}$ such that α_0 is the unique index in $\mathcal{A}_{n_0}$ for which $x \in F_{\alpha_0}$, which gives $N_{n_0} = 1$. For $n > n_0$, and $\beta \in \mathcal{N}_{n,x}$, by Lemma 3.9 (4), there is $\alpha \in \mathcal{A}_{n_0}$ such that $\beta \in \mathcal{A}_{\alpha,n-n_0}$. Then we have $x \in F_\beta \subset F_\alpha$, which gives $\alpha = \alpha_0$ by the uniqueness of α_0. Thus $\mathcal{N}_{n,x} \subset \mathcal{A}_{\alpha_0,n-n_0}$, which is finite by Lemma 3.9 (6). □

Remark 3.11. We provide an example to show that the uniqueness condition in (RF4.a) is critical for the locally finite property in Proposition 3.10.

Take a countable infinite subset $E \subset \mathbb{S}^1$ and let

$$X := \{\alpha x \in \mathbb{R}^2 : x \in E, \alpha \geqslant 0\},$$

which is equipped with the intrinsic distance

$$d(\alpha x, \beta y) := \begin{cases} \alpha + \beta & \text{if } x \neq y, \\ |\alpha - \beta| & \text{if } x = y. \end{cases}$$

For each $n \in \mathbb{Z}$, $k \geqslant 0$ and $x \in E$, we define subsets of X by

$$F_{n,k,x} := \{\alpha x : \alpha \in [k2^{-n}, (k+1)2^{-n}]\} \quad \text{and} \quad V_{n,k,x} := \{k2^{-n}x, (k+1)2^{-n}x\}.$$

Then for each $n \in \mathbb{Z}$, we define $\mathcal{A}_n := \{(n,k,x) : k \geqslant 0, x \in E\}$, $\mathcal{A} := \bigcup_{n\in\mathbb{Z}} \mathcal{A}_n$. It is not hard to check that $\{F_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ satisfy all conditions for a finitely ramified structure except (FR4.a), since for all $n \in \mathbb{Z}$, the n-cell containing $x = (0,0)$ is not unique in $\mathcal{A}_n$. Moreover, the neighborhood $\mathcal{N}_{x,n}$ for $x = (0,0)$ is always infinite for all $n \in \mathbb{Z}$.

For each $x \in X$ and $n \in \mathbb{Z}$, we define the n-level closed neighborhood of x by

$$F_n(x) := \bigcup_{\alpha\in\mathcal{N}_{n,x}} F_\alpha$$

and let $U_n(x)$ be the interior of $F_n(x)$. Let $V_* := \bigcup_{\alpha\in\mathcal{A}} V_\alpha$ and

$$\mathcal{C}_X := \{U_n(x)\}_{n\in\mathbb{Z},x\in V_*}.$$

Then by [Typ, Proposition 2.10], $\mathcal{C}_X$ is a fundamental neighborhood system of X.

Corollary 3.12. *$\mathcal{C}_X$ is a measure determining class of X.*

Proof. Take two Radon measures μ, ν on X such that $\mu(U_n(x)) = \nu(U_n(x))$ for all $x \in V_*$ and $n \in \mathbb{Z}$. Since $\mathcal{C}_X$ is a fundamental neighborhood system on X, for each $x \in V_*$, we have

$$\mu(\{x\}) = \lim_{n\to\infty} \mu(U_n(x)) = \lim_{n\to\infty} \nu(U_n(x)) = \nu(\{x\}).$$

