Nonlinear Renormalization and p-Resistance Forms on the Golden Ratio Sierpiński Gasket

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Abstract

Let G be the golden ratio Sierpiński gasket, with basic contraction ratio $\rho = (\sqrt{5}-1)/2$ and three-point boundary V_0. A countable cell decomposition adapted to its exact overlaps defines a renormalization operator $\mathcal{R}_{p,r}$ on p-resistance forms on V_0. For $p \in (1,\infty)$ and $r \in (\rho^{p-1}, 1)$, we prove the joint continuity of $\mathcal{R}_{p,r}$. It has a unique eigenvalue $\lambda_p(r)$ and, up to a positive scalar, a unique eigenform $\mathcal{D}_{p,r}$, which is necessarily symmetric. This eigenform yields a regular strongly local self-similar p-resistance form $(\mathcal{E}_{p,r}, \mathcal{F}_{p,r})$ on G, unique up to a positive scalar. There is a unique $r_p \in (\rho^{p-1}, 1)$ with $\lambda_p(r_p) = r_p^2$; for the corresponding standard form $(\mathcal{E}_p, \mathcal{F}_p)$, its effective p-resistance $R_{\mathcal{E}_p}$ satisfies $R_{\mathcal{E}_p}(x,y) \asymp |x-y|^{\log r_p / \log \rho}$. Finally, $p \mapsto r_p^{1/(p-1)}$ is non-increasing and continuous; consequently, $p \mapsto r_p$ is strictly decreasing and continuous.

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1 Introduction

The object of this paper is the nonlinear energy theory of the *golden ratio Sierpiński gasket* shown in Figure 1. Its defining maps are simple, but its *exact overlaps* rule out the usual finite-graph renormalization scheme.

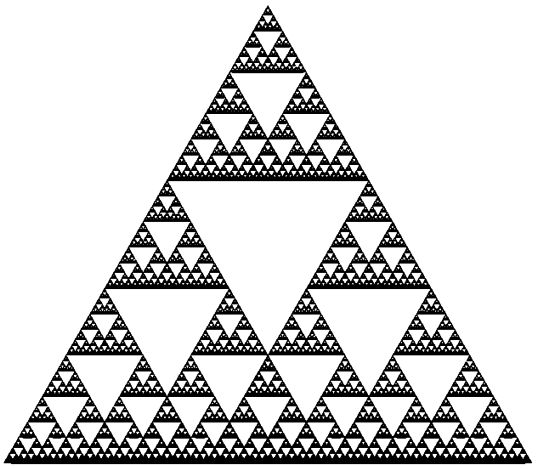


Figure 1: The golden ratio Sierpiński gasket G.

Analysis on fractals has two classical starting points. In the late 1980s and early 1990s, the constructions of Brownian motion on the Sierpiński gasket by Goldstein, Kusuoka, and Barlow–Perkins opened the subject [13, 24, 5]; Lindstrøm then extended this picture to nested fractals [27], while Kigami's compatible graph energies gave a direct analytic construction on *post-critically finite* (p.c.f.) self-similar sets [19, 20]. A second basic model is the Sierpiński carpet. Beginning in the same period, Barlow and Bass constructed Brownian motion on infinitely ramified carpets [2, 3], whereas Kusuoka and Zhou constructed the corresponding Dirichlet form [25]. These are two equivalent descriptions of the same symmetric Markov process: a regular symmetric Dirichlet form determines its associated Hunt process, and conversely [12]. In 2010, Barlow, Bass, Kumagai, and Teplyaev proved the corresponding uniqueness theorem [4]. More recently, for unconstrained Sierpiński carpets, which generalize the classical carpets by allowing small cells to lie off a fixed grid, Cao and the second author developed a direct analytic method establishing the existence of Dirichlet forms, and subsequently proved uniqueness and convergence of the associated resistance forms [10, 11]. Taken together, the gasket and the carpet illustrate the two basic geometric regimes in fractal analysis—finite and infinite ramification—and have long served as its reference examples.

Resistance forms, introduced and developed by Kigami, give a pointwise language for energy on these spaces [20, 21]. In the quadratic case, a regular resistance form yields a regular Dirichlet form and the corresponding diffusion. Recently, Kajino and Shimizu introduced *p-resistance forms* as nonlinear L^p-counterparts [16, 18]. When $p = 2$, the two notions agree. For $p \in (1, \infty)$, a p-resistance form is a p-energy form with the additional properties needed for pointwise analysis: a complete energy space, point separation, finite effective p-resistance, and a contraction principle. These properties give traces and harmonic extensions, as well as pointwise resistance estimates. In the fractal setting considered here, we further require the p-resistance forms to be self-similar.

From this viewpoint, a p-energy on a fractal plays the role of the classical Sobolev energy $\int |\nabla u|^p$. It is the basic tool for p-harmonic functions and nonlinear potential theory [14, 6]. On general metric spaces, related first-order theories include Korevaar–Schoen energies and Newtonian Sobolev spaces [23, 32]. On self-similar sets, early work on p.c.f. sets appears in [15, 34], followed by a general construction in [7]. Recent work reaches more general infinitely

ramified fractals and compact metric spaces [22, 33, 29, 17]. The golden ratio Sierpiński gasket is a nonuniform variant of the classical gasket and lies just outside the familiar finite-graph setting. Let q_0, q_1, q_2 be the vertices of an equilateral triangle, with q_0 at the top, and put $\rho = (\sqrt{5}-1)/2$. The set G is generated by one similitude of ratio ρ^2 fixing q_0 and two similitudes of ratio ρ fixing q_1 and q_2; denote these maps by F_0, F_1, F_2, respectively. Their coordinates are given in Section 3. For a finite word $w = w_1 \cdots w_m$, write $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$. The identity $F_{122}G = F_{211}G$ is an *exact overlap*: two different words describe the same cell. It prevents the usual reduction to a finite boundary graph. At the same time, the overlap has *finite type*, so one can organize the geometry by a countable family of cells with controlled boundary intersections [1, 31, 30, 26, 28, 8]. In the quadratic case, this observation was used in [9] to construct a local resistance form with dense continuous domain, characterize it by a countable self-similarity relation, and derive resistance and heat-kernel estimates.

In this paper, for the golden ratio Sierpiński gasket, we develop the nonlinear counterpart by combining the countable-cell geometry of [9] with the p-resistance framework in [16, 18]. The passage from 2 to p is not formal. On the three-point boundary V_0, a p-resistance form need not be representable by a finite weighted graph, and harmonic extension is nonlinear. Moreover, the renormalized trace is obtained by minimizing a countable cell energy. We therefore work directly with the full cone of p-resistance forms on V_0. Compactness of the minimizing extensions, together with uniform tail bounds, yields joint continuity of the renormalization operator. The normalized operator preserves a compact convex subset of this cone, so Schauder's theorem yields a boundary eigenform. For uniqueness, we compare the extremal ratios of two boundary eigenforms and use a one-vertex Perron–Frobenius asymptotic along the nested cells $F_0^n G$. Finally, the boundary eigenpair defines compatible finite-level forms whose monotone limit is a global self-similar p-resistance form on G.

Here is the basic construction. Let $V_0 = \{q_0, q_1, q_2\}$, and let W_1 be the representative word set from [9] (see also Definition 3.2): it contains one word for each cell in the countable decomposition. For a word w, $|w|$ denotes its length. For a p-resistance form $\mathcal{D}$ on V_0 and $r > 0$, define

$$\Psi_{p,r}\mathcal{D}(f) := \sum_{w \in W_1} r^{1-|w|}\mathcal{D}(f \circ F_w|_{V_0}),$$

and let $\mathcal{R}_{p,r}\mathcal{D}$ be its trace to V_0. We call $\mathcal{R}_{p,r}$ the *renormalization operator*. We first establish its continuity under simultaneous variation of p, r, and the p-resistance form $\mathcal{D}$ on V_0. We then prove that, for every $p \in (1, \infty)$ and $r \in (\rho^{p-1}, 1)$, the operator $\mathcal{R}_{p,r}$ has an eigenform $\mathcal{D}_{p,r}$, unique up to a positive scalar, satisfying $\mathcal{R}_{p,r}\mathcal{D}_{p,r} = \lambda_p(r)\mathcal{D}_{p,r}$ for some $\lambda_p(r) > 0$. Every eigenform is symmetric under the reflection that exchanges q_1 and q_2.

The eigenform on V_0 determines a compatible sequence of finite-level energies and hence a p-resistance form $(\mathcal{E}_{p,r}, \mathcal{F}_{p,r})$ on G. Up to a positive scalar, this global form is characterized by the cellwise domain relation and the weighted energy identity associated with the countable iterated function system $\{F_w\}_{w \in W_1}$.

The renormalization factor r_p, characterized by $\lambda_p(r_p) = r_p^2$, is called *standard* because, for this choice, the energy weight of every cell is the same fixed power of its geometric contraction ratio. For the associated standard form, we prove a two-sided power-law estimate for the effective resistance and establish the monotonicity and continuity of the standard renormalization factor with respect to p.

All functions in this paper are real-valued. For a p-resistance form $(\mathcal{E}, \mathcal{F})$, the symbol $R_{\mathcal{E}}$ denotes its effective p-resistance, defined precisely in Definition 2.1.

The main results are stated below. Let $\mathcal{M}_p(V_0)$ denote the set of p-resistance forms on V_0, and let ρ_w denote the contraction ratio of F_w. A form on the compact set G is *regular* when its domain is uniformly dense in $C(G)$. It is *strongly local* when energy contributions supported in separate regions do not interact; the exact identity used here is (6.27).

Theorem 1. *Fix $p \in (1,\infty)$ and $r \in (\rho^{p-1},1)$.*

(a) *The operator $\mathcal{R}_{p,r}$ has a unique eigenvalue $\lambda_p(r) > 0$. There exists a p-resistance form $\mathcal{D}_{p,r} \in \mathcal{M}_p(V_0)$, unique up to a positive scalar, such that $\mathcal{R}_{p,r}\mathcal{D}_{p,r} = \lambda_p(r)\mathcal{D}_{p,r}$. Moreover, $\mathcal{D}_{p,r}$ is symmetric.*

(b) *Set $\gamma_0 = \lambda_p(r)$, $\gamma_1 = \gamma_2 = r$, and $\gamma_w = \prod_{j=1}^{|w|}\gamma_{w_j}$. Up to a positive scalar, there is a unique p-resistance form $(\mathcal{E}_{p,r},\mathcal{F}_{p,r})$ on G such that*

$$\mathcal{F}_{p,r} = \Big\{ f \in C(G) : f\circ F_w \in \mathcal{F}_{p,r} \quad \forall\, w \in W_1,\ \sum_{w\in W_1} \gamma_w^{-1}\mathcal{E}_{p,r}(f\circ F_w) < \infty \Big\},$$

and $\mathcal{E}_{p,r}(f) = \sum_{w\in W_1}\gamma_w^{-1}\mathcal{E}_{p,r}(f\circ F_w)$ for every $f \in \mathcal{F}_{p,r}$. This form is regular and strongly local.

Theorem 2. *For every $p \in (1,\infty)$, there exists a unique $r_p \in (\rho^{p-1},1)$ such that $\lambda_p(r_p) = r_p^2$. Let $(\mathcal{E}_p,\mathcal{F}_p)$ be the form in Theorem 1 corresponding to $r = r_p$, and set $\theta_p = \log r_p/\log\rho$. Then $\gamma_w = \rho_w^{\theta_p}$ for every $w \in W_1$, and there exist constants $c_1, c_2 > 0$ such that*

$$c_1|x-y|^{\theta_p} \le R_{\mathcal{E}_p}(x,y) \le c_2|x-y|^{\theta_p}, \qquad x,y \in G.$$

For $p = 2$, the eigenform and global form constructed in Theorems 1 and 2 coincide with those constructed in [9].

The dependence of the standard renormalization factor on the exponent is the content of the following theorem.

Theorem 3. *The map $p \mapsto r_p^{1/(p-1)}$ is non-increasing and continuous on $(1,\infty)$. Consequently, $p \mapsto r_p$ is continuous and strictly decreasing, and hence so is $p \mapsto \lambda_p = r_p^2$.*

The monotonicity argument compares the standard cell energies at two different exponents by means of harmonic oscillation estimates. Continuity is proved sequentially: as the exponent varies, the corresponding normalized boundary eigenforms are precompact in the topology of local uniform convergence on $\mathcal{L}(V_0)$, and the varying-exponent continuity of the renormalization operator allows us to pass to the limit in the standard eigenvalue equation.

We organize the paper as follows. Section 2 reviews the p-resistance facts used later. Section 3 describes the geometry of the golden ratio Sierpiński gasket and constructs the countable level-one form. Section 4 defines the renormalization operator and proves its joint continuity. Section 5 constructs an eigenform, estimates its eigenvalue, and determines r_p. Section 6 extends the boundary eigenpair to a global form and proves the resistance estimate. Section 7 proves uniqueness of the boundary eigenform and of the global form and completes the proofs of Theorems 1 and 2, while Section 8 studies the dependence of the standard renormalization factor on p and proves Theorem 3. Appendix A establishes the one-vertex asymptotic needed for uniqueness. We write $A \asymp B$ if $cB \le A \le CB$ for some constants $c, C > 0$ independent of the variables under consideration. Unless stated otherwise, constants denoted by c, C, c_i, or C_i are positive, depend only on the fixed parameters; c and C may change from line to line.

Before closing this section, we point out that the golden ratio Sierpiński gasket is the $n = 3$ member of a natural family. If $n \ge 4$, $\rho \in (0,1)$ is the root of $\rho^n - 2\rho + 1 = 0$, and the contraction ratios of F_0, F_1, F_2 are $1-\rho, \rho, \rho$, respectively, then the resulting gasket has the exact overlap $F_1F_2^{n-1}G = F_2F_1^{n-1}G$, which generalizes the exact overlap above; see [9]. The nonlinear construction for the $n = 3$ case can be naturally extends to this family.

2 p-Resistance forms

We recall, without proof, only the facts on p-resistance forms used below. The statements have been shortened or combined when convenient; they are contained in [18, Section 2], with complete

proofs in [16]. Throughout this section, $p \in (1,\infty)$ is fixed. For real-valued functions, $u \wedge v$ and $u \vee v$ denote the pointwise minimum and maximum. We write $u^+ := u \vee 0$, $u^- := -(u \wedge 0)$, and $\mathrm{sgn}(a) := |a|^{-1}a$, with $\mathrm{sgn}(0) = 0$. For a function u on a nonempty set A, put

$$\|u\|_{\infty,A} := \sup_{x\in A} |u(x)|, \qquad \mathrm{osc}_A(u) := \sup_{x,y\in A} |u(x) - u(y)|.$$

We omit the set A from the norm when the domain is clear. We write $\mathbf{1}_A$ for the constant-one function on A and, for $x \in A$, $\mathbf{1}_x^A$ for the indicator of $\{x\}$ in A.

Definition 2.1 (p-resistance form)**.** Let K be a nonempty set, let $\mathcal{F}_p \subset \mathbb{R}^K$ be a linear subspace, and let $\mathcal{E}_p : \mathcal{F}_p \to [0,\infty)$. The pair $(\mathcal{E}_p, \mathcal{F}_p)$ is called a *p-resistance form* on K if the following conditions hold.

(RF1)$_p$ $\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$ and $\{u \in \mathcal{F}_p : \mathcal{E}_p(u) = 0\} = \mathbb{R}\mathbf{1}_K$.

(RF2)$_p$ The quotient normed space $(\mathcal{F}_p/\mathbb{R}\mathbf{1}_K, \mathcal{E}_p^{1/p})$ is complete.

(RF3)$_p$ For every pair of distinct points $x, y \in K$, there exists $u \in \mathcal{F}_p$ such that $u(x) \neq u(y)$.

(RF4)$_p$ The *effective p-resistance* between $x, y \in K$ is finite:

$$R_{\mathcal{E}_p}(x,y) := \sup_{u\in\mathcal{F}_p\setminus\mathbb{R}\mathbf{1}_K} \frac{|u(x)-u(y)|^p}{\mathcal{E}_p(u)} < \infty. \tag{2.1}$$

(RF5)$_p$ The *generalized contraction property* holds: if $q \in (0,p]$, $r \in [p,\infty]$, $m,n \in \mathbb{N}$, and $T = (T_1,\dots,T_n) : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $T(0) = 0$ and

$$\|T(x) - T(y)\|_{\ell^r} \le \|x - y\|_{\ell^q}, \qquad x, y \in \mathbb{R}^m,$$

then, for every $u = (u_1,\dots,u_m) \in \mathcal{F}_p^m$, one has $T(u) := (T_1(u),\dots,T_n(u)) \in \mathcal{F}_p^n$ and

$$\left\| \left(\mathcal{E}_p(T_k(u))^{1/p}\right)_{k=1}^n \right\|_{\ell^r} \le \left\| \left(\mathcal{E}_p(u_j)^{1/p}\right)_{j=1}^m \right\|_{\ell^q}.$$

Remark 2.2. If V is a nonempty finite set, then $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on V if and only if $\mathcal{F}_p = \mathbb{R}^V$ and $\mathcal{E}_p$ satisfies (RF1)$_p$ and (RF5)$_p$; see [18, Example 2.2(1)].

By the definition in (2.1),

$$|u(x) - u(y)|^p \le R_{\mathcal{E}_p}(x,y)\mathcal{E}_p(u), \qquad u \in \mathcal{F}_p, \quad x, y \in K. \tag{2.2}$$

Lemma 2.3. *Let $\mathcal{E}_{p,1}, \mathcal{E}_{p,2} : \mathcal{F}_p \to [0,\infty)$ and $a_1, a_2 \in [0,\infty)$. If $\mathcal{E}_{p,1}^{1/p}$ and $\mathcal{E}_{p,2}^{1/p}$ are seminorms, then so is $(a_1\mathcal{E}_{p,1} + a_2\mathcal{E}_{p,2})^{1/p}$. If both forms satisfy* (RF5)$_p$*, then so does their weighted sum.*

Henceforth in this section, $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on a nonempty set K.

Proposition 2.4. *Let $u, v \in \mathcal{F}_p$.*

(1) *If $\varphi \in C(\mathbb{R})$ is 1-Lipschitz, then $\varphi(u) \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi(u)) \le \mathcal{E}_p(u)$. In particular, $u^+ \wedge 1$, $|u|$, u^+, and u^- belong to $\mathcal{F}_p$ and have energy at most $\mathcal{E}_p(u)$.*

(2) *If u and v are bounded, then $uv \in \mathcal{F}_p$ and $\mathcal{E}_p(uv)^{1/p} \le \|v\|_{\infty,K}\mathcal{E}_p(u)^{1/p} + \|u\|_{\infty,K}\mathcal{E}_p(v)^{1/p}$.*

(3) *The following p-Clarkson inequalities hold:*

$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \ge 2\left(\mathcal{E}_p(u)^{1/(p-1)} + \mathcal{E}_p(v)^{1/(p-1)}\right)^{p-1}, \qquad 1 < p \le 2, \tag{2.3}$$

$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \le 2\left(\mathcal{E}_p(u)^{1/(p-1)} + \mathcal{E}_p(v)^{1/(p-1)}\right)^{p-1}, \qquad 2 \le p < \infty. \tag{2.4}$$

Theorem 2.5. *For every $u, v \in \mathcal{F}_p$, the limit in (2.5)*

$$\mathcal{E}_p(u;v) := \frac{1}{p}\,\frac{d}{dt}\mathcal{E}_p(u+tv)\Big|_{t=0} \tag{2.5}$$

exists, $\mathcal{E}_p(u;\cdot)$ is linear, $\mathcal{E}_p(u;u) = \mathcal{E}_p(u)$, and $\mathcal{E}_p(u;\mathbf{1}_K) = 0$. Moreover, for $u, u_1, u_2, v \in \mathcal{F}_p$ and $a \in \mathbb{R}$,

$$t \longmapsto \mathcal{E}_p(u+tv;v) \text{ is strictly increasing} \iff v \notin \mathbb{R}\mathbf{1}_K, \tag{2.6}$$

$$\mathcal{E}_p(au;v) = \operatorname{sgn}(a)|a|^{p-1}\mathcal{E}_p(u;v), \qquad \mathcal{E}_p(u+a\mathbf{1}_K;v) = \mathcal{E}_p(u;v), \tag{2.7}$$

$$|\mathcal{E}_p(u;v)| \le \mathcal{E}_p(u)^{(p-1)/p}\mathcal{E}_p(v)^{1/p}. \tag{2.8}$$

Let $\alpha_p := \frac{1}{p} \wedge \frac{p-1}{p}$. There exists $c_p > 0$ such that

$$|\mathcal{E}_p(u_1;v) - \mathcal{E}_p(u_2;v)| \le c_p\big(\mathcal{E}_p(u_1) \vee \mathcal{E}_p(u_2)\big)^{(p-1-\alpha_p)/p}\mathcal{E}_p(u_1-u_2)^{\alpha_p/p}\mathcal{E}_p(v)^{1/p}. \tag{2.9}$$

Proposition 2.6. *Let $u_1, u_2, v \in \mathcal{F}_p$. If $((u_2-u_1)\wedge v)(x) = 0$ for every $x \in K$, then $\mathcal{E}_p(u_1;v) \ge \mathcal{E}_p(u_2;v)$.*

Definition 2.7 ($\mathcal{E}_p$-harmonic functions)**.** Let $B \subset K$. A function $h \in \mathcal{F}_p$ is called *$\mathcal{E}_p$-harmonic on $K \setminus B$* if it minimizes $\mathcal{E}_p$ among functions with the same restriction to B, or equivalently, if $\mathcal{E}_p(h;v) = 0$ for every $v \in \mathcal{F}_p$ with $v|_B = 0$. We write $\mathcal{H}_{\mathcal{E}_p,B} := \{h \in \mathcal{F}_p : h \text{ is } \mathcal{E}_p\text{-harmonic on } K\setminus B\}$.

Theorem 2.8. *Let $\varnothing \ne B \subset K$, set $\mathcal{F}_p|_B := \{u|_B : u \in \mathcal{F}_p\}$, and define*

$$\mathcal{E}_p|_B(u) := \inf\{\mathcal{E}_p(v) : v \in \mathcal{F}_p,\ v|_B = u\}, \qquad u \in \mathcal{F}_p|_B. \tag{2.10}$$

Then $(\mathcal{E}_p|_B, \mathcal{F}_p|_B)$ is a p-resistance form on B, called the trace *of $(\mathcal{E}_p, \mathcal{F}_p)$ on B, and $R_{\mathcal{E}_p|_B} = R_{\mathcal{E}_p}|_{B\times B}$. For every $u \in \mathcal{F}_p|_B$, there is a unique $h_B^{\mathcal{E}_p}[u] \in \mathcal{F}_p$ such that $h_B^{\mathcal{E}_p}[u]|_B = u$ and $\mathcal{E}_p\big(h_B^{\mathcal{E}_p}[u]\big) = \mathcal{E}_p|_B(u)$. Consequently, $\mathcal{H}_{\mathcal{E}_p,B} = h_B^{\mathcal{E}_p}(\mathcal{F}_p|_B)$ and*

$$h_B^{\mathcal{E}_p}[au+b\mathbf{1}_B] = ah_B^{\mathcal{E}_p}[u] + b\mathbf{1}_K, \qquad a,b \in \mathbb{R}, \tag{2.11}$$

$$\mathcal{E}_p|_B(u|_B;v|_B) = \mathcal{E}_p(u;v), \qquad u \in \mathcal{H}_{\mathcal{E}_p,B},\ v \in \mathcal{F}_p. \tag{2.12}$$

If $\varnothing \ne A \subset B$, then $(\mathcal{E}_p|_B)|_A = \mathcal{E}_p|_A$ and $h_B^{\mathcal{E}_p} \circ h_A^{\mathcal{E}_p|_B} = h_A^{\mathcal{E}_p}$.

Proposition 2.9. *Let $\varnothing \ne B \subset K$ and $u, v \in \mathcal{H}_{\mathcal{E}_p,B}$. If $u \le v$ on B, then $u \le v$ on K.*

For a nonempty set $B \subset K$ and $x \in K$, define

$$R_{\mathcal{E}_p}(x,B) := \sup\left\{\mathcal{E}_p(u)^{-1} : u \in \mathcal{F}_p,\ u(x) = 1,\ u|_B = 0\right\}, \tag{2.13}$$

where, as usual, the supremum of the empty set is understood to be 0.

Proposition 2.10. *The function $\widehat{R}_{\mathcal{E}_p}(x,y) := R_{\mathcal{E}_p}(x,y)^{1/(p-1)}$, $x, y \in K$, is a metric on K. Moreover, let $\varnothing \ne B \subset K$, let $h \in \mathcal{H}_{\mathcal{E}_p,B}$ with $\|h\|_{\infty,B} < \infty$, and assume $0 < R_{\mathcal{E}_p}(x,B) < \infty$. Then*

$$|h(x) - h(y)| \le \frac{R_{\mathcal{E}_p}(x,y)^{1/(p-1)}}{R_{\mathcal{E}_p}(x,B)^{1/(p-1)}}\operatorname{osc}_B(h), \qquad y \in K.$$

The following completion theorem is an equivalent formulation of [16, Theorem 6.22].

Theorem 2.11. *Let $(\widetilde{K}, \widetilde{d})$ be the completion of $(K, \widehat{R}_{\mathcal{E}_p})$, and set $\widetilde{\mathcal{F}}_p := \{u \in C(\widetilde{K}) : u|_K \in \mathcal{F}_p\}$ and $\widetilde{\mathcal{E}}_p(u) := \mathcal{E}_p(u|_K)$ for $u \in \widetilde{\mathcal{F}}_p$. Then $(\widetilde{\mathcal{E}}_p, \widetilde{\mathcal{F}}_p)$ is a p-resistance form on $\widetilde{K}$, $\widehat{R}_{\widetilde{\mathcal{E}}_p} = \widetilde{d}$, and the restriction map $\widetilde{\mathcal{F}}_p \to \mathcal{F}_p$ is a linear isomorphism.*

Proposition 2.12. *Let $u \in \mathcal{F}_p$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$.*

(1) *If $u_n(x) - u_n(y) \to u(x) - u(y)$ for every $x, y \in K$, then $\mathcal{E}_p(u) \le \liminf_{n\to\infty} \mathcal{E}_p(u_n)$.*

(2) *One has $\mathcal{E}_p(u_n - u) \to 0$ if and only if $\limsup_{n\to\infty} \mathcal{E}_p(u_n) \le \mathcal{E}_p(u)$ and $u_n(x) - u_n(y) \to u(x) - u(y)$ for every $x, y \in K$.*

Proposition 2.13. *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ be 1-Lipschitz and converge pointwise to the identity. Then, for every $u \in \mathcal{F}_p$, $\varphi_n(u) \in \mathcal{F}_p$ and $\mathcal{E}_p(u - \varphi_n(u)) \to 0$.*

3 The golden ratio Sierpiński gasket

The geometric framework used in this section, namely the finite-type structure and the representative countable cell decomposition, is taken from [9, Sections 2–3]. We first review the geometric properties needed below and then establish the associated level-one p-resistance results. The proofs of Propositions 3.5 and 3.6 adapt the corresponding arguments in the quadratic case to arbitrary $p \in (1, \infty)$.

3.1 Exact overlaps and finite type

Let $q_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $q_1 = (0, 0)$, $q_2 = (1, 0)$, and $\rho := \frac{\sqrt{5}-1}{2}$. Define the contracting similitudes

$$F_0(x) = \rho^2(x - q_0) + q_0, \qquad F_1(x) = \rho(x - q_1) + q_1, \qquad F_2(x) = \rho(x - q_2) + q_2. \tag{3.1}$$

By Hutchinson's theorem, there is a unique nonempty compact set $G \subset \mathbb{R}^2$ such that

$$G = F_0G \cup F_1G \cup F_2G. \tag{3.2}$$

We call G the *golden ratio Sierpiński gasket*. We equip G with the Euclidean metric $d(x, y) := |x - y|$. Thus (3.1) fixes the maps and (3.2) fixes the set throughout the paper.

We begin by fixing the word notation and recalling the finite-type property. Let K be a self-similar set generated by an iterated function system $\{F_i\}_{i=0}^{N-1}$, where F_i has contraction ratio ρ_i. For $m \ge 1$, a *word of length m* is a string $w = w_1w_2 \cdots w_m$ with $w_i \in \{0, 1, \ldots, N - 1\}$, and we write $|w| = m$. The empty word is denoted by $\varnothing$, and the set of all finite words by $\widetilde{W}_*$. Define $F_w := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_{|w|}}$ and $F_\varnothing := \mathrm{id}_K$. Also set $\rho_w := \prod_{j=1}^{|w|} \rho_{w_j}$, $\rho_\varnothing := 1$, and $\rho_* := \min_{0\le i<N} \rho_i$.

Definition 3.1 (Finite type)**.** The self-similar set K is said to be of *finite type* if only finitely many similitudes $h = F_w^{-1} \circ F_v$, with $w, v \in \widetilde{W}_*$, satisfy $F_wK \cap F_vK \neq \varnothing$ and $\rho_h \in (\rho_*, \rho_*^{-1})$, where ρ_h is the similarity ratio of h.

Since $1 - \rho = \rho^2$, a direct calculation gives $F_{122} = F_{211}$, and hence $F_{122}G = F_{211}G$. As verified in [9, Section 2], the resulting iterated function system is of finite type; see [9, Definition 2.1] and the subsequent discussion. The resulting essentially non-overlapping graph-directed realization satisfies the graph-directed open set condition. Consequently, the Hausdorff dimension $d_H := \dim_H G$ of G is $(\log \eta)/(-2 \log \rho) \approx 1.6824$, where η is the largest root of $x^3 - 6x^2 + 5x - 1 = 0$. Moreover, the d_H-dimensional Hausdorff measure of G is positive and finite; see [9, 30].

3.2 Countable cells and graph

For finite words $w, w' \in \widetilde{W}_*$, their concatenation is denoted by ww'. If $w = w_1 \cdots w_m$ and $0 \le \ell \le m$, set $[w]_\ell := w_1 \cdots w_\ell$ and $[w]_0 := \varnothing$.

Definition 3.2 (Representative words and cells)**.** Fix a *representative word set* $W_1 \subset \bigcup_{n=0}^{\infty} \{1, 2\}^n \times \{0\}$ such that

(1) for every $w \in \bigcup_{n=0}^{\infty}\{1,2\}^n \times \{0\}$, there exists $w' \in W_1$ with $F_w = F_{w'}$;

(2) if $w, w' \in W_1$ are distinct, then $F_w \neq F_{w'}$.

For $n \geq 1$, set $W_{1,n} := \{w \in W_1 : |w| = n\}$. For $m \geq 2$, define $W_m := W_1^m = \{w^{(1)} \cdots w^{(m)} : w^{(j)} \in W_1\}$. Finally, let $V_0 := \{q_0, q_1, q_2\}$ and $V_m := \bigcup_{w \in W_m} F_w V_0$ for $m \geq 1$, and denote by $\overline{V}_m$ the Euclidean closure of V_m.

We call $F_w G$, $w \in W_1$, a *level-one cell*. The *cell-incidence graph* has vertex set W_1 and joins two words when their cells intersect.

For distinct $x, y \in V_1$, declare $x \sim y$ if there exists $w \in W_1$ such that $x, y \in F_w V_0$. This defines an infinite graph $(V_1, \sim)$; we call it the *level-one graph*; see Figure 2. Its lower part contains a sequence of finite horizontal chains whose vertices lie in V_1 and accumulate on the bottom segment $\overline{V}_1 \setminus V_1$.

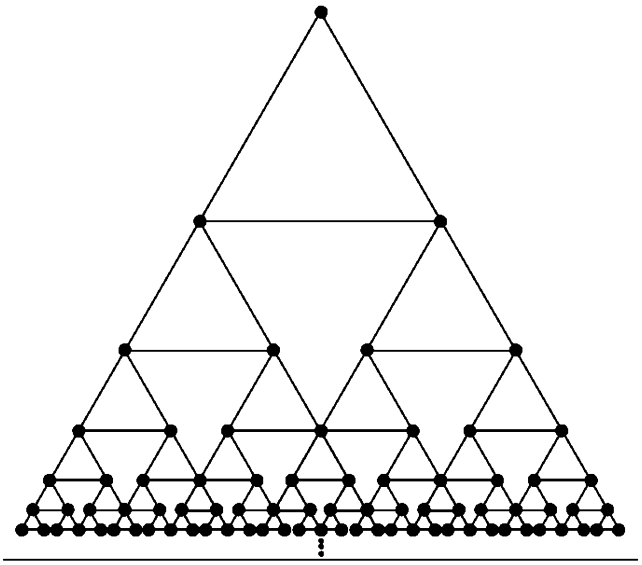


Figure 2: The infinite graph $(V_1, \sim)$. The bottom line, which is not part of V_1, represents $\overline{V}_1 \setminus V_1$.

The following geometric properties, proved in [9, Section 3], will be used repeatedly. We list them here without proof.

Proposition 3.3. *The following assertions hold.*

(1) *The family $\{F_w\}_{w \in W_1}$ forms a countable iterated function system for the golden ratio Sierpiński gasket G, with $G = \overline{\bigcup_{w \in W_1} F_w G}$, where the closure is taken in the Euclidean topology. Furthermore, distinct level-one cells intersect only at their boundary vertices: whenever $w \neq w'$ in W_1, $F_w G \cap F_{w'} G = F_w V_0 \cap F_{w'} V_0$. The cell-incidence graph is connected and locally finite.*

(2) *If $N_n := \#W_{1,n}$, then there exist constants $c, C > 0$ such that $c\rho^{-n} \leq N_n \leq C\rho^{-n}$ for all $n \geq 1$.*

(3) *For every $n \geq 1$, if $x, y \in V_1$ are distinct and $d(x,y) < \rho^{n+2}$, then there exist $w, w' \in \{1,2\}^n$ such that $x \in F_w V_1$, $y \in F_{w'} V_1$, and $F_w V_1 \cap F_{w'} V_1 \neq \varnothing$.*

3.3 Level-one form and resistance

Fix $p \in (1, \infty)$, and let $\mathcal{M}_p(V_0)$ denote the set of all p-resistance forms on V_0. For $\mathcal{D} \in \mathcal{M}_p(V_0)$, let $R_{\mathcal{D}}$ denote its effective p-resistance and set $\widehat{R}_{\mathcal{D}} := R_{\mathcal{D}}^{1/(p-1)}$. By the first assertion of Proposition 2.10, $\widehat{R}_{\mathcal{D}}$ defines a metric on V_0. Set $\widehat{R}_0^*(\mathcal{D}) := \max_{x,y \in V_0} \widehat{R}_{\mathcal{D}}(x,y)$.

Definition 3.4 (Level-one energy)**.** For $r > 0$ and $\mathcal{D} \in \mathcal{M}_p(V_0)$, define

$$\Psi_{p,r}\mathcal{D}(f) := \sum_{w \in W_1} r^{-|w|+1}\mathcal{D}\big(f \circ F_w|_{V_0}\big), \tag{3.3}$$

and set

$$\mathcal{F}^{(1)}_{p,r,\mathcal{D}} := \mathrm{Dom}(\Psi_{p,r}\mathcal{D}) := \left\{f \in \mathbb{R}^{V_1} : \Psi_{p,r}\mathcal{D}(f) < \infty\right\}.$$

Whenever a boundary form is applied to $f \circ F_w$, its restriction to V_0 is understood.

Proposition 3.5. *For every $r > 0$ and $\mathcal{D} \in \mathcal{M}_p(V_0)$, $\big(\Psi_{p,r}\mathcal{D}, \mathcal{F}^{(1)}_{p,r,\mathcal{D}}\big)$ is a p-resistance form on V_1.*

Proof. By Lemma 2.3, every finite partial sum of (3.3) has the seminorm and generalized contraction properties; these properties pass to $\Psi_{p,r}\mathcal{D}$ by monotone convergence. Vanishing energy forces constancy on each cell boundary and hence, by connectedness of the cell-incidence graph, on V_1. Thus $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ hold.

For $x, y \in V_1$, choose a finite chain $x = x_0, \dots, x_m = y$ with $\{x_{i-1}, x_i\} \subset F_{w_i}V_0$. Then

$$|f(x) - f(y)| \le \left(\max_{a,b \in V_0} R_{\mathcal{D}}(a,b)\right)^{1/p} \sum_{i=1}^{m} r^{(|w_i|-1)/p}\Psi_{p,r}\mathcal{D}(f)^{1/p}.$$

Hence $(\mathrm{RF4})_p$ holds.