Note that for each $\alpha \in \mathcal{A}_n, n \in \mathbb{Z}$, there is $x \in V_* \cap (F_\alpha \backslash V_\alpha)$, which gives $F_\alpha = F_n(x)$, thus $\mathrm{int}F_\alpha \in \mathcal{C}_X$ for all $\alpha \in \mathcal{A}$. Define $U_\alpha := F_\alpha \backslash V_\alpha$ for all $\alpha \in \mathcal{A}$ and then we have

$$\mu(U_\alpha) = \mu(\mathrm{int}F_\alpha) - \sum_{x \in V_\alpha \backslash \partial F_\alpha} \mu(\{x\}) = \nu(\mathrm{int}F_\alpha) - \sum_{x \in V_\alpha \backslash \partial F_\alpha} \nu(\{x\}) = \nu(U_\alpha).$$

Fix $\alpha_0 \in \mathcal{A}$ and define

$$\mathcal{G}_{\alpha_0} := \{B \in \mathcal{B}(F_{\alpha_0}) : \mu(B) = \nu(B)\},$$

which is clearly a λ-system. Define $\mathcal{C}^{(1)} := \{U_\alpha : \alpha \in \mathcal{A}, F_\alpha \subset F_{\alpha_0}\}$ and $\mathcal{C}^{(0)} := \{\{x\} : x \in V_* \cap F_{\alpha_0}\}$. As proved above, we have $\mathcal{C}^{(1)} \cup \mathcal{C}^{(0)} \subset \mathcal{G}_{\alpha_0}$. Moreover, $\mathcal{C}^{(1)} \cup \mathcal{C}^{(0)} \cup \{\varnothing\}$ is a π-system generating $\mathcal{B}(F_{\alpha_0})$, thus by monotone class theorem, we have $\mathcal{G}_{\alpha_0} = \mathcal{B}(F_{\alpha_0})$. In particular, we have $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}(U_{\alpha_0})$.

Then for general $B \in \mathcal{B}(X)$, we have

$$\mu(B) = \sum_{\alpha \in \mathcal{A}_0} \mu(B \cap U_\alpha) + \sum_{x \in \bigcup_{\alpha \in \mathcal{A}_0} V_\alpha} \mu(B \cap \{x\}) = \sum_{\alpha \in \mathcal{A}_0} \nu(B \cap U_\alpha) + \sum_{x \in \bigcup_{\alpha \in \mathcal{A}_0} V_\alpha} \nu(B \cap \{x\}) = \nu(B).$$

Thus $\mathcal{C}_X$ is a measure determining class. □

3.2.3 Inverse regularity theorem for finitely ramified fractals

In order to apply Theorem 1.2 to finitely ramified fractals with the measure determining class $\mathcal{C}_X$, we need to prove that each $U_n(x) \in \mathcal{C}_X$ is trace admissible and admits an interior cutoff sequence. We will mainly follow the arguments in [Typ, Section 3].

In this part, we assume $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ is a local regular resistance form on X, where the corresponding resistance distance space (X, R) is a proper finitely ramified fractal with cells $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ and boundaries $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ with index set $\mathcal{A} = \bigcup_{n \in \mathbb{Z}} \mathcal{A}_n$. Let m be a Radon measure on (X, R) such that $0 < m(B(x, r)) < \infty$ for any $x \in X$ and $r > 0$. Then write $(\mathcal{E}, \mathcal{F})$ for the Dirichlet form on (X, R, m) induced by $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$, so that $(X, d, m, \mathcal{E}, \mathcal{F})$ is a MMD space.

For each $\alpha \in \mathcal{A}$, let $(\mathcal{E}_{\mathrm{res},\alpha}, \mathcal{F}_{\mathrm{res},\alpha})$ be the restriction of $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ on F_α. Then $(\mathcal{E}_{\mathrm{res},\alpha}, \mathcal{F}_{\mathrm{res},\alpha})$ coincides with the local Dirichlet space $(\mathcal{E}_{\mathrm{int}F_\alpha}, \mathcal{F}(\mathrm{int}F_\alpha))$. For each $U_n(x) \in \mathcal{C}_X$, we define

$$\mathcal{F}_{\mathrm{res},n,x} := \{u \in C(F_n(x)) : u|_{F_\beta} \in \mathcal{F}_{\mathrm{res},\beta} \text{ for all } \beta \in \mathcal{N}_{n,x}\}$$

and

$$\mathcal{E}_{\mathrm{res},n,x}(u, v) := \sum_{\beta \in \mathcal{N}_{n,x}} \mathcal{E}_{\mathrm{res},\beta}(u, v) \quad \text{for all } u, v \in \mathcal{F}_{\mathrm{res},n,x}.$$