To verify $(\mathrm{RF2})_p$, let $\{f_n\}$ be Cauchy in the energy seminorm and normalize each f_n by requiring $f_n(o) = 0$ at a fixed $o \in V_1$. The preceding estimate makes $\{f_n(x)\}$ Cauchy for every $x \in V_1$; set $f(x) := \lim_{n\to\infty} f_n(x)$. Continuity of $\mathcal{D}$ on $\mathbb{R}^{V_0}$ and Fatou's lemma yield $\Psi_{p,r}\mathcal{D}(f_n - f) \le \liminf_{k\to\infty} \Psi_{p,r}\mathcal{D}(f_n - f_k)$, so f has finite energy and $f_n \to f$ in the quotient norm. Finally, local finiteness gives $\mathbf{1}_x^{V_1} \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ for every $x \in V_1$; these functions separate points. Thus $(\mathrm{RF2})_p$ and $(\mathrm{RF3})_p$ hold. □

Let $R_{p,r,\mathcal{D}}$ be the effective p-resistance associated with $\Psi_{p,r}\mathcal{D}$, and set $\widehat{R}_{p,r,\mathcal{D}} := R^{1/(p-1)}_{p,r,\mathcal{D}}$. We also write $s := r^{1/(p-1)}$.

Proposition 3.6. *Let $\mathcal{D} \in \mathcal{M}_p(V_0)$ and $0 < r < 1$. Every $f \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ admits a unique continuous extension to $\overline{V}_1$. More precisely, there exists $C > 0$ such that*

$$|f(x) - f(y)| \le C\,\Psi_{p,r}\mathcal{D}(f)^{1/p} d(x,y)^{\frac{\log r}{p \log \rho}}, \qquad x, y \in V_1. \tag{3.4}$$

If, in addition, $\rho^{p-1} < r < 1$, then $\big(\Psi_{p,r}\mathcal{D}, \mathcal{F}^{(1)}_{p,r,\mathcal{D}}\big)$ extends canonically to a p-resistance form on $\overline{V}_1$, and the associated effective p-resistance satisfies

$$R_{p,r,\mathcal{D}}(x,y) \le \left(\frac{4\widehat{R}^*_0(\mathcal{D})}{s^3(1-s)}\right)^{p-1} d(x,y)^{\frac{\log r}{\log \rho}}. \tag{3.5}$$

Proof. For $w \in W_{1,n}$ and $x, y \in F_wV_0$, the weight of the cell yields $\widehat{R}_{p,r,\mathcal{D}}(x,y) \le \widehat{R}^*_0(\mathcal{D})s^{n-1}$. Summing this estimate along the nested vertical chains gives

$$\widehat{R}_{p,r,\mathcal{D}}(x,y) < \frac{2s^n}{1-s}\widehat{R}^*_0(\mathcal{D}), \qquad x, y \in F_wV_1,$$

for $w \in \{1,2\}^n$. To prove (3.5), we first take $x, y \in V_1$. The case $x = y$ is trivial. The same vertical-chain argument with the empty prefix gives

$$\widehat{R}_{p,r,\mathcal{D}}(x,y) < \frac{2}{1-s}\widehat{R}^*_0(\mathcal{D}), \qquad x, y \in V_1.$$

If $d(x,y) \geq \rho^3$, then $d(x,y)^{\log s/\log \rho} \geq s^3$, so this estimate implies (3.5). We may therefore suppose that $x \neq y$, $d(x,y) < \rho^3$, and choose $n \geq 1$ so that $\rho^{n+3} \leq d(x,y) < \rho^{n+2}$. Then Proposition 3.3(3) yields $w, w' \in \{1,2\}^n$ with $x \in F_w V_1$, $y \in F_{w'} V_1$, and $F_w V_1 \cap F_{w'} V_1 \neq \varnothing$. Hence

$$\widehat{R}_{p,r,\mathcal{D}}(x,y) \leq \frac{4s^n}{1-s} \widehat{R}_0^*(\mathcal{D}).$$

Raising this estimate to the $(p-1)$st power proves (3.5) on V_1. The resistance inequality (2.2) then gives (3.4); consequently, every finite-energy function extends uniquely and continuously to $\overline{V}_1$.

Let (X, d_X) be the metric completion of $(V_1, \widehat{R}_{p,r,\mathcal{D}})$. By Theorem 2.11, the level-one form extends to a p-resistance form on X. Under the assumption $\rho^{p-1} < r < 1$, the estimate (3.5) makes the identity $(V_1, d) \to (V_1, \widehat{R}_{p,r,\mathcal{D}})$ uniformly continuous, and hence it extends to a continuous map $\theta : (\overline{V}_1, d) \to (X, d_X)$.

To prove that θ is injective, let $\xi \neq \eta$ and choose a linear function $\ell : \mathbb{R}^2 \to \mathbb{R}$ such that $\ell(\xi) \neq \ell(\eta)$. Since every F_w, $w \in W_{1,n}$, has similarity ratio ρ^{n+1}, Proposition 3.3(2) gives

$$\begin{aligned}\Psi_{p,r}\mathcal{D}(\ell|_{V_1}) &= \sum_{n\geq 1} r^{-n+1} \# W_{1,n} \rho^{p(n+1)} \mathcal{D}(\ell|_{V_0}) \\ &\leq C_1 r \rho^p \mathcal{D}(\ell|_{V_0}) \sum_{n\geq 1} \left(\frac{\rho^{p-1}}{r}\right)^n < \infty.\end{aligned}$$

The continuous extension of $\ell|_{V_1}$ to X separates $\theta(\xi)$ and $\theta(\eta)$, so θ is injective. Since $\overline{V}_1$ is compact and X is Hausdorff, θ is a homeomorphism onto its compact image. After identifying V_1 with its canonical image in X, we have $V_1 \subset \theta(\overline{V}_1)$ and hence $X = \overline{V_1}^{d_X} \subset \theta(\overline{V}_1) \subset X$. Thus $\theta(\overline{V}_1) = X$. Identifying functions in $\mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ with their continuous extensions to $\overline{V}_1$, we conclude that $\left(\Psi_{p,r}\mathcal{D}, \mathcal{F}^{(1)}_{p,r,\mathcal{D}}\right)$ is a p-resistance form on $\overline{V}_1$. Finally, by density, (3.5) extends to $\overline{V}_1$. □

Remark 3.7. The lower bound $r > \rho^{p-1}$ is sharp for point separation after continuous extension. Indeed, suppose that $r \leq \rho^{p-1}$, and set $A_n := W_{1,n} \cup W_{1,n+1}$ and $V_{A_n} := \bigcup_{w\in A_n} F_w V_0$. For $f \in \mathbb{R}^{V_{A_n}}$, set

$$\mathcal{D}_{A_n}(f) := \sum_{w\in A_n} r^{1-|w|} \mathcal{D}(f \circ F_w|_{V_0}).$$

The family A_n induces a connected finite subgraph. Hence $\left(\mathcal{D}_{A_n}, \mathbb{R}^{V_{A_n}}\right)$ is a p-resistance form by Lemma 2.3 and Remark 2.2. Let R_{A_n} denote its effective p-resistance, and set $\widehat{R}_{A_n} := R_{A_n}^{1/(p-1)}$. The finite-chain geometry underlying Proposition 3.3 (see [9, Section 3]) joins $F_1^n q_0$ to $F_2^n q_0$ through at most $C_1 \rho^{-n}$ cell sides from A_n. Each such side $\{x, y\}$ satisfies $\widehat{R}_{A_n}(x,y) \leq C_2 r^{n/(p-1)}$. Thus the triangle inequality gives

$$R_{A_n}(F_1^n q_0, F_2^n q_0) \leq C_3 r^n \rho^{-n(p-1)}.$$

Let $f \in C(\overline{V}_1)$ satisfy $f|_{V_1} \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ and $f(q_1) \neq f(q_2)$. Then $F_1^n q_0 \to q_1$ and $F_2^n q_0 \to q_2$. Hence there exists $\delta > 0$ such that $|f(F_1^n q_0) - f(F_2^n q_0)|^p \geq \delta$ for all sufficiently large n. Consequently,

$$\mathcal{D}_{A_n}(f) \geq \delta R_{A_n}(F_1^n q_0, F_2^n q_0)^{-1} \geq C_3^{-1} \delta r^{-n} \rho^{n(p-1)} \geq C_3^{-1} \delta.$$

Since each cell occurs in at most two consecutive families A_n,

$$2\Psi_{p,r}\mathcal{D}(f) \geq \sum_{n\geq 1} \mathcal{D}_{A_n}(f) = \infty,$$

which is a contradiction. Thus the continuous finite-energy functions on $\overline{V}_1$ fail to separate q_1 and q_2.

4 The renormalization operator

Fix $p \in (1,\infty)$ and write $\mathcal{L}(V_0) := \mathbb{R}^{V_0}$. We identify $u \in \mathcal{L}(V_0)$ with $u = (u(q_0), u(q_1), u(q_2)) \in \mathbb{R}^3$ and set $\mathbf{1} := \mathbf{1}_{V_0} = (1,1,1)$. We first place the p-resistance forms on V_0 in an ambient Banach space. Its topology is used for continuity of the renormalization operator and, in Section 5, for the Schauder fixed-point argument.

4.1 Boundary-form topology

Definition 4.1 (Boundary-form space)**.** Define

$$\mathscr{B}_p(V_0) := \left\{ \Phi \in C(\mathbb{R}^3) : \begin{array}{l} \Phi(cu) = |c|^p \Phi(u), \\ \Phi(u + t\mathbf{1}) = \Phi(u) \end{array} \text{ for all } c,t \in \mathbb{R},\ u \in \mathbb{R}^3 \right\}.$$

Fix the base point q_0 and let $\mathcal{H}_0 := \{u \in \mathcal{L}(V_0) : u(q_0) = 0\}$. For $\Phi \in \mathscr{B}_p(V_0)$, define

$$\|\Phi\|_{\mathscr{B}_p} := \sup\{|\Phi(u)| : u \in \mathcal{H}_0,\ \operatorname{osc}_{V_0}(u) = 1\}. \tag{4.1}$$

For the norm in (4.1), invariance under addition of constants and p-homogeneity give

$$\|\Phi\|_{\mathscr{B}_p} = \sup_{u \notin \mathbb{R}\mathbf{1}} \frac{|\Phi(u)|}{\operatorname{osc}_{V_0}(u)^p},$$

so the norm is independent of the choice of the base point.

Proposition 4.2. *The following assertions hold.*

(1) $(\mathscr{B}_p(V_0), \|\cdot\|_{\mathscr{B}_p})$ *is a Banach space.*

(2) *For* $\Phi_n, \Phi \in \mathscr{B}_p(V_0)$, $\|\Phi_n - \Phi\|_{\mathscr{B}_p} \to 0$ *if and only if* $\Phi_n \to \Phi$ *uniformly on every compact subset of* $\mathbb{R}^3$.

Proof. Let $S := \{u \in \mathcal{H}_0 : \operatorname{osc}_{V_0}(u) = 1\}$. The restriction map $J\Phi = \Phi|_S$ is an isometric embedding of $\mathscr{B}_p(V_0)$ into $C(S)$, and its image is closed under uniform convergence. Hence $\mathscr{B}_p(V_0)$ is complete.

For every compact set $K \subset \mathbb{R}^3$,

$$\sup_{u \in K} |\Phi_n(u) - \Phi(u)| \le \|\Phi_n - \Phi\|_{\mathscr{B}_p} \sup_{u \in K} \operatorname{osc}_{V_0}(u)^p.$$

Thus norm convergence implies local uniform convergence. Conversely, local uniform convergence implies uniform convergence on the compact set S, and hence convergence in $\|\cdot\|_{\mathscr{B}_p}$. □

Recall from Section 3.3 that $\mathcal{M}_p(V_0)$ denotes the set of all p-resistance forms on V_0. Under the identification $\mathcal{L}(V_0) \simeq \mathbb{R}^3$, every $\mathcal{D} \in \mathcal{M}_p(V_0)$ belongs to $\mathscr{B}_p(V_0)$: indeed, $\mathcal{D}^{1/p}$ is a seminorm on the finite-dimensional space $\mathcal{L}(V_0)$, so $\mathcal{D}$ is continuous, while p-homogeneity and invariance under addition of constants follow from the resistance-form axioms. Thus $\mathcal{M}_p(V_0)$ is a convex cone in $\mathscr{B}_p(V_0)$ by Lemma 2.3. We equip it with the induced metric $d_{\mathscr{B}_p}(\mathcal{D}_1, \mathcal{D}_2) := \|\mathcal{D}_1 - \mathcal{D}_2\|_{\mathscr{B}_p}$.

4.2 Definition and domain

We now define the renormalization operator $\mathcal{R}_{p,r}$ and describe the domain properties needed for the continuity argument.

Let $r \in (\rho^{p-1}, 1)$ and $\mathcal{D} \in \mathcal{M}_p(V_0)$. By Proposition 3.6, we identify every $f \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ with its unique continuous extension to $\overline{V}_1$; under this identification, $\big(\Psi_{p,r}\mathcal{D}, \mathcal{F}^{(1)}_{p,r,\mathcal{D}}\big)$ is a p-resistance form on $\overline{V}_1$. The *renormalization operator* is its trace to V_0:

$$\mathcal{R}_{p,r}\mathcal{D} := (\Psi_{p,r}\mathcal{D})|_{V_0}.$$

Equivalently, for $u \in \mathcal{L}(V_0)$,

$$(\mathcal{R}_{p,r}\mathcal{D})(u) := \inf\left\{\Psi_{p,r}\mathcal{D}(h) : h \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}},\ h|_{V_0} = u\right\}.$$

Proposition 4.3. *Let $\mathcal{D} \in \mathcal{M}_p(V_0)$ and $\rho^{p-1} < r_1 < r_2 < 1$.*

(1) *The domain $\mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ depends only on r, and $\mathcal{F}^{(1)}_{p,r_1,\mathcal{D}} \subset \mathcal{F}^{(1)}_{p,r_2,\mathcal{D}}$.*

(2) *For every $f \in \mathcal{F}^{(1)}_{p,r_2,\mathcal{D}}$ and every $\varepsilon > 0$, there exists $g \in \mathcal{F}^{(1)}_{p,r_1,\mathcal{D}}$ such that*

$$g|_{V_0} = f|_{V_0}, \qquad \Psi_{p,r_2}\mathcal{D}(f-g) < \varepsilon.$$

In addition, g may be chosen so that, for some $A > 0$, $\operatorname{osc}_{F_wV_0} g \le A\rho^{|w|}$ for all sufficiently long $w \in W_1$.

(3) *The space $\mathcal{F}^{(1)}_{p,r_1,\mathcal{D}}$ is dense in $C(\overline{V}_1)$.*

Proof. (1) Since $\mathcal{D}^{1/p}$ induces a norm on $\mathcal{L}(V_0)/\mathbb{R}\mathbf{1}$, finite dimensionality gives constants $c_0, C_0 > 0$ such that

$$c_0\Big(|h(q_0)-h(q_1)|^p + |h(q_0)-h(q_2)|^p\Big) \le \mathcal{D}(h) \le C_0\Big(|h(q_0)-h(q_1)|^p + |h(q_0)-h(q_2)|^p\Big) \tag{4.2}$$

for every $h \in \mathcal{L}(V_0)$. Thus any two forms in $\mathcal{M}_p(V_0)$ are multiplicatively comparable, so $\mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ depends only on r. If $r_1 < r_2$, then $r_2^{1-|w|} \le r_1^{1-|w|}$ for every $w \in W_1$, and hence $\mathcal{F}^{(1)}_{p,r_1,\mathcal{D}} \subset \mathcal{F}^{(1)}_{p,r_2,\mathcal{D}}$.

(2) For $h : V_1 \to \mathbb{R}$, set

$$E_r^{\ge n}(h) := \sum_{\ell=n}^{\infty}\sum_{w\in W_{1,\ell}} r^{1-\ell}\mathcal{D}(h\circ F_w|_{V_0}).$$

Fix $\delta > 0$, to be determined later. Given $f \in \mathcal{F}^{(1)}_{p,r_2,\mathcal{D}}$, choose $n \ge 3$ so large that $E_{r_2}^{\ge n}(f) < \delta$. List the points $\{F_wq_0 : w \in W_{1,n}\}$ from left to right as $p_1, \dots, p_N$, and set

$$\Delta_n(f) := |f(q_1)-f(p_1)|^p + \sum_{i=1}^{N-1}|f(p_i)-f(p_{i+1})|^p + |f(p_N)-f(q_2)|^p.$$

The level-one geometry gives cell-edge paths between consecutive p_i's, using only cells in $W_{1,n}\cup W_{1,n+1}$, with uniformly bounded length and overlap. Hence finite Hölder's inequality and (4.2) yield

$$r_2^{-n}\sum_{i=1}^{N-1}|f(p_i)-f(p_{i+1})|^p \le C_1\sum_{\ell=n}^{n+1}\sum_{w\in W_{1,\ell}} r_2^{1-\ell}\mathcal{D}(f\circ F_w|_{V_0}), \tag{4.3}$$

where we used $r_2^{-n} \le r_2^{-1}r_2^{1-\ell}$ for $\ell \in \{n, n+1\}$.

For the end increments, note that $p_1 = F_1^{n-1}q_0$ and $p_N = F_2^{n-1}q_0$. The successive edges of the chains $F_1^mq_0 \to q_1$ and $F_2^mq_0 \to q_2$, $m \ge n-1$, are sides of cells in $W_{1,m+1}$. Thus weighted Hölder's inequality and (4.2) give

$$\begin{aligned} r_2^{-n}|f(q_1)-f(p_1)|^p &= r_2^{-n}\left|\sum_{m=n-1}^{\infty}\big(f(F_1^{m+1}q_0)-f(F_1^mq_0)\big)\right|^p \\ &\le r_2^{-n}\left(\sum_{m=n-1}^{\infty} r_2^{-m}\left|f(F_1^{m+1}q_0)-f(F_1^mq_0)\right|^p\right)\left(\sum_{m=n-1}^{\infty} r_2^{m/(p-1)}\right)^{p-1} \end{aligned}$$

$$\le \frac{c_0^{-1} r_2^{-1}}{\bigl(1 - r_2^{1/(p-1)}\bigr)^{p-1}} E_{r_2}^{\ge n}(f),$$

and the same estimate holds with (q_1, p_1) replaced by (q_2, p_N). Combining these estimates with (4.3), we obtain $r_2^{-n}\Delta_n(f) \le C_2 E_{r_2}^{\ge n}(f)$.

Let φ_n depend only on the horizontal coordinate and interpolate linearly between the values of f at $q_1, p_1, \dots, p_N, q_2$. Define

$$g(z) = \begin{cases} f(z), & z \in \bigcup_{\ell<n} \bigcup_{w\in W_{1,\ell}} \{F_w q_0\}, \\ \varphi_n(z), & z \in \overline{V}_1 \setminus \bigcup_{\ell<n} \bigcup_{w\in W_{1,\ell}} \{F_w q_0\}. \end{cases}$$

The exceptional set on which g is not given by φ_n is finite and lies away from the bottom-line accumulation set. Thus $g \in C(\overline{V}_1)$, and $g|_{V_0} = f|_{V_0}$. To estimate the tail, note that the consecutive interpolation intervals have lengths comparable to ρ^n. Hence, for every $\ell \ge n$, each such interval meets at most $C_3\rho^{n-\ell}$ cells from $W_{1,\ell}$, while every such cell has horizontal diameter at most $C_3\rho^{\ell}$. Since φ_n is affine between consecutive interpolation points, for $\ell \ge n$, finite Hölder's inequality gives

$$\sum_{w\in W_{1,\ell}} \operatorname{osc}_{F_w V_0}(g)^p \le C_4 \rho^{(p-1)(\ell-n)} \Delta_n(f). \tag{4.4}$$

Combining (4.4) with (4.2), and using $\rho^{p-1} < r_i$, we obtain

$$E_{r_i}^{\ge n}(g) \le C_5 \Delta_n(f) \sum_{\ell=n}^{\infty} r_i^{1-\ell} \rho^{(p-1)(\ell-n)} \le C_6 r_i^{-n} \Delta_n(f), \qquad i \in \{1, 2\}.$$

So,

$$E_{r_1}^{\ge n}(g) \le C_6 r_1^{-n} \Delta_n(f) < \infty, \qquad E_{r_2}^{\ge n}(g) \le C_6 r_2^{-n} \Delta_n(f) \le C_2 C_6 \delta.$$

Thus $g \in \mathcal{F}^{(1)}_{p,r_1,\mathcal{D}}$. Since $f - g$ vanishes on the first $n-1$ cell levels (the lower vertices of such a cell occur among the preceding top vertices or the level-n knots, where $\varphi_n = f$), Minkowski's inequality gives

$$\Psi_{p,r_2}\mathcal{D}(f-g)^{1/p} \le E_{r_2}^{\ge n}(f)^{1/p} + E_{r_2}^{\ge n}(g)^{1/p} \le \bigl(1 + (C_2C_6)^{1/p}\bigr)\delta^{1/p}.$$

Choosing δ sufficiently small proves the approximation assertion in (2). On the tail, g depends only on the horizontal coordinate and is piecewise linear. Hence $\operatorname{osc}_{F_w V_0} g \le A\rho^{|w|}$ for some $A > 0$ and all sufficiently long $w \in W_1$, which proves the additional assertion in (2).

(3) If $f \in C(\overline{V}_1)$, carry out the same construction without assuming finite energy. The first $n-1$ levels contribute only finitely many terms, while (4.4) and $\rho^{p-1} < r_1$ show that the tail has finite r_1-energy. Hence $g \in \mathcal{F}^{(1)}_{p,r_1,\mathcal{D}}$. Moreover, the construction satisfies $\|f - g\|_\infty \le 2\omega_f(C_7\rho^n)$, where ω_f is the modulus of continuity of f. Letting $n \to \infty$ proves (3). □

4.3 Continuity

We next establish continuity of $\mathcal{R}_{p,r}$ under simultaneous variation of the exponent p, the weight parameter r, and the resistance form $\mathcal{D}$. This result is not needed for the fixed-p eigenform construction in Section 5, but it will be used in Section 8 to prove continuity of the standard renormalization factor r_p with respect to p. For varying exponents p, convergence of resistance forms on V_0 is understood as local uniform convergence on $\mathcal{L}(V_0)$, which is independent of p; see Proposition 4.2. It is convenient to write $r = s^{p-1}$ with $s \in (\rho, 1)$.

Theorem 4.4. *Let $p_n \to p \in (1,\infty)$ and $s_n \to s \in (\rho,1)$, and set $r_n = s_n^{p_n-1}$ and $r = s^{p-1}$. Suppose that $\mathcal{D}_n \in \mathcal{M}_{p_n}(V_0)$, $\mathcal{D} \in \mathcal{M}_p(V_0)$, and $\mathcal{D}_n \to \mathcal{D}$. Then*

$$\mathcal{R}_{p_n,r_n}\mathcal{D}_n \longrightarrow \mathcal{R}_{p,r}\mathcal{D}$$

locally uniformly on $\mathcal{L}(V_0)$.

Proof. The argument combines compactness of the minimizing extensions with a uniform tail estimate, needed because the exponent p varies.

Write $\Psi_n := \Psi_{p_n,r_n}\mathcal{D}_n$, $\Psi := \Psi_{p,r}\mathcal{D}$, and denote their traces by $\mathcal{R}_n$ and $\mathcal{R}$. On the compact set $S := \{v \in \mathcal{H}_0 : \operatorname{osc}_{V_0}(v) = 1\}$, local uniform convergence and the positivity of $\mathcal{D}$ give $0 < c_1 \le C_1 < \infty$ such that $c_1 \le \mathcal{D}_n \le C_1$ on S for all sufficiently large n. Hence, by p_n-homogeneity and invariance under addition of constants,

$$c_1 \operatorname{osc}_{V_0}(v)^{p_n} \le \mathcal{D}_n(v) \le C_1 \operatorname{osc}_{V_0}(v)^{p_n}, \qquad v \in \mathcal{L}(V_0). \tag{4.5}$$

Fix $u \in \mathcal{L}(V_0)$ and $\varepsilon > 0$, and choose $r_0 \in (\rho^{p-1}, r)$. Let f be the Ψ-minimizing extension of u, so that $\Psi(f) = \mathcal{R}(u)$. By Proposition 4.3(2), there is an extension g of u, chosen with the additional tail property stated there, such that

$$g \in \mathcal{F}^{(1)}_{p,r_0,\mathcal{D}}, \qquad \Psi(g) \le \Psi(f) + \varepsilon = \mathcal{R}(u) + \varepsilon,$$

and, for some $A > 0$ and $N \ge 1$, $\operatorname{osc}_{F_wV_0}(g) \le A\rho^{|w|}$ whenever $w \in W_1$ and $|w| \ge N$. Using (4.5), this estimate, and $\#W_{1,\ell} \le C_2\rho^{-\ell}$, we obtain

$$\sum_{w\in W_{1,\ell}} r_n^{1-\ell}\mathcal{D}_n(g\circ F_w|_{V_0}) \le C_3 r_n \left(\frac{\rho}{s_n}\right)^{(p_n-1)\ell}, \qquad \ell \ge N.$$

For each fixed ℓ, the corresponding level sum converges to $\sum_{w\in W_{1,\ell}} r^{1-\ell}\mathcal{D}(g\circ F_w|_{V_0})$, while $(\rho/s_n)^{p_n-1} \to (\rho/s)^{p-1} < 1$. Thus the level sums are bounded by $C_4\eta^\ell$ for some $\eta \in (0,1)$, and dominated convergence over $\ell \in \mathbb{N}$ yields $\Psi_n(g) \to \Psi(g)$. Since $\mathcal{R}_n(u) \le \Psi_n(g)$,

$$\limsup_{n\to\infty} \mathcal{R}_n(u) \le \mathcal{R}(u). \tag{4.6}$$

Let h_n be the Ψ_n-minimizing extension of u. By the weak comparison principle in Proposition 2.9,

$$\min_{V_0} u \le h_n \le \max_{V_0} u \qquad \text{on } \overline{V}_1,$$

so $\{h_n\}$ is uniformly bounded, while (4.6) and $p_n \to p$ give $K := \sup_n \Psi_n(h_n)^{1/p_n} < \infty$. The lower bound in (4.5) gives $\widehat{R}_0^*(\mathcal{D}_n) \le c_1^{-1/(p_n-1)}$, and hence $\sup_n \widehat{R}_0^*(\mathcal{D}_n) < \infty$. Applying Proposition 3.6 with $s_n = r_n^{1/(p_n-1)}$ yields

$$|h_n(x) - h_n(y)| \le \left(\frac{4\widehat{R}_0^*(\mathcal{D}_n)}{s_n^3(1-s_n)}\right)^{\frac{p_n-1}{p_n}} \Psi_n(h_n)^{1/p_n} d(x,y)^{\frac{\log r_n}{p_n \log \rho}}$$

for $x, y \in \overline{V}_1$. Since $p_n \to p$, $s_n \to s$, and $d(x,y) \le 1$, the preceding bounds imply that there exist constants $C_5 > 0$ and $\beta > 0$ such that

$$|h_n(x) - h_n(y)| \le C_5 d(x,y)^\beta, \qquad x, y \in \overline{V}_1.$$

By the Arzelà–Ascoli theorem, after passing to a subsequence realizing the lower limit of $\Psi_n(h_n)$, h_n converges uniformly to an extension h of u. For each fixed $w \in W_1$, $r_n^{1-|w|}\mathcal{D}_n(h_n \circ F_w|_{V_0}) \to r^{1-|w|}\mathcal{D}(h\circ F_w|_{V_0})$. Fatou's lemma yields

$$\mathcal{R}(u) \le \Psi(h) \le \liminf_{n\to\infty} \Psi_n(h_n) = \liminf_{n\to\infty} \mathcal{R}_n(u).$$

Together with (4.6), this proves $\mathcal{R}_n(u) \longrightarrow \mathcal{R}(u)$ for every $u \in \mathcal{L}(V_0)$.

It remains to upgrade the convergence to local uniform convergence. Set $\mathcal{N}_n := \mathcal{R}_n^{1/p_n}$ and $\mathcal{N} := \mathcal{R}^{1/p}$. These are seminorms on $\mathcal{L}(V_0)$. If e_0, e_1, e_2 denote the standard basis of $\mathcal{L}(V_0) \simeq \mathbb{R}^3$, then the pointwise convergence above and $p_n \to p$ imply $\sup_n \max_{0\le i\le 2} \mathcal{N}_n(e_i) < \infty$. Consequently, $\{\mathcal{N}_n\}$ is equicontinuous, since

$$|\mathcal{N}_n(u) - \mathcal{N}_n(v)| \le \mathcal{N}_n(u - v) \le C_6 \sum_{i=0}^{2} |u(q_i) - v(q_i)|.$$

Thus $\mathcal{N}_n \to \mathcal{N}$ locally uniformly on $\mathcal{L}(V_0)$. Since $p_n \to p$, it follows that $\mathcal{R}_n = \mathcal{N}_n^{p_n} \to \mathcal{N}^p = \mathcal{R}$ locally uniformly on $\mathcal{L}(V_0)$. □

Corollary 4.5. *For $p \in (1, \infty)$, the map*

$$\mathcal{R}_p : (\rho^{p-1}, 1) \times \mathcal{M}_p(V_0) \longrightarrow \mathcal{M}_p(V_0), \qquad (r, \mathcal{D}) \longmapsto \mathcal{R}_{p,r}\mathcal{D},$$

is jointly continuous with respect to the usual topology in r and the metric $d_{\mathscr{B}_p}$ in $\mathcal{D}$.

Proof. Let $r_n \to r$ and $\mathcal{D}_n \to \mathcal{D}$ in $d_{\mathscr{B}_p}$. Set $p_n \equiv p$, $s_n := r_n^{1/(p-1)}$, and $s := r^{1/(p-1)}$. Then $s_n \to s$, and Theorem 4.4 gives local uniform convergence $\mathcal{R}_{p,r_n}\mathcal{D}_n \to \mathcal{R}_{p,r}\mathcal{D}$ on $\mathcal{L}(V_0)$. The conclusion follows from Proposition 4.2(2). □

5 Renormalization eigenforms

We construct eigenforms of the nonlinear renormalization operator $\mathcal{R}_{p,r}$ on the symmetric cone of p-resistance forms on V_0. A compact convex invariant set for $\mathcal{R}_{p,r}$ gives existence by the Schauder fixed-point theorem. We then prove uniqueness of the eigenvalue, establish its dependence on r, and identify the unique standard renormalization factor.

Throughout this section, fix $p \in (1, \infty)$ and $r \in (\rho^{p-1}, 1)$. For $\mathcal{D} \in \mathcal{M}_p(V_0)$, we retain the notation $R_{\mathcal{D}}$ for its effective p-resistance and $C_{\mathcal{D}}(x, y) = R_{\mathcal{D}}(x, y)^{-1}$ for the corresponding *conductance*, where $x, y \in V_0$ and $x \ne y$.

5.1 Existence and eigenvalues

Let $\sigma : G \to G$ be the reflection in the vertical symmetry axis, so that $\sigma(q_0) = q_0$, $\sigma(q_1) = q_2$, and $\sigma(q_2) = q_1$ on V_0. We call a form *symmetric* when it is invariant under this reflection, and set

$$\mathcal{M}_p^{\mathrm{sym}}(V_0) := \{\mathcal{D} \in \mathcal{M}_p(V_0) : \mathcal{D}(u \circ \sigma|_{V_0}) = \mathcal{D}(u) \text{ for every } u \in \mathcal{L}(V_0)\}.$$

Both the countable level-one cell structure and the weights are invariant under σ. Consequently, $\mathcal{R}_{p,r}(\mathcal{M}_p^{\mathrm{sym}}(V_0)) \subseteq \mathcal{M}_p^{\mathrm{sym}}(V_0)$.

Set

$$u_* := \left(\frac{1}{2}, 0, 1\right) \in \mathcal{L}(V_0), \tag{5.1}$$

so that $u_*(q_0) = 1/2$, $u_*(q_1) = 0$, and $u_*(q_2) = 1$. For $\mathcal{D} \in \mathcal{M}_p(V_0)$, define the *normalized renormalization operator*

$$\widetilde{\mathcal{R}}_{p,r}\mathcal{D} := \frac{\mathcal{R}_{p,r}\mathcal{D}}{(\mathcal{R}_{p,r}\mathcal{D})(u_*)}. \tag{5.2}$$

Then $(\widetilde{\mathcal{R}}_{p,r}\mathcal{D})(u_*) = 1$. For $c > 0$, define

$$\mathcal{K}_c := \left\{\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0) : \mathcal{D}(u_*) = 1,\ C_{\mathcal{D}}(q_0, q_1) \ge c\right\}. \tag{5.3}$$

The set $\mathcal{K}_c$ will serve as the compact invariant set in the fixed-point argument.

Lemma 5.1. *For every $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$,*

$$C_{\mathcal{R}_{p,r}\mathcal{D}}(q_0, q_1) \geq \left(1 - r^{1/(p-1)}\right)^{p-1} C_{\mathcal{D}}(q_0, q_1). \tag{5.4}$$

Proof. By the definition of conductance and the p-homogeneity of $\mathcal{D}$,

$$\mathcal{D}(v) \geq C_{\mathcal{D}}(q_0, q_1)|v(q_0) - v(q_1)|^p, \qquad v \in \mathbb{R}^{V_0}. \tag{5.5}$$

Let $f \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$ satisfy $f(q_0) = 0$ and $f(q_1) = 1$. Applying (5.5) to the cells $F_{1^n 0}G$, $n \geq 0$, yields

$$\Psi_{p,r}\mathcal{D}(f) \geq C_{\mathcal{D}}(q_0, q_1) \sum_{n=0}^{\infty} r^{-n} \left|f(F_1^{n+1}q_0) - f(F_1^n q_0)\right|^p.$$

Since $F_1^n q_0 \to q_1$, the weighted Hölder inequality gives

$$1 = |f(q_1) - f(q_0)|^p \leq \left(\sum_{n=0}^{\infty} r^{-n} \left|f(F_1^{n+1}q_0) - f(F_1^n q_0)\right|^p\right) \left(\sum_{n=0}^{\infty} r^{n/(p-1)}\right)^{p-1}.$$

Therefore,

$$\sum_{n=0}^{\infty} r^{-n} \left|f(F_1^{n+1}q_0) - f(F_1^n q_0)\right|^p \geq \left(1 - r^{1/(p-1)}\right)^{p-1}.$$

Taking the infimum over all admissible f proves (5.4). □

Lemma 5.2. *There exists a constant $b_p > 0$, depending only on p, such that*

$$(\mathcal{R}_{p,r}\mathcal{D})(u_*) \leq b_p \frac{r\rho^{p-1}}{r - \rho^{p-1}} C_{\mathcal{D}}(q_0, q_1)$$

for every $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$.

Proof. We first note that

$$\mathcal{D}(v) \leq 2^p C_{\mathcal{D}}(q_0, q_1)|v(q_0) - v(q_1)|^p \tag{5.6}$$

whenever $v(q_1) = v(q_2)$.

Indeed, let h minimize $\mathcal{D}$ subject to $h(q_0) = 0$ and $h(q_1) = 1$. By the contraction property, we may assume $0 \leq h \leq 1$. Writing $h(q_2) = t \in [0, 1]$, symmetry and the triangle inequality for $\mathcal{D}^{1/p}$ yield

$$(1 + t)\mathcal{D}(0, 1, 1)^{1/p} = \mathcal{D}(h + h \circ \sigma)^{1/p} \leq 2C_{\mathcal{D}}(q_0, q_1)^{1/p}.$$

Thus

$$\mathcal{D}(1, 0, 0) = \mathcal{D}(0, 1, 1) \leq 2^p C_{\mathcal{D}}(q_0, q_1),$$

which implies (5.6).