Then $(\mathcal{E}_{\mathrm{res},n,x}, \mathcal{F}_{\mathrm{res},n,x})$ is a resistance form on $F_n(x)$, which coincides with the local Dirichlet space $(\mathcal{E}_{U_n(x)}, \mathcal{F}(U_n(x)))$.

Lemma 3.13. *Let $U_n(x) \in \mathcal{C}_X$ with $n \in \mathbb{Z}, x \in V_*$.*

(1) $\overline{\dim}_B(\partial U_n(x)) = 0$.

(2) $U_n(x)$ *is an extension domain. In particular,* $U_n(x)$ *is trace admissible.*

(3) $U_n(x)$ *admits an interior cutoff sequence.*

Proof. (1) This simply follows from that $\partial U_n(x)$ is a finite set.

(2) Since $\partial U_n(x)$ is a finite set, we can fix a linear operator $T : \mathcal{F}_{\mathrm{res},n,x} \to \mathcal{F}$ such that $Tu|_{F_n(x)} = u$ and $\mathrm{im}T$ is a finite dimensional space. In particular, we have $Tu = 0$ if and only if $u = 0$, thus T is a bounded operator. Thus $U_n(x)$ is an extension domain. Combining Proposition 2.8 and Corollary 3.7, $U_n(x)$ is trace admissible.

(3) For each $y \in \partial U_n(x)$ and $j \geqslant 1$, we define

$$V_{y,j} := U_{n+j}(y) \cap U_n(x)$$

and let $V_j := \bigcup_{y\in\partial U_n(x)} V_{y,j}$. When j is large enough, V_j is a disjoint union of $V_{y,j}$ such that each distinct $V_{y,j}, V_{y',j}$ have positive distance. A similar argument as before shows that each $V_{y,j}$ is a trace admissible open set; thus, V_j is a trace admissible domain when $j \geqslant 1$. In particular, $\{V_j\}_{j\geqslant 0}$ gives an interior cutoff sequence of U with $V_0 := U_n(x)$. □

Combining all the discussions above, we reach the following inverse regularity theorem for resistance forms on finitely ramified fractals.

Theorem 3.14. *Let* $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$ *be a local regular resistance form on* X *and* R *be the resistance distance. Assume* (X, R) *is a metric doubling finitely ramified fractal. Let* m *be a Radon measure on* X *such that* $0 < m(B(x, r)) < \infty$ *for all* $x \in X, r > 0$ *and write* $(\mathcal{E}, \mathcal{F})$ *for the Dirichlet form on* (X, R, m) *induced by* $(\mathcal{E}_{\mathrm{res}}, \mathcal{F}_{\mathrm{res}})$*. Then for any* $\epsilon > 0$*, we have*

$$\mathrm{Dom}\Delta \cap C^{1+\epsilon}(X) \subset \mathbb{R}\mathbf{1}_X.$$

Proof. Note that a finitely ramified fractal is connected and thus uniformly perfect. Then Proposition 3.5 implies that $(X, R, m, \mathcal{E}, \mathcal{F})$ has the capacity upper bound with index $\sigma = 1$. Then, by combining Corollary 3.12 and Lemma 3.13, we apply Theorem 1.2 with $\gamma = 0$ to derive the desired result. □

Proof of Theorem 1.1. This follows by Theorem 3.14 and that the Euclidean distance d and resistance distance R satisfy $d^{\log(5/3)/\log 2} \asymp R$. □

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