Let $f(x, y) = x$, so that $f|_{V_0} = u_*$. Define the horizontal-edge relation on V_1 by

$$x \sim_{\mathrm{h}} y \quad \Longleftrightarrow \quad \{x, y\} = \{F_w q_1, F_w q_2\} \quad \text{for some } w \in W_1,$$

and let $[x]_{\mathrm{h}}$ be the corresponding connected component. Set

$$g(x) := \frac{1}{\#[x]_{\mathrm{h}}} \sum_{y \in [x]_{\mathrm{h}}} f(y).$$

Then g is constant on every horizontal edge. Since the horizontal components have diameters tending to zero at the boundary, g extends continuously to $\overline{V}_1$, with $g|_{V_0} = f|_{V_0} = u_*$. Since $f(x, y) = x$ and g is obtained by averaging f over horizontal components, we have

$$\operatorname{osc}_{F_w V_0} g \leq c_1 \rho^n, \qquad w \in W_{1,n}. \tag{5.7}$$

Since $g(F_w q_1) = g(F_w q_2)$, it follows from (5.6) and (5.7) that

$$\mathcal{D}(g \circ F_w|_{V_0}) \le 2^p c_1^p C_{\mathcal{D}}(q_0, q_1)\rho^{pn}, \qquad w \in W_{1,n}.$$

By Proposition 3.3, $\#W_{1,n} \le c_2\rho^{-n}$. We therefore obtain

$$\Psi_{p,r}\mathcal{D}(g) \le 2^p c_1^p c_2 C_{\mathcal{D}}(q_0, q_1) \sum_{n=1}^{\infty} r^{-n+1}\rho^{(p-1)n} = 2^p c_1^p c_2 \frac{r\rho^{p-1}}{r - \rho^{p-1}} C_{\mathcal{D}}(q_0, q_1).$$

Hence $g \in \mathcal{F}^{(1)}_{p,r,\mathcal{D}}$, and the variational characterization of the trace gives $(\mathcal{R}_{p,r}\mathcal{D})(u_*) \le \Psi_{p,r}\mathcal{D}(g)$. The result follows with $b_p := 2^p c_1^p c_2$. □

Proposition 5.3. *There exists $c_{p,r} > 0$ such that $C_{\widetilde{\mathcal{R}}_{p,r}\mathcal{D}}(q_0, q_1) \ge c_{p,r}$ for every $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$. Consequently, $\widetilde{\mathcal{R}}_{p,r}(\mathcal{K}_{c_{p,r}}) \subseteq \mathcal{K}_{c_{p,r}}$.*

Proof. By the definition of $\widetilde{\mathcal{R}}_{p,r}$ and the homogeneity of conductance,

$$C_{\widetilde{\mathcal{R}}_{p,r}\mathcal{D}}(q_0, q_1) = \frac{C_{\mathcal{R}_{p,r}\mathcal{D}}(q_0, q_1)}{(\mathcal{R}_{p,r}\mathcal{D})(u_*)}.$$

Therefore, Lemmas 5.1 and 5.2 give

$$C_{\widetilde{\mathcal{R}}_{p,r}\mathcal{D}}(q_0, q_1) \ge \frac{1}{b_p}\bigl(1 - r^{1/(p-1)}\bigr)^{p-1} \frac{r - \rho^{p-1}}{r\rho^{p-1}}.$$

Thus one may take

$$c_{p,r} := \frac{1}{b_p}\bigl(1 - r^{1/(p-1)}\bigr)^{p-1} \frac{r - \rho^{p-1}}{r\rho^{p-1}} > 0.$$

Since the normalized operator preserves reflection symmetry and satisfies $(\widetilde{\mathcal{R}}_{p,r}\mathcal{D})(u_*) = 1$, the invariance of $\mathcal{K}_{c_{p,r}}$ follows. □

Lemma 5.4. *Let $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ satisfy $\mathcal{D}(u_*) = 1$. Then, for every $v \in \mathcal{L}(V_0)$,*

$$\frac{1}{2} C_{\mathcal{D}}(q_0, q_1)^{1/p} \operatorname{osc}_{V_0}(v) \le \mathcal{D}(v)^{1/p} \le 2 \operatorname{osc}_{V_0}(v). \tag{5.8}$$

Proof. By the definition of conductance and reflection symmetry,

$$\mathcal{D}(v) \ge C_{\mathcal{D}}(q_0, q_1) \max_{i=1,2} |v(q_0) - v(q_i)|^p.$$

Since $\operatorname{osc}_{V_0}(v) \le 2\max_{i=1,2}|v(q_0) - v(q_i)|$, the lower bound follows.

For the upper bound, set $M := \operatorname{osc}_{V_0}(v)$. If $M > 0$, choose a piecewise linear function $\phi : \mathbb{R} \to \mathbb{R}$, constant outside $[0, 1]$, such that

$$\phi(0) = 0, \qquad \phi\left(\frac{1}{2}\right) = v(q_0) - v(q_1), \qquad \phi(1) = v(q_2) - v(q_1).$$

Then $\operatorname{Lip}(\phi) \le 2M$ and $v - v(q_1)\mathbf{1}_{V_0} = \phi \circ u_*$. By the contraction property in Proposition 2.4,

$$\mathcal{D}(v)^{1/p} = \mathcal{D}(\phi \circ u_*)^{1/p} \le 2M\mathcal{D}(u_*)^{1/p} = 2M.$$

The case $M = 0$ is immediate. □

The estimate (5.8) will be used below to obtain compactness.

Proposition 5.5. *For every $c > 0$, the set $\mathcal{K}_c$ is compact and convex in $\mathscr{B}_p(V_0)$.*

Proof. Convexity. Let $\mathcal{D}_t = t\mathcal{D}_1 + (1-t)\mathcal{D}_2$, where $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{K}_c$ and $t \in [0,1]$. Then $\mathcal{D}_t \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$, $\mathcal{D}_t(u_*) = 1$, and

$$C_{\mathcal{D}_t}(q_0, q_1) \geq tC_{\mathcal{D}_1}(q_0, q_1) + (1-t)C_{\mathcal{D}_2}(q_0, q_1) \geq c.$$

Thus $\mathcal{K}_c$ is convex.

Closedness. Suppose that $\mathcal{D}_n \in \mathcal{K}_c$ and $\mathcal{D}_n \to \mathcal{D}$ in $\mathscr{B}_p(V_0)$. The seminorm property, $(\mathrm{RF5})_p$, reflection symmetry, and normalization pass to the limit. By Lemma 5.4,

$$\mathcal{D}_n(v) \geq 2^{-p} c \operatorname{osc}_{V_0}(v)^p, \qquad v \in \mathcal{L}(V_0),$$

so $\mathcal{D}$ is nondegenerate. Moreover, if $v(q_0) = 0$ and $v(q_1) = 1$, then $\mathcal{D}_n(v) \geq c$; passing to the limit and taking the infimum gives $C_{\mathcal{D}}(q_0, q_1) \geq c$. Hence $\mathcal{D} \in \mathcal{K}_c$, and $\mathcal{K}_c$ is closed.

Relative compactness. For $\mathcal{D} \in \mathcal{K}_c$, set $\mathcal{N}_{\mathcal{D}} := \mathcal{D}^{1/p}$. By Lemma 5.4,

$$\mathcal{N}_{\mathcal{D}}(v) \leq 2 \operatorname{osc}_{V_0}(v), \qquad v \in \mathcal{L}(V_0).$$

Hence

$$|\mathcal{N}_{\mathcal{D}}(u) - \mathcal{N}_{\mathcal{D}}(v)| \leq \mathcal{N}_{\mathcal{D}}(u - v) \leq 2 \operatorname{osc}_{V_0}(u - v).$$

On the compact set $S := \{v \in \mathcal{H}_0 : \operatorname{osc}_{V_0}(v) = 1\}$, we have $\mathcal{N}_{\mathcal{D}} \leq 2$, and therefore

$$|\mathcal{D}(u) - \mathcal{D}(v)| \leq p 2^p \operatorname{osc}_{V_0}(u - v), \qquad u, v \in S.$$

Thus the restrictions of the forms in $\mathcal{K}_c$ to S are uniformly bounded and equicontinuous. The Arzelà–Ascoli theorem and Proposition 4.2(2) imply relative compactness in $\mathscr{B}_p(V_0)$. Together with closedness, this proves compactness. □

Proposition 5.6. *Fix p and r. Suppose that*

$$\mathcal{R}_{p,r}\mathcal{D} = \lambda\mathcal{D}, \qquad \mathcal{R}_{p,r}\mathcal{D}' = \lambda'\mathcal{D}',$$

where $\mathcal{D}, \mathcal{D}' \in \mathcal{M}_p(V_0)$. Then $\lambda = \lambda'$. Hence the eigenvalue depends only on p and r and may be denoted by $\lambda_p(r)$.

Proof. Let $\theta := \max_{v \notin \mathbb{R}\mathbf{1}} \frac{\mathcal{D}'(v)}{\mathcal{D}(v)}$. The quotient $\mathcal{L}(V_0)/\mathbb{R}\mathbf{1}$ is finite dimensional, so the maximum is attained. Choose $u \notin \mathbb{R}\mathbf{1}$ such that $\mathcal{D}'(u) = \theta\mathcal{D}(u)$, and let h be the $\Psi_{p,r}\mathcal{D}$-minimizing extension of u. Then

$$\lambda'\mathcal{D}'(u) \leq \Psi_{p,r}\mathcal{D}'(h) \leq \theta\Psi_{p,r}\mathcal{D}(h) = \theta\lambda\mathcal{D}(u).$$

Hence $\lambda' \leq \lambda$. Interchanging $\mathcal{D}$ and $\mathcal{D}'$ gives $\lambda \leq \lambda'$, and therefore $\lambda = \lambda'$. □

Theorem 5.7. *For every $p \in (1, \infty)$ and $r \in (\rho^{p-1}, 1)$, there exist $\mathcal{D}_{p,r} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ and $\lambda_p(r) > 0$ such that*

$$\mathcal{R}_{p,r}\mathcal{D}_{p,r} = \lambda_p(r)\mathcal{D}_{p,r}. \tag{5.9}$$

The eigenform may be normalized by $\mathcal{D}_{p,r}(u_*) = 1$.

Proof. Let $c := c_{p,r}$ be the constant from Proposition 5.3. To see that $\mathcal{K}_c$ is nonempty, take $\mathcal{D}_0(u) := \sum_{0 \leq i < j \leq 2} |u(q_i) - u(q_j)|^p$, which belongs to $\mathcal{M}_p^{\mathrm{sym}}(V_0)$. Then Proposition 5.3 gives $\widetilde{\mathcal{R}}_{p,r}\mathcal{D}_0 \in \mathcal{K}_c$. By Propositions 5.3 and 5.5 and Corollary 4.5, the normalized operator $\widetilde{\mathcal{R}}_{p,r}$ is a continuous self-map of the compact convex set $\mathcal{K}_c$. The Schauder fixed-point theorem therefore yields $\mathcal{D}_{p,r} \in \mathcal{K}_c$ such that $\widetilde{\mathcal{R}}_{p,r}\mathcal{D}_{p,r} = \mathcal{D}_{p,r}$. With $\lambda_p(r) := (\mathcal{R}_{p,r}\mathcal{D}_{p,r})(u_*)$, this fixed-point identity is equivalent to (5.9). □

Proposition 5.8. *For fixed p, the function $r \mapsto \lambda_p(r)$ is continuous and non-increasing on $(\rho^{p-1}, 1)$.*

Proof. Monotonicity. Let $r_1 \le r_2$, and choose eigenforms $\mathcal{D}_i$ such that

$$\mathcal{R}_{p,r_i}\mathcal{D}_i = \lambda_p(r_i)\mathcal{D}_i, \qquad i = 1, 2.$$

Set $\theta := \max_{v \notin \mathbb{R}\mathbf{1}} \frac{\mathcal{D}_2(v)}{\mathcal{D}_1(v)}$, and choose u attaining the maximum. Let h be the $\Psi_{p,r_1}\mathcal{D}_1$-minimizing extension of u. Since $r_1^{1-|w|} \ge r_2^{1-|w|}$ for every $w \in W_1$, we have

$$\lambda_p(r_2)\mathcal{D}_2(u) \le \Psi_{p,r_2}\mathcal{D}_2(h) \le \theta\Psi_{p,r_1}\mathcal{D}_1(h) = \theta\lambda_p(r_1)\mathcal{D}_1(u).$$

Since $\mathcal{D}_2(u) = \theta\mathcal{D}_1(u)$, it follows that $\lambda_p(r_2) \le \lambda_p(r_1)$.

Continuity. Let $r_n \to r$, and choose a compact interval I contained in $(\rho^{p-1}, 1)$ that contains r and all sufficiently large r_n. The explicit bound in Proposition 5.3 yields $c_0 > 0$ such that normalized eigenforms for all parameters in I may be chosen in the compact set $\mathcal{K}_{c_0}$. Let $\mathcal{D}_n$ be such eigenforms, normalized by $\mathcal{D}_n(u_*) = 1$.

Since $\mathcal{K}_{c_0}$ is compact by Proposition 5.5, every subsequence of $\{\mathcal{D}_n\}$ has a further subsequence $\{\mathcal{D}_{n_k}\}$ converging to some $\mathcal{D} \in \mathcal{K}_{c_0}$. By the joint continuity of $\mathcal{R}_p$,

$$\lambda_p(r_{n_k}) = (\mathcal{R}_{p,r_{n_k}}\mathcal{D}_{n_k})(u_*) \longrightarrow (\mathcal{R}_{p,r}\mathcal{D})(u_*) =: \lambda.$$

Passing to the limit in

$$\mathcal{R}_{p,r_{n_k}}\mathcal{D}_{n_k} = \lambda_p(r_{n_k})\mathcal{D}_{n_k}$$

gives $\mathcal{R}_{p,r}\mathcal{D} = \lambda\mathcal{D}$. By Proposition 5.6, $\lambda = \lambda_p(r)$. Thus every subsequence contains a further subsequence along which the eigenvalues converge to $\lambda_p(r)$, and consequently $\lambda_p(r_n) \to \lambda_p(r)$. □

5.2 Eigenvalue estimates and the standard factor

The adjective "standard" refers to compatibility between geometric scale and energy weight. The W_1-cells F_0G, $F_{10}G$, and $F_{20}G$ have contraction ratios ρ^2, ρ^3, ρ^3 and weights $\lambda_p(r), \lambda_p(r)r, \lambda_p(r)r$, respectively. Requiring these weights to match the contraction ratios leads to

$$\lambda_p(r) = r^2. \tag{5.10}$$

Equivalently, with $\theta = \log r / \log \rho$, every cell weight is its contraction ratio raised to the power θ. This is the compatibility condition used in the quadratic construction of [9, Section 5]. Thus (5.10) is precisely the scale-matching equation that selects the standard renormalization factor.

Fix $r \in (\rho^{p-1}, 1)$ and choose $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ such that $\mathcal{R}_{p,r}\mathcal{D} = \lambda_p(r)\mathcal{D}$. Write $R_0 := R_{\mathcal{D}}$ and $R_1 := R_{\Psi_{p,r}\mathcal{D}}$ for the effective p-resistances associated with $\mathcal{D}$ and $\Psi_{p,r}\mathcal{D}$, respectively. Let $A := \{0, 20, 110, 120, 210, 1120, 1210\} \subset W_1$ be the family of red cells in Figure 3. Set $V_A := \bigcup_{w\in A} F_w V_0$. For $f \in \mathbb{R}^{V_A}$, define $\mathcal{D}_A(f) := \sum_{w\in A} r^{1-|w|}\mathcal{D}(f \circ F_w|_{V_0})$. Since A is connected, Lemma 2.3 and Remark 2.2 imply that $(\mathcal{D}_A, \mathbb{R}^{V_A})$ is a p-resistance form. Let R_A denote its effective p-resistance.

Lemma 5.9. *Define*

$$B_p(r) := 2 + r^{1/(p-1)} + 4r^{2/(p-1)} + 2r^{3/(p-1)}$$

and

$$\beta_p(r) := \left[r^{-1} + B_p(r)^{1-p}\right]^{-1/(p-1)}. \tag{5.11}$$

Then $0 < \beta_p(r) < r^{1/(p-1)}$, *and*

$$\lambda_p(r) \ge \left[\frac{1}{1 - r^{1/(p-1)}} + \beta_p(r) - r^{1/(p-1)}\right]^{1-p} > \left(1 - r^{1/(p-1)}\right)^{p-1}. \tag{5.12}$$

Moreover,

$$\lambda_p(r) \le \frac{2r^{-1}}{\left(1 + (2r^{-1})^{1/(p-1)}\right)^{p-1}} < 1.$$

Proof. Upper bound. Let $u(q_0) = 0$ and $u(q_1) = u(q_2) = 1$. For $t \in [0,1]$, define $f_t \in \mathbb{R}^{V_1}$ by

$$f_t(q_0) = 0, \qquad f_t(F_1 q_0) = f_t(F_2 q_0) = t,$$

and $f_t = 1$ at all remaining vertices of V_1. Then

$$\Psi_{p,r}\mathcal{D}(f_t) = \big(t^p + 2r^{-1}(1-t)^p\big)\mathcal{D}(u).$$

Hence, by the variational characterization of the trace,

$$\lambda_p(r) \le \min_{0\le t\le 1}\big\{t^p + 2r^{-1}(1-t)^p\big\} = \frac{2r^{-1}}{\big(1 + (2r^{-1})^{1/(p-1)}\big)^{p-1}} < 1.$$

Lower bound. Set $s := r^{1/(p-1)}$, $a := F_1 q_0$, and $b := F_1^2 q_0$. The finite cell family A and the edge chain from b to a used below are shown in Figure 3.

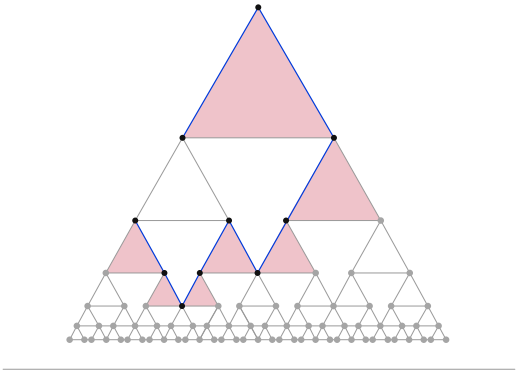


Figure 3: The finite cell family $A = \{0, 20, 110, 120, 210, 1120, 1210\}$ (red) and an edge chain (blue) joining $b = F_1^2 q_0$ to $a = F_1 q_0$.

For every f with $f(a) = 0$ and $f(b) = 1$,

$$\Psi_{p,r}\mathcal{D}(f) \ge r^{-1}\mathcal{D}(f \circ F_{10}|_{V_0}) + \mathcal{D}_A(f),$$

and, by the definition of R_0,

$$\mathcal{D}(f \circ F_{10}|_{V_0}) \ge R_0(q_0, q_1)^{-1}.$$

If $w \in A$, $|w| = k$, and $i \in \{1, 2\}$, then the definition of $\mathcal{D}_A$ and reflection symmetry imply

$$R_A(F_w q_0, F_w q_i)^{1/(p-1)} \le s^{k-1} R_0(q_0, q_1)^{1/(p-1)}.$$

The blue chain in Figure 3 uses $2, 1, 4, 2$ cell sides from cells of lengths $1, 2, 3, 4$, respectively. Hence the triangle inequality for $R_A^{1/(p-1)}$ gives

$$R_A(a, b)^{1/(p-1)} \le \big(2 + s + 4s^2 + 2s^3\big) R_0(q_0, q_1)^{1/(p-1)} = B_p(r) R_0(q_0, q_1)^{1/(p-1)}.$$

Consequently,

$$\mathcal{D}_A(f) \ge B_p(r)^{1-p} R_0(q_0, q_1)^{-1}.$$

Taking the infimum over all such f yields

$$R_1(a, b)^{1/(p-1)} \le \beta_p(r) R_0(q_0, q_1)^{1/(p-1)}. \tag{5.13}$$

In particular, $0 < \beta_p(r) < r^{1/(p-1)} = s$.

Similarly, for every $n \geq 0$,

$$R_1(F_1^n q_0, F_1^{n+1} q_0)^{1/(p-1)} \leq s^n R_0(q_0, q_1)^{1/(p-1)},$$

where (5.13) gives the sharper factor $\beta_p(r)$ in place of s when $n = 1$. Summing these estimates along the chain $q_0, F_1 q_0, F_1^2 q_0, \ldots \longrightarrow q_1$ gives

$$R_1(q_0, q_1)^{1/(p-1)} \leq R_0(q_0, q_1)^{1/(p-1)} \left(\frac{1}{1-s} + \beta_p(r) - s \right). \tag{5.14}$$

The eigenvalue equation and trace compatibility give

$$R_1(q_0, q_1) = \lambda_p(r)^{-1} R_0(q_0, q_1).$$

Substitution into (5.14) yields

$$\lambda_p(r) \geq \left(\frac{1}{1-s} + \beta_p(r) - s \right)^{1-p}.$$

Since $\beta_p(r) < s$ and $1 - p < 0$, this lower bound is strictly larger than $(1-s)^{p-1}$. □

Theorem 5.10. *For every $p \in (1, \infty)$, there exists a unique $r_p \in (\rho^{p-1}, 1)$ such that*

$$\lambda_p(r_p) = r_p^2. \tag{5.15}$$

Consequently, there exists $\mathcal{D}_p \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ satisfying $\mathcal{R}_{p,r_p} \mathcal{D}_p = r_p^2 \mathcal{D}_p$.

Proof. Let $r_0 = \rho^{p-1}$. By (5.11), β_p extends continuously to r_0 and $\beta_p(r_0) < \rho$. Letting $r \downarrow r_0$ in (5.12) gives

$$\liminf_{r \downarrow r_0} \lambda_p(r) \geq \left[\frac{1}{1-\rho} + \beta_p(r_0) - \rho \right]^{1-p} > (1-\rho)^{p-1} = \rho^{2(p-1)} = r_0^2.$$

Hence $\lambda_p(r) > r^2$ for all $r > r_0$ sufficiently close to r_0. On the other hand,

$$\limsup_{r \uparrow 1} \lambda_p(r) \leq \frac{2}{(1 + 2^{1/(p-1)})^{p-1}} < 1 = \lim_{r \uparrow 1} r^2.$$

By the continuity assertion in Proposition 5.8, the functions $\lambda_p(r)$ and r^2 intersect at some $r_p \in (r_0, 1)$. Since λ_p is non-increasing whereas $r \mapsto r^2$ is strictly increasing, this intersection is unique.

Finally, applying Theorem 5.7 with $r = r_p$ gives $\mathcal{D}_p \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ satisfying $\mathcal{R}_{p,r_p} \mathcal{D}_p = \lambda_p(r_p) \mathcal{D}_p = r_p^2 \mathcal{D}_p$. □

The identity (5.15) will be imposed in the global construction to make energy weights depend only on geometric scale.

6 Global p-resistance forms

Following Kigami's compatible-energy construction on p.c.f. self-similar sets [20] and the quadratic construction of Cao and the second author for the golden ratio Sierpiński gasket G [9] , we define compatible energies on the finite-level vertex sets V_m and then pass to their limit to obtain the global form on G .

Fix $p \in (1, \infty)$ and $r \in (\rho^{p-1}, 1)$. By Theorem 5.7 , choose $\mathcal{D} \in \mathcal{M}_p(V_0)$ and $\lambda > 0$ such that

$$\mathcal{R}_{p,r} \mathcal{D} = \lambda \mathcal{D}. \tag{6.1}$$

By Proposition 5.6, necessarily $\lambda = \lambda_p(r)$, and Lemma 5.9 gives $0 < \lambda < 1$.

Set

$$\gamma_0 := \lambda, \qquad \gamma_1 = \gamma_2 := r, \qquad \gamma_w := \prod_{j=1}^{|w|} \gamma_{w_j} \quad (w \in \widetilde{W}_*), \tag{6.2}$$

with $\gamma_\varnothing := 1$. Moreover,

$$\gamma_{w\tau} = \gamma_w \gamma_\tau, \qquad w, \tau \in \widetilde{W}_*.$$

If $w \in W_1$, then

$$\gamma_w = \lambda r^{|w|-1}. \tag{6.3}$$

More generally, if $w \in W_m$, then $\gamma_w = \lambda^m r^{|w|-m}$.

6.1 Compatible level forms

Definition 6.1 (Compatible level forms)**.** For $m \geq 0$, define

$$\mathcal{D}^{(m)}(f) := \sum_{w \in W_m} \gamma_w^{-1} \mathcal{D}\big(f \circ F_w|_{V_0}\big), \tag{6.4}$$

and

$$\mathcal{F}^{(m)} := \Big\{ f \in C(\overline{V}_m) : \mathcal{D}^{(m)}(f) < \infty \Big\}.$$

For $m = 0$, we set $\mathcal{D}^{(0)} = \mathcal{D}$ and $\mathcal{F}^{(0)} = \mathbb{R}^{V_0}$.

For $m = 1$, (6.3) gives $\mathcal{D}^{(1)} = \lambda^{-1} \Psi_{p,r} \mathcal{D}$. Hence, Theorem 5.7 gives

$$\mathcal{D}^{(1)}\Big|_{V_0} = \lambda^{-1} \mathcal{R}_{p,r} \mathcal{D} = \mathcal{D}. \tag{6.5}$$

Theorem 6.2. *For every $m \geq 0$, $\big(\mathcal{D}^{(m)}, \mathcal{F}^{(m)}\big)$ is a regular p-resistance form on $\overline{V}_m$. Let*

$$R_m(x,y) := \sup \left\{ \frac{|f(x) - f(y)|^p}{\mathcal{D}^{(m)}(f)} : f \in \mathcal{F}^{(m)} \setminus \mathbb{R}\mathbf{1}_{\overline{V}_m} \right\}$$

denote its effective p-resistance. Moreover,

$$\mathcal{D}^{(m+1)}\Big|_{\overline{V}_m} = \mathcal{D}^{(m)}. \tag{6.6}$$

Consequently, for $n \geq m$ and $x, y \in \overline{V}_m$,

$$R_n(x,y) = R_m(x,y). \tag{6.7}$$

Proof. We prove by induction on $m \geq 1$ the following statement:

$$(\mathrm{P})_m : \quad \begin{aligned} &\big(\mathcal{D}^{(m)}, \mathcal{F}^{(m)}\big) \text{ is a regular } p\text{-resistance form on } \overline{V}_m, \\ &\mathcal{D}^{(m)}\Big|_{\overline{V}_{m-1}} = \mathcal{D}^{(m-1)}. \end{aligned}$$

For $m = 1$, Proposition 3.6 , together with (6.3), shows that $\big(\mathcal{D}^{(1)}, \mathcal{F}^{(1)}\big)$ is a p-resistance form on $\overline{V}_1$. Its regularity follows from Proposition 4.3 (3), and the trace identity in $(\mathrm{P})_1$ is (6.5).

Fix $m \geq 1$ and assume $(\mathrm{P})_m$. We prove $(\mathrm{P})_{m+1}$. Since $W_{m+1} = W_m W_1$ and $\gamma_{w\tau} = \gamma_w \gamma_\tau$,

$$\mathcal{D}^{(m+1)}(f) = \sum_{w \in W_m} \gamma_w^{-1} \mathcal{D}^{(1)}\big((f \circ F_w)|_{\overline{V}_1}\big). \tag{6.8}$$

The level-one trace identity (6.5) and the trace inequality in Theorem 2.8 yield

$$\mathcal{D}^{(m+1)}(f) \geq \mathcal{D}^{(m)}(f|_{\overline{V}_m}). \tag{6.9}$$

We now verify that $(\mathcal{D}^{(m+1)}, \mathcal{F}^{(m+1)})$ is a p-resistance form. By Lemma 2.3 , every finite partial sum in (6.8) has the seminorm and generalized contraction properties; these pass to $\mathcal{D}^{(m+1)}$ by monotone convergence. If $\mathcal{D}^{(m+1)}(f) = 0$, then (6.9) and $(\mathrm{P})_m$ show that f is constant on $\overline{V}_m$. Since the summands in (6.8) are nonnegative, each of them vanishes. The base case $(\mathrm{P})_1$ then makes f constant, with the same value, on every $F_w\overline{V}_1$, $w \in W_m$. Thus $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ hold.

We next verify $(\mathrm{RF2})_p$. Fix $o \in V_0$, and let $f_k(o) = 0$ with

$$\mathcal{D}^{(m+1)}(f_k - f_\ell) \longrightarrow 0.$$

By (6.9) and the completeness of the $\mathcal{D}^{(m)}$-energy norm , $f_k|_{\overline{V}_m}$ converges in the energy norm to some $\widetilde{f} \in \mathcal{F}^{(m)}$, normalized by $\widetilde{f}(o) = 0$; applying (2.2) to $\mathcal{D}^{(m)}$ then gives pointwise convergence of f_k to $\widetilde{f}$. For every $w \in W_m$,

$$\mathcal{D}^{(1)}\big(((f_k - f_\ell) \circ F_w)|_{\overline{V}_1}\big) \leq \gamma_w \mathcal{D}^{(m+1)}(f_k - f_\ell),$$

so $f_k \circ F_w$ is Cauchy in the level-one energy seminorm. Its restrictions to V_0 converge, through the convergence on $F_wV_0 \subset \overline{V}_m$, to $\widetilde{f} \circ F_w|_{V_0}$. Completeness of the level-one form therefore gives a function $\widetilde{f}_w \in \mathcal{F}^{(1)}$ to which $f_k \circ F_w$ converges, with $\widetilde{f}_w|_{V_0} = \widetilde{f} \circ F_w|_{V_0}$. Hence the functions $\widetilde{f}_w$ glue with $\widetilde{f}$ to a function f.

Now we prove $f \in C(\overline{V}_{m+1})$. Define

$$V'_{m+1} := \bigcup_{w \in W_m} F_w([0,1] \times \{0\}) \subset \overline{V}_{m+1},$$

which gives (non-disjoint) decomposition $\overline{V}_{m+1} = V_{m+1} \cup V'_{m+1}$ and satisfies the iteration relation

$$V'_{m+1} = \Big(\bigcup_{w \in W_m} F_w(V'_1) \Big) \cup V'_m. \tag{6.10}$$

Moreover, this gives

$$\overline{V}_{m+1} = V_{m+1} \cup V'_{m+1} = \Big(\bigcup_{w \in W_m} F_w(\overline{V}_1) \Big) \cup V'_m. \tag{6.11}$$

Clearly f is continuous at each $x \in V_{m+1} \setminus V'_{m+1}$ since such x is isolated in $\overline{V}_{m+1}$. Moreover, if $x \in F_w(V'_1)$ for some $w \in W_m$, then there is $r > 0$ such that $B(x,r) \cap \overline{V}_{m+1} \subset B(x,r) \cap F_w(\overline{V}_1)$, thus the continuity of f at x follows from the continuity of $\tilde{f}_w$ on $F_w(\overline{V}_1)$. By (6.10), it remains to prove that f is continuous at each $x \in V'_m$.

Let $x \in V'_m$. For each $r > 0$ and $y \in (B(x,r) \setminus \{x\}) \cap \overline{V}_{m+1}$, there are only two cases by (6.11):

(1) $y \in V'_m$, in which case $|f(x) - f(y)| \leq \mathrm{osc}_{\overline{V}_m \cap B(x,r)}(\tilde{f})$.

(2) $y \in F_w(\overline{V}_1)$ for some $w \in W_m$, in which case

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(F_w(q_0))| + |f(F_w(q_0)) - f(y)| \\ &\leq \mathrm{osc}_{\overline{V}_m \cap B\left(x, r + \mathrm{diam}(F_w(\overline{V}_1))\right)}(\tilde{f}) + \mathrm{osc}_{F_w(\overline{V}_1)}(\tilde{f}_w) \\ &\leq \mathrm{osc}_{\overline{V}_m \cap B(x, Ar)}(\tilde{f}) + \mathrm{osc}_{F_w(\overline{V}_1)}(\tilde{f}_w) \end{aligned}$$

for some constant $A \geq 1$. Here the third line follows from a geometric observation similar to Whitney decomposition that, for each $x \in V_m'$, $(B(x,r) \setminus \{x\}) \cap F_w(\overline{V}_1) \neq \emptyset$ implies $\operatorname{diam}(F_w(\overline{V}_1)) \leq A'r$ for some constant $A' \geq 1$. To bound those correlated $\operatorname{osc}_{F_w(\overline{V}_1)}(\tilde{f}_w)$ when r is small, let $M = \sup_k \mathcal{D}^{(m+1)}(f_k) < \infty$. Fatou's lemma and Proposition 3.6 give a constant $C_1 > 0$ such that

$$\operatorname{osc}_{F_w\overline{V}_1}(f)^p \leq C_1 M \gamma_w. \tag{6.12}$$

For every $a > 0$, only finitely many $w \in W_m$ satisfy $\gamma_w \geq a$. Note that for every $w \in W_m$, there exists $r > 0$ such that $B(x,r) \cap F_w(\overline{V}_1) = \emptyset$. Therefore we can pick $r > 0$ small enough such that $\operatorname{osc}_{F_w(\overline{V}_1)}(\tilde{f}_w) < a$ for all $w \in W_m$ satisfying $B(x,r) \cap F_w(\overline{V}_1) \neq \emptyset$ by the finiteness before. This further gives

$$|f(x) - f(y)| \leq \operatorname{osc}_{\overline{V}_m \cap B(x,Ar)}(\tilde{f}) + a$$

in the second case for every $r > 0$ small enough. Since $\tilde{f}$ is continuous on $\overline{V}_m$, combining the two cases, we reach to

$$\lim_{r\to 0} \operatorname{osc}_{\overline{V}_{m+1}\cap B(x,r)}(f) \leq \lim_{r\to 0} \operatorname{osc}_{\overline{V}_m \cap B(x,Ar)}(\tilde{f}) + a = a.$$

By letting $a \to 0$, it follows the claim that f is continuous at every $x \in V_m'$. At this stage, we have proved the desired $f \in C(\overline{V}_{m+1})$.

Another application of Fatou's lemma gives

$$\mathcal{D}^{(m+1)}(f_k - f) \leq \liminf_{\ell\to\infty} \mathcal{D}^{(m+1)}(f_k - f_\ell),$$

so $f_k \to f$ in the energy norm. This proves $(\mathrm{RF2})_p$.

We now verify $(\mathrm{RF3})_p$. Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be linear. For $\tau \in W_{1,n}$, the similarity ratio of F_τ is ρ^{n+1}. Hence the p-homogeneity and constant invariance of $\mathcal{D}$ give , for every $w \in W_m$,

$$\mathcal{D}\big((\ell \circ F_w \circ F_\tau)|_{V_0}\big) = \rho^{p(n+1)} \mathcal{D}\big((\ell \circ F_w)|_{V_0}\big),$$

and therefore

$$\mathcal{D}^{(1)}\big((\ell \circ F_w)|_{\overline{V}_1}\big) = \Lambda_{p,r,\lambda} \mathcal{D}\big((\ell \circ F_w)|_{V_0}\big),$$

where

$$\Lambda_{p,r,\lambda} := \lambda^{-1} \sum_{n=1}^{\infty} \#W_{1,n}\, r^{1-n} \rho^{p(n+1)} \leq C_2 \sum_{n=1}^{\infty} \left(\frac{\rho^{p-1}}{r}\right)^n < \infty \quad \big(\#W_{1,n} \asymp \rho^{-n},\ r > \rho^{p-1}\big).$$

The same computation with $F_w = \mathrm{id}$ gives $\mathcal{D}^{(1)}(\ell|_{\overline{V}_1}) = \Lambda_{p,r,\lambda} \mathcal{D}(\ell|_{V_0}) < \infty$, so $\ell|_{\overline{V}_1} \in \mathcal{F}^{(1)}$. Assume inductively that $\ell|_{\overline{V}_m} \in \mathcal{F}^{(m)}$. Summing the preceding identity over $w \in W_m$ and using (6.8), we obtain

$$\mathcal{D}^{(m+1)}(\ell|_{\overline{V}_{m+1}}) = \Lambda_{p,r,\lambda} \mathcal{D}^{(m)}(\ell|_{\overline{V}_m}) < \infty.$$

Thus $\ell|_{\overline{V}_{m+1}} \in \mathcal{F}^{(m+1)}$. Given $x \neq y$, choose ℓ with $\ell(x) \neq \ell(y)$. This proves $(\mathrm{RF3})_p$.

It remains to verify $(\mathrm{RF4})_p$. Let $x, y \in \overline{V}_{m+1}$ and $f \in \mathcal{F}^{(m+1)}$. If $x \in \overline{V}_m$, let $x' = x$; otherwise, choose $w \in W_m$ and $x' \in F_w V_0$ such that $x, x' \in F_w \overline{V}_1$. Then

$$|f(x) - f(x')|^p \leq R_1(F_w^{-1}x, F_w^{-1}x') \mathcal{D}^{(1)}(f \circ F_w) \leq C_3 \gamma_w \mathcal{D}^{(m+1)}(f),$$

where $C_3 := \sup_{a,b\in\overline{V}_1} R_1(a,b) < \infty$. Define y' in the same manner.

By the induction hypothesis $(\mathrm{P})_m$, $R_m(x', y') < \infty$, and (6.9) gives

$$|f(x') - f(y')|^p \leq R_m(x', y') \mathcal{D}^{(m+1)}(f).$$

Together with the analogous estimate for $|f(y)-f(y')|^p$, the triangle inequality yields

$$|f(x)-f(y)| \le C(x,y)\mathcal{D}^{(m+1)}(f)^{1/p}$$

for some $C(x,y)<\infty$. Thus $(\mathrm{RF4})_p$ follows.

Furthermore, by Proposition 2.4 (2), $\mathcal{F}^{(m+1)}$ is a subalgebra containing the constants and separating points ; hence the Stone–Weierstrass theorem shows that $(\mathcal{D}^{(m+1)},\mathcal{F}^{(m+1)})$ is a regular p-resistance form .

It remains to prove the trace identity in $(\mathrm{P})_{m+1}$. For $f\in\mathcal{F}^{(m)}$, use Theorem 2.8 to take the $\mathcal{D}^{(1)}$-minimizing extension in every level-m cell and glue these extensions to a function Hf. The cell boundaries agree by construction. Moreover, the resistance estimate in Proposition 3.6 gives a constant $C_4>0$ such that

$$\operatorname{osc}_{F_w\overline{V}_1}(Hf)^p \le C_4\mathcal{D}(f\circ F_w|_{V_0}) \le C_4\gamma_w\mathcal{D}^{(m)}(f), \qquad w\in W_m.$$

Since $\gamma_w\to 0$ outside finite subfamilies of W_m, the function Hf is continuous at the accumulation points. Its cellwise energy sum is finite, so $Hf\in\mathcal{F}^{(m+1)}$, and

$$\mathcal{D}^{(m+1)}(Hf)=\sum_{w\in W_m}\gamma_w^{-1}\mathcal{D}(f\circ F_w|_{V_0})=\mathcal{D}^{(m)}(f). \tag{6.13}$$

Combining (6.13) with (6.9) proves

$$\mathcal{D}^{(m+1)}\Big|_{\overline{V}_m}=\mathcal{D}^{(m)}.$$

Thus $(\mathrm{P})_{m+1}$ holds. Together with $\mathcal{D}\in\mathcal{M}_p(V_0)$, the induction proves the regularity assertion and (6.6) for every $m\ge 0$. Finally, repeated use of (6.6) and the preservation of effective resistance under traces in Theorem 2.8 yield (6.7). □

Let $V_*:=\bigcup_{m=0}^{\infty}\overline{V}_m$. By (6.6),

$$\mathcal{D}^{(m+1)}(f|_{\overline{V}_{m+1}})\ge\mathcal{D}^{(m)}(f|_{\overline{V}_m})$$

for every function f on V_*. By (6.7), if $x,y\in V_*$ and $x,y\in\overline{V}_m\subset\overline{V}_n$, then $R_m(x,y)=R_n(x,y)$. Thus, for $x,y\in V_*$, we may choose any m such that $x,y\in\overline{V}_m$ and define

$$\widetilde{R}(x,y):=R_m(x,y).$$

We next establish a uniform resistance estimate on V_*, which will be used to pass to the global limit. Recall that $\widetilde{W}_*$ is the set of finite words for the original iterated function system $\{F_0,F_1,F_2\}$. For the resistance estimates below, set

$$\theta_{p,r}:=\min\left\{\frac{\log r}{\log\rho},\frac{\log\lambda}{2\log\rho}\right\}>0. \tag{6.14}$$

By (6.14),

$$r\le\rho^{\theta_{p,r}}, \qquad \lambda\le\rho^{2\theta_{p,r}}.$$

In (6.2), the letter 0 has weight λ and contraction ratio ρ^2, whereas the letters 1 and 2 have weight r and contraction ratio ρ. Multiplying these bounds along a word gives

$$\gamma_w\le\rho_w^{\theta_{p,r}}, \qquad w\in\widetilde{W}_*. \tag{6.15}$$

Lemma 6.3. *For $w\in\widetilde{W}_*$, let $\nu(w):=\#\{1\le j\le|w|:w_j=0\}$. Then, for every $f\in\mathcal{F}^{(\nu(w)+1)}$,*

$$\mathcal{D}^{(1)}\big((f\circ F_w)|_{\overline{V}_1}\big)\le\gamma_w\mathcal{D}^{(\nu(w)+1)}(f). \tag{6.16}$$

Consequently, there exists $C>0$ such that, for all $x,y\in F_w\overline{V}_1$,

$$R_{\nu(w)+1}(x,y)\le Cd(x,y)^{\theta_{p,r}}. \tag{6.17}$$

Proof. For each $\tau \in W_1$, split $w\tau$ immediately after each occurrence of 0 and replace every resulting block by its representative in W_1. This produces a word

$$v(\tau) \in W_{\nu(w)+1} \qquad \text{such that} \qquad F_{v(\tau)} = F_{w\tau}.$$

Since $\nu(v(\tau)) = \nu(w\tau)$ and $\rho_u = \rho^{|u|+\nu(u)}$ for every word u, the identity $F_{v(\tau)} = F_{w\tau}$ implies $|v(\tau)| = |w\tau|$. Consequently,

$$\gamma_{v(\tau)} = \gamma_{w\tau} = \gamma_w \gamma_\tau.$$

Moreover, the words $v(\tau)$ are distinct as τ varies, since F_w is injective and W_1 is a representative word set. Therefore,

$$\begin{aligned}\mathcal{D}^{(1)}\big((f \circ F_w)|_{\overline{V}_1}\big) &= \sum_{\tau \in W_1} \gamma_\tau^{-1} \mathcal{D}(f \circ F_{w\tau}|_{V_0}) \\ &= \gamma_w \sum_{\tau \in W_1} \gamma_{v(\tau)}^{-1} \mathcal{D}(f \circ F_{v(\tau)}|_{V_0}) \\ &\le \gamma_w \mathcal{D}^{(\nu(w)+1)}(f),\end{aligned}$$

which proves (6.16).

Now set $x' := F_w^{-1}x$ and $y' := F_w^{-1}y$. Since $\mathcal{D}^{(1)} = \lambda^{-1}\Psi_{p,r}\mathcal{D}$, the corresponding effective resistances satisfy $R_1(x', y') = \lambda R_{p,r,\mathcal{D}}(x', y')$. Hence, by (3.5),

$$R_1(x', y') \le C_1 d(x', y')^{\frac{\log r}{\log \rho}}.$$

Taking the supremum in the definition of effective resistance and using (6.16), we obtain

$$R_{\nu(w)+1}(x, y) \le \gamma_w R_1(x', y') \le C_1 \gamma_w d(x', y')^{\frac{\log r}{\log \rho}}.$$

Finally, since $d(x', y') \le 1$, $\theta_{p,r} \le \frac{\log r}{\log \rho}$, and $\gamma_w \le \rho_w^{\theta_{p,r}}$ by (6.15),

$$\gamma_w d(x', y')^{\frac{\log r}{\log \rho}} \le \rho_w^{\theta_{p,r}} d(x', y')^{\theta_{p,r}} = d(x, y)^{\theta_{p,r}},$$

which proves (6.17).

□

Proposition 6.4. *There exists $C > 0$ such that*

$$\widetilde{R}(x, y) \le C d(x, y)^{\theta_{p,r}}, \qquad x, y \in V_*. \tag{6.18}$$

Proof. We first establish the following uniform estimate for the resistance diameter of each cell:

$$\widetilde{R}(x, y) \le C_1 \rho_w^{\theta_{p,r}}, \qquad x, y \in F_w G \cap V_*, \quad w \in \widetilde{W}_*. \tag{6.19}$$

Suppose that $y \in F_w \overline{V}_1$ and $x \in F_w \overline{V}_n$. Along the nested cell chain from $F_w \overline{V}_1$ to x, choose words $v_0, \dots, v_{n-1} \in \widetilde{W}_*$ and points

$$y = x_0, x_1, \dots, x_n = x$$

such that

$$x_j, x_{j+1} \in F_{wv_j} \overline{V}_1, \qquad \rho_{v_j} \le \rho^j, \qquad 0 \le j < n.$$

By Lemma 6.3,

$$\widetilde{R}(x_j, x_{j+1}) \le C_2 \rho_{wv_j}^{\theta_{p,r}} \le C_2 (\rho_w \rho^j)^{\theta_{p,r}}.$$

Therefore,

$$\widetilde{R}(x,y)^{1/(p-1)} \le C_2 \rho_w^{\theta_{p,r}/(p-1)} \sum_{j=0}^{\infty} \rho^{j\theta_{p,r}/(p-1)} \le C_3 \rho_w^{\theta_{p,r}/(p-1)}.$$

A general pair $x, y \in F_w G \cap V_*$ can be joined through a point $z \in F_w \overline{V}_1$, proving (6.19).

Now let $x \neq y$ in V_*. Choose $w, w' \in \widetilde{W}_*$ such that

$$x \in F_w G, \qquad y \in F_{w'} G, \qquad \rho d(x,y) \le \rho_w, \rho_{w'} < \rho^{-1} d(x,y).$$

By Proposition 3.3, there is a constant $K \in \mathbb{N}$, depending only on G, and a chain of words in $\widetilde{W}_*$

$$w = w^{(0)}, w^{(1)}, \dots, w^{(k)} = w', \qquad k \le K,$$

satisfying

$$\min\{\rho_w, \rho_{w'}\} \le \rho_{w^{(i)}} < \rho^{-2} \min\{\rho_w, \rho_{w'}\}, \qquad 0 \le i \le k,$$
$$F_{w^{(i-1)}} \overline{V}_1 \cap F_{w^{(i)}} \overline{V}_1 \neq \varnothing, \qquad 1 \le i \le k.$$

For each $1 \le i \le k$, choose a point in this intersection,

$$z_i \in F_{w^{(i-1)}} \overline{V}_1 \cap F_{w^{(i)}} \overline{V}_1, \qquad 1 \le i \le k,$$

and set $z_0 = x$ and $z_{k+1} = y$. By (6.19),

$$\widetilde{R}(z_i, z_{i+1}) \le C_1 \rho_{w^{(i)}}^{\theta_{p,r}} \le C_4 d(x,y)^{\theta_{p,r}}, \qquad 0 \le i \le k.$$

Since k is uniformly bounded, the triangle inequality for $\widetilde{R}^{1/(p-1)}$ yields

$$\widetilde{R}(x,y)^{1/(p-1)} \le C_5 d(x,y)^{\theta_{p,r}/(p-1)}.$$

Raising to the power $p-1$ proves (6.18). □

6.2 Global construction and self-similarity

Having constructed compatible finite-level forms, we now pass to their global limit.

Definition 6.5 (Global form)**.** For every $f \in C(G)$, the sequence $\{\mathcal{D}^{(m)}(f|_{\overline{V}_m})\}_{m \ge 0}$ is nondecreasing by (6.9). Define

$$\mathcal{F}_{p,r,\mathcal{D}} := \left\{ f \in C(G) : \lim_{m \to \infty} \mathcal{D}^{(m)}(f|_{\overline{V}_m}) < \infty \right\}, \tag{6.20}$$

and, for $f \in \mathcal{F}_{p,r,\mathcal{D}}$, set

$$\mathcal{E}_{p,r,\mathcal{D}}(f) := \lim_{m \to \infty} \mathcal{D}^{(m)}(f|_{\overline{V}_m}). \tag{6.21}$$

For the remainder of this subsection, write $(\mathcal{E}, \mathcal{F}) := (\mathcal{E}_{p,r,\mathcal{D}}, \mathcal{F}_{p,r,\mathcal{D}})$.

Proposition 6.6. *The domain $\mathcal{F}$ is dense in $C(G)$ with respect to the uniformly convergence topology.*

Proof. Let $f \in C(G)$ and $\varepsilon > 0$. Choose m so large that the oscillation of f on every level-m cell is less than ε. By Theorem 6.2, there exists $g \in \mathcal{F}^{(m)}$ such that $\|g - f|_{\overline{V}_m}\|_\infty < \varepsilon$. Extend g successively using the minimizing p-harmonic extensions. Equation (6.6) keeps every finite-level energy equal to $\mathcal{D}^{(m)}(g)$. Let $\widetilde{g} : V_* \to \mathbb{R}$ denote the resulting function. By Proposition 6.4, there is $C_1 > 0$ such that

$$|\widetilde{g}(x) - \widetilde{g}(y)|^p \le C_1 \mathcal{D}^{(m)}(g) d(x,y)^{\theta_{p,r}}, \qquad x, y \in V_*,$$

so $\widetilde{g}$ extends uniquely to a continuous function on G, again denoted by $\widetilde{g}$. The finite-level energy identity then gives $\widetilde{g} \in \mathcal{F}$. The weak comparison principle Proposition 2.9 on each level-m cell, together with this approximation, yields $\|\widetilde{g} - f\|_\infty \le 2\varepsilon$. Thus $\mathcal{F}$ is dense in $C(G)$. □

Theorem 6.7. *The pair $(\mathcal{E},\mathcal{F})$ is a regular p-resistance form on G. For every $m \geq 0$,*

$$\mathcal{E}|_{\overline{V}_m} = \mathcal{D}^{(m)}. \tag{6.22}$$

Moreover, there exists $C > 0$ such that

$$R_{\mathcal{E}}(x,y) \leq C\, d(x,y)^{\theta_{p,r}}, \qquad x,y \in G. \tag{6.23}$$

Proof. Fix $C_1 > 0$ such that the estimate in Proposition 6.4 reads $\widetilde{R}(x,y) \leq C_1 d(x,y)^{\theta_{p,r}}$ for $x,y \in V_*$. For $f \in \mathcal{F}$, the sequence $\mathcal{D}^{(m)}(f|_{\overline{V}_m})$ increases to $\mathcal{E}(f)$. Since each $(\mathcal{D}^{(m)})^{1/p}$ is a seminorm, $\mathcal{F}$ is a linear space containing the constants, and homogeneity and the triangle inequality pass to the limit. If $\mathcal{E}(f) = 0$, then f is constant on every $\overline{V}_m$, hence on the dense set V_* and therefore on G. This proves $(\mathrm{RF1})_p$. Applying the generalized contraction property in Theorem 6.2 at each finite level and letting $m \to \infty$ proves $(\mathrm{RF5})_p$. Moreover, Proposition 6.6 shows that $\mathcal{F}$ is dense in $C(G)$, which implies that $\mathcal{F}$ separates points, proving $(\mathrm{RF3})_p$.

We next prove completeness $(\mathrm{RF2})_p$. Let $\{f_n\} \subset \mathcal{F}$ be Cauchy in the energy seminorm. After adding constants, assume that $f_n(q_0) = 0$ for every n. For $x \in V_*$, choose m such that $x \in \overline{V}_m$. Then

$$|f_n(x) - f_k(x)|^p \leq R_m(x,q_0)\mathcal{D}^{(m)}\big((f_n - f_k)|_{\overline{V}_m}\big) \leq R_m(x,q_0)\mathcal{E}(f_n - f_k),$$

so $\{f_n(x)\}$ is Cauchy. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for $x \in V_*$. Since $\{f_n\}$ is energy Cauchy, $C_2 := \sup_{n\geq 1} \mathcal{E}(f_n) < \infty$. For $x,y \in V_*$, choose m such that $x,y \in \overline{V}_m$. By Proposition 6.4,

$$|f(x) - f(y)|^p = \lim_{n\to\infty} |f_n(x) - f_n(y)|^p \leq \widetilde{R}(x,y)C_2 \leq C_1 C_2 d(x,y)^{\theta_{p,r}},$$

Hence f is uniformly continuous on V_* and extends uniquely to a continuous function on G, still denoted by f.

For each fixed m, the sequence $f_n|_{\overline{V}_m}$ is Cauchy in the $\mathcal{D}^{(m)}$-seminorm. Completeness in Theorem 6.2 and pointwise convergence therefore give $f|_{\overline{V}_m} \in \mathcal{F}^{(m)}$.

Let $\varepsilon > 0$ and choose N such that $\mathcal{E}(f_k - f_n) \leq \varepsilon$ whenever $k,n \geq N$. For $n \geq N$ and every m, Proposition 2.12(1) gives

$$\mathcal{D}^{(m)}\big((f - f_n)|_{\overline{V}_m}\big) \leq \liminf_{k\to\infty} \mathcal{D}^{(m)}\big((f_k - f_n)|_{\overline{V}_m}\big) \leq \varepsilon.$$

Letting $m \to \infty$ yields $\mathcal{E}(f - f_n) \leq \varepsilon$. Thus $f - f_n \in \mathcal{F}$ for every $n \geq N$; since $f_n \in \mathcal{F}$, it follows that $f \in \mathcal{F}$. Moreover, $f_n \to f$ in the energy seminorm, which proves $(\mathrm{RF2})_p$.

We now prove $(\mathrm{RF4})_p$ together with (6.23). Let $x,y \in G$, and choose $x_n, y_n \in V_*$ such that $x_n \to x$ and $y_n \to y$. For each n, choose m_n with $x_n, y_n \in \overline{V}_{m_n}$. Then, for every $f \in \mathcal{F}$,

$$\begin{aligned} |f(x_n) - f(y_n)|^p &\leq R_{m_n}(x_n,y_n)\mathcal{D}^{(m_n)}(f|_{\overline{V}_{m_n}}) \\ &\leq \widetilde{R}(x_n,y_n)\mathcal{E}(f) \\ &\leq C_1 \mathcal{E}(f) d(x_n,y_n)^{\theta_{p,r}}. \end{aligned}$$

Letting $n \to \infty$ gives

$$|f(x) - f(y)|^p \leq C_1 \mathcal{E}(f) d(x,y)^{\theta_{p,r}}.$$

Taking the supremum over nonconstant $f \in \mathcal{F}$ proves (6.23), and hence $(\mathrm{RF4})_p$. Moreover, Proposition 6.6 gives regularity.

It remains to identify the finite-level traces. If $f \in \mathcal{F}$ and $f|_{\overline{V}_m} = g$, then

$$\mathcal{D}^{(m)}(g) \leq \mathcal{E}(f),$$

so $\mathcal{E}|_{\overline{V}_m}(g) \geq \mathcal{D}^{(m)}(g)$. Conversely, for $g \in \mathcal{F}^{(m)}$, successively apply the minimizing p-harmonic extension at every higher level. This produces a function $\widetilde{g}$ on V_* such that

$$\mathcal{D}^{(n)}(\widetilde{g}|_{\overline{V}_n}) = \mathcal{D}^{(m)}(g), \qquad n \geq m.$$

By Proposition 6.4, $\widetilde{g}$ extends continuously to G; the extension belongs to $\mathcal{F}$ and has energy $\mathcal{D}^{(m)}(g)$. Therefore $\mathcal{E}|_{\overline{V}_m}(g) \leq \mathcal{D}^{(m)}(g)$, which proves (6.22). □

We next establish the compatibility of the finite-level and global two-variable energies. This will yield the two-variable self-similarity identity later in Theorem 6.9, which also establishes the energy self-similarity and strong locality of the global form. For each $m \geq 0$ and $f, g \in \mathcal{F}^{(m)}$, we use the two-variable notation $\mathcal{D}^{(m)}(f;g)$ from (2.5).

Theorem 6.8. *For $f \in \mathcal{F}$ and $m \geq 0$, let $H_m f := h^{\mathcal{E}}_{\overline{V}_m}[f|_{\overline{V}_m}]$ be the $\mathcal{E}$-harmonic extension of $f|_{\overline{V}_m}$. Then*

$$\mathcal{E}(H_m f - f) \longrightarrow 0. \tag{6.24}$$

Consequently, for all $f, g \in \mathcal{F}$,

$$\mathcal{E}(f;g) = \lim_{m\to\infty} \mathcal{D}^{(m)}\big(f|_{\overline{V}_m}; g|_{\overline{V}_m}\big). \tag{6.25}$$

Proof. By (6.22),

$$\mathcal{E}(H_m f) = \mathcal{D}^{(m)}(f|_{\overline{V}_m}) \uparrow \mathcal{E}(f).$$

Moreover, for every $x \in V_*$, one has $H_m f(x) = f(x)$ for all sufficiently large m. Since $q_0 \in \overline{V}_m$ for every m, the uniform bound on $\mathcal{E}(H_m f)$, together with (6.23), shows that $\{H_m f\}_m$ is uniformly bounded and equicontinuous on G. By the Arzelà–Ascoli theorem, every subsequence has a uniformly convergent further subsequence. Any such limit agrees with f on the dense set V_*, and hence equals f on G. Thus $H_m f \to f$ uniformly on G. Now Proposition 2.12(2) , together with $\mathcal{E}(H_m f) \uparrow \mathcal{E}(f)$, implies (6.24).

Finally, (6.22) and (2.12) give $\mathcal{D}^{(m)}(f|_{\overline{V}_m}; g|_{\overline{V}_m}) = \mathcal{E}(H_m f; g)$. Applying (2.9) and (6.24) to the right-hand side proves (6.25). □

Theorem 6.9. *The global form $(\mathcal{E}, \mathcal{F})$ satisfies the following self-similarity properties:*

(a) $$\mathcal{F} = \left\{ f \in C(G) : f \circ F_w \in \mathcal{F} \quad \forall\, w \in W_1,\ \sum_{w\in W_1} \gamma_w^{-1} \mathcal{E}(f \circ F_w) < \infty \right\}.$$

(b) *For every $f \in \mathcal{F}$, $\mathcal{E}(f) = \sum_{w\in W_1} \gamma_w^{-1} \mathcal{E}(f \circ F_w)$.*

Moreover, for $f, g \in \mathcal{F}$,

$$\mathcal{E}(f;g) = \sum_{w\in W_1} \gamma_w^{-1} \mathcal{E}(f \circ F_w; g \circ F_w), \tag{6.26}$$

and the series converges absolutely. The form is strongly local, in the sense that if $f, g, h \in \mathcal{F}$ and $(f - c_1)(g - c_2) = 0$ on G for some $c_1, c_2 \in \mathbb{R}$, then

$$\mathcal{E}(f + g + h) + \mathcal{E}(h) = \mathcal{E}(f + h) + \mathcal{E}(g + h). \tag{6.27}$$

Proof. For $m \geq 0$, the finite-level recursion can be written as

$$\mathcal{D}^{(m+1)}\big(f|_{\overline{V}_{m+1}}\big) = \sum_{w\in W_1} \gamma_w^{-1} \mathcal{D}^{(m)}\big((f \circ F_w)|_{\overline{V}_m}\big). \tag{6.28}$$

If $f \in \mathcal{F}$, then for each $w \in W_1$, (6.28) gives $\mathcal{D}^{(m)}\big((f \circ F_w)|_{\overline{V}_m}\big) \leq \gamma_w \mathcal{D}^{(m+1)}(f|_{\overline{V}_{m+1}}) \leq \gamma_w \mathcal{E}(f)$, so $f \circ F_w \in \mathcal{F}$. Letting $m \to \infty$ in (6.28) and using monotone convergence then gives (b) and the summability required in (a). Conversely, if $f \in C(G)$ satisfies the conditions on the right-hand

side of (a), then $\mathcal{D}^{(m+1)}(f|_{\overline{V}_{m+1}}) \le \sum_{w\in W_1} \gamma_w^{-1}\mathcal{E}(f\circ F_w)$ for every m, by (6.28). Hence $f\in\mathcal{F}$. This proves (a)–(b).

We next prove the two-variable self-similarity identity (6.26). For every finite $J\subset W_1$ and $u\in\mathcal{F}$, set

$$T_J(u) := \left(\sum_{w\in W_1\setminus J} \gamma_w^{-1}\mathcal{E}(u\circ F_w)\right)^{1/p}.$$

By (a), $T_J(u)\to 0$ as $J\uparrow W_1$. The p-triangle inequality, followed by the discrete Minkowski inequality, gives

$$T_J(f+tg) \le T_J(f) + |t|T_J(g).$$

Moreover, the finite-level version of (2.8), discrete Hölder, and the trace formula (6.22), which gives $\mathcal{D}^{(m)}(h|_{\overline{V}_m}) \le \mathcal{E}(h)$ for $h\in\mathcal{F}$, imply that, for $u,v\in\mathcal{F}$,

$$\sum_{w\in W_1\setminus J} \gamma_w^{-1}\left|\mathcal{D}^{(m)}\big((u\circ F_w)|_{\overline{V}_m}; (v\circ F_w)|_{\overline{V}_m}\big)\right| \le T_J(u)^{p-1}T_J(v).$$

Substitute $f+tg$ for f in (6.28). For each summand on the right-hand side, the limit in (2.5) at $t=0$ exists, and, after division by p, gives the corresponding two-variable energy. The preceding estimate with $u=f+tg$ and $v=g$ has tails converging to zero uniformly for t in a neighborhood of zero. Since the series itself converges at $t=0$ by (6.28), we may pass to the limit in the difference quotient term by term, obtaining

$$\mathcal{D}^{(m+1)}\big(f|_{\overline{V}_{m+1}}; g|_{\overline{V}_{m+1}}\big) = \sum_{w\in W_1} \gamma_w^{-1}\mathcal{D}^{(m)}\big((f\circ F_w)|_{\overline{V}_m}; (g\circ F_w)|_{\overline{V}_m}\big). \tag{6.29}$$

For each $w\in W_1$, $\mathcal{D}^{(m)}\big((f\circ F_w)|_{\overline{V}_m}; (g\circ F_w)|_{\overline{V}_m}\big) \to \mathcal{E}(f\circ F_w; g\circ F_w)$ by Theorem 6.8. The same tail estimate, now with $u=f$ and $v=g$, is uniform in m and permits the limit $m\to\infty$ to be taken term by term in (6.29), yielding (6.26). Fatou's lemma applied to the same estimate with $J=\varnothing$ gives absolute convergence of the series in (6.26).

Finally, we prove the strong locality of $(\mathcal{E},\mathcal{F})$. Since $\mathcal{E}$ is invariant under addition of constants, it suffices to consider $f,g,h\in\mathcal{F}$ with $fg=0$. For $n\ge 1$, set $\varphi_n(t) := t-(n^{-1}\wedge(t\vee(-n^{-1})))$, $f_n := \varphi_n\circ f$, and $g_n := \varphi_n\circ g$. These functions are 1-Lipschitz and converge pointwise to the identity, so Proposition 2.13 gives $\mathcal{E}(f-f_n)+\mathcal{E}(g-g_n)\to 0$.

Since $f,g\in C(G)$ and $fg=0$, for fixed n, the compact sets $\operatorname{supp} f_n$ and $\operatorname{supp} g_n$ are disjoint. Since $\rho_w\le\rho^m$ for $w\in W_m$, every sufficiently fine level-m cell meets at most one of them. Hence, for all sufficiently large m and every $w\in W_m$, either f_n or g_n vanishes identically on F_wG, and hence

$$\mathcal{D}((f_n+g_n+h)\circ F_w|_{V_0}) + \mathcal{D}(h\circ F_w|_{V_0}) = \mathcal{D}((f_n+h)\circ F_w|_{V_0}) + \mathcal{D}((g_n+h)\circ F_w|_{V_0}).$$

Multiplying by γ_w^{-1}, summing over W_m, and then letting $m\to\infty$, (6.4) and (6.21) give

$$\mathcal{E}(f_n+g_n+h)+\mathcal{E}(h) = \mathcal{E}(f_n+h)+\mathcal{E}(g_n+h).$$

Finally, letting $n\to\infty$ and using Proposition 2.13 proves (6.27). □

6.3 Standard resistance estimates

We now specialize the preceding construction to the standard renormalization factor $r=r_p$, for which $\lambda=\lambda_p(r_p)=r_p^2$. Choose a corresponding eigenform $\mathcal{D}_p\in\mathcal{M}_p^{\mathrm{sym}}(V_0)$, write $(\mathcal{E}_p,\mathcal{F}_p) :=$

$(\mathcal{E}_{p,r_p,\mathcal{D}_p}, \mathcal{F}_{p,r_p,\mathcal{D}_p})$, and set $\theta_p := \log r_p / \log \rho$. For this choice, the elementary weights are $\gamma_0 = r_p^2$ and $\gamma_1 = \gamma_2 = r_p$. Hence, for every $w \in \widetilde{W}_*$,

$$\gamma_w = \rho_w^{\theta_p}. \tag{6.30}$$

In particular, $\theta_{p,r_p} = \theta_p$ and, for $w \in W_1$, $\gamma_w = r_p^{|w|+1}$. By Theorems 6.7 and 6.9, $(\mathcal{E}_p, \mathcal{F}_p)$ is a regular, strongly local, self-similar p-resistance form, and satisfies

$$R_{\mathcal{E}_p}(x,y) \le C d(x,y)^{\theta_p}, \qquad x, y \in G. \tag{6.31}$$

Theorem 6.9 and (6.30) yield the following counterpart of [9, Theorem 6.6].

Corollary 6.10. *The standard form $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies*

$$\mathcal{F}_p = \left\{ f \in C(G) : f \circ F_w \in \mathcal{F}_p \quad \forall\, w \in W_1, \ \sum_{w \in W_1} \rho_w^{-\theta_p} \mathcal{E}_p(f \circ F_w) < \infty \right\}.$$

Moreover, for every $f \in \mathcal{F}_p$, $\mathcal{E}_p(f) = \sum_{w \in W_1} \rho_w^{-\theta_p} \mathcal{E}_p(f \circ F_w)$. For $f, g \in \mathcal{F}_p$,

$$\mathcal{E}_p(f;g) = \sum_{w \in W_1} \rho_w^{-\theta_p} \mathcal{E}_p(f \circ F_w; g \circ F_w),$$

and the series converges absolutely.

Proof. The assertions follow from Theorem 6.9 and (6.30). □

For a nonempty set $A \subset G$ and $x \in G \setminus A$, we use the notation $R_{\mathcal{E}_p}(x, A)$ introduced in (2.13). For $s > 0$, write $B_s(x) := \{y \in G : d(x,y) < s\}$ and $B_s^c(x) := G \setminus B_s(x)$.

For a nonempty word $w \in \widetilde{W}_*$, let w^- denote the word obtained by deleting its last symbol. For $t \in (0,1)$, define the family of cells stopped at scale t by

$$\mathscr{C}_t := \{F_w G : w \in \widetilde{W}_* \setminus \{\varnothing\},\ \rho_w \le t < \rho_{w^-}\},$$

where repeated copies of the same cell are identified. Each $F_w G \in \mathscr{C}_t$ is said to be a *stopping cell* stopped at t. For $U \subset G$, let $\mathring{U}$ and $\partial_G U$ denote its interior and boundary relative to G.

Lemma 6.11. *For $x \in G$ and $s \in (0,1)$, define*

$$U_0(x,s) := \bigcup_{\substack{K \in \mathscr{C}_{s\rho^2} \\ x \in K}} K, \qquad U_1(x,s) := \bigcup_{\substack{K \in \mathscr{C}_{s\rho^2} \\ K \cap U_0(x,s) \neq \varnothing}} K.$$

Then the following assertions hold.

(1) $x \in U_0(x,s) \subset \mathring{U}_1(x,s) \subset U_1(x,s) \subset B_s(x)$.

(2) *There exists a finite family $\{(U_{i,0}, U_{i,1})\}_{i=1}^M$ such that for every x and s there are $i \in \{1, \dots, M\}$ and a cell- and boundary-preserving similitude*

$$\psi_{x,s} : U_1(x,s) \longrightarrow U_{i,1}, \qquad \psi_{x,s}(U_0(x,s)) = U_{i,0}.$$

Each $U_{i,0}, U_{i,1}$ is said to be a model *of $U_0(x,s), U_1(x,s)$ respectively. Let $a_{x,s}$ denote the similarity ratio of $\psi_{x,s}^{-1}$. Then there exist constants $c_0, C_0 > 0$ such that*

$$c_0 s \le a_{x,s} \le C_0 s.$$

Proof. Set $t = s\rho^2$. The stopping family $\mathscr{C}_t$ covers G, and every $F_wG \in \mathscr{C}_t$ satisfies

$$t\rho^2 < \rho_w \le t.$$

Hence $x \in U_0(x,s)$. Since $\mathscr{C}_t$ is finite, the cells disjoint from $U_0(x,s)$ have positive distance from $U_0(x,s)$, and therefore $U_0(x,s) \subset \mathring{U}_1(x,s)$. If $y \in U_1(x,s)$, there are $K, K_0 \in \mathscr{C}_t$ such that $y \in K$, $x \in K_0$, and $K \cap K_0 \neq \varnothing$. Thus

$$d(x,y) \le \operatorname{diam} K_0 + \operatorname{diam} K \le 2s\rho^2 < s,$$

which proves (1).

For (2), if $F_wG, F_vG \in \mathscr{C}_t$ intersect, then

$$\rho^2 < \frac{\rho_v}{\rho_w} < \rho^{-2},$$

so the finite-type property in Definition 3.1 gives only finitely many possible relative similitudes $F_w^{-1} \circ F_v$. Fix $K_* \in \mathscr{C}_t$ containing x. Every cell forming $U_1(x,s)$ is connected to K_* by at most two cell intersections, and every cell meeting $U_1(x,s)$ by at most three. Hence only finitely many normalized cell- and boundary-configurations can occur, independently of x and s.

Choose one representative for each model. Writing $K_* = F_uG$, we have

$$s\rho^4 < \rho_u \le s\rho^2.$$

Since the scales of the cells corresponding to K_* in the representative models form a fixed finite set, the similarity ratio $a_{x,s}$ of $\psi_{x,s}^{-1}$ satisfies $c_0 s \le a_{x,s} \le C_0 s$. □

Recall that for each $w \in \widetilde{W}_*$, $\nu(w)$ denotes the number of occurrences of 0 in w. For each $\tau \in W_1$, choose a representative $v_\tau \in W_{\nu(w)+1}$ such that $F_{v_\tau} = F_{w\tau}$, and set $A := \{v_\tau : \tau \in W_1\}$. Then we have the following $W_{\nu(w)+1}$-cell decomposition of F_wG:

$$F_wG = \overline{\bigcup_{v\in A} F_vG}.$$

Moreover, for each $f \in \mathcal{F}_p$, it satisfies

$$\rho_w^{-\theta_p}\mathcal{E}_p(f\circ F_w) = \rho_w^{-\theta_p}\sum_{\tau\in W_1}\rho_\tau^{-\theta_p}\mathcal{E}_p(f\circ F_w\circ F_\tau) = \sum_{v\in A}\rho_v^{-\theta_p}\mathcal{E}_p(f\circ F_v). \tag{6.32}$$

As a consequence, for each $t > 0$, since $\mathscr{C}_t$ is finite, there exists $m \ge 1$ such that

$$\mathcal{E}_p(f) = \sum_{v\in W_m}\rho_v^{-\theta_p}\mathcal{E}_p(f\circ F_v) = \sum_{K\in\mathscr{C}_t}\sum_{v\in A_K}\rho_v^{-\theta_p}\mathcal{E}_p(f\circ F_v) = \sum_{K\in\mathscr{C}_t}\rho_{w_K}^{-\theta_p}\mathcal{E}_p(f\circ F_{w_K}), \tag{6.33}$$

where $w_K \in \widetilde{W}_*$ such that $F_{w_K}G = K$, and A_K is the W_m-cell decomposition of K as above for each $K \in \mathscr{C}_t$, which clearly satisfies $A_K \cap A_{K'} = \emptyset$ for all distinct $K, K' \in \mathscr{C}_t$ and $W_m = \bigcup_{K\in\mathscr{C}_t} A_K$.

The proof of the next proposition adapts the argument in [9, Proposition 7.3] to the p-resistance setting.

Proposition 6.12. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 d(x,y)^{\theta_p} \le R_{\mathcal{E}_p}(x,y) \le c_2 d(x,y)^{\theta_p}, \qquad x, y \in G. \tag{6.34}$$

Moreover, there exist $c_3, c_4 > 0$ such that, for every $x \in G$ and $s \in (0,1)$ with $B_s^c(x) \neq \varnothing$,

$$c_3 s^{\theta_p} \le R_{\mathcal{E}_p}\big(x, B_s^c(x)\big) \le c_4 s^{\theta_p}. \tag{6.35}$$

Proof. The upper bound follows from (6.31). To prove the lower bound, fix $x \in G$ and $s \in (0,1)$, and write $U_j := U_j(x,s)$, $j = 0,1$. Choose from Lemma 6.11 a corresponding model and a cell- and boundary-preserving similitude

$$\psi_{x,s} : U_1 \longrightarrow U_{i,1}, \qquad \psi_{x,s}(U_0) = U_{i,0},$$

and write $a = a_{x,s}$. Then $c_0 s \le a \le C_0 s$, uniformly in x and s.

For each pair of $U_{i,0}$ and $U_{i,1}$, choose $h_i \in C(G)$ such that $0 \le h_i \le 1$, $h_i = 1$ on $U_{i,0}$, and $h_i = 0$ on $G \setminus \mathring{U}_{i,1}$. By regularity in Theorem 6.7 , take $g_i \in \mathcal{F}_p$ with $\|g_i - h_i\|_\infty < 1/4$ and set a cutoff function of $U_{i,0}, U_{i,1}$ by

$$\varphi_i := \big(2(g_i - 1/4)\big)^+ \wedge 1.$$

Indeed, the generalized contraction property in Theorem 6.7 gives $\varphi_i \in \mathcal{F}_p$, $0 \le \varphi_i \le 1$, $\varphi_i|_{U_{i,0}} = 1$, and $\varphi_i|_{G\setminus \mathring{U}_{i,1}} = 0$. In fact, the strict approximation bound and compactness show that $\operatorname{supp}\varphi_i$ is a compact subset of $\mathring{U}_{i,1}$. Define

$$f_{x,s}(z) := \begin{cases} \varphi_i(\psi_{x,s}(z)), & z \in U_1, \\ 0, & z \in G \setminus U_1. \end{cases}$$

Since $\psi_{x,s}$ preserves the relative boundary, $\operatorname{supp} f_{x,s}$ is a compact subset of $\mathring{U}_1$. Hence $f_{x,s} \in C(G)$, $f_{x,s}(x) = 1$, and $f_{x,s}|_{B_s^c(x)} = 0$.

It remains to estimate the energy upper bound of $f_{x,s}$. Since $\operatorname{supp} f_{x,s}$ is a compact subset of $\mathring{U}_1$ and the diameters of the cells in W_m tend uniformly to zero, for all sufficiently large m every W_m-cell meeting $\partial_G U_1$ has zero energy contribution. Hence every cell contributing nontrivially to $\mathcal{D}^{(m)}(f_{x,s}|_{\overline{V}_m})$ is contained in a stopping cell forming U_1.

Write $M_\varphi := \max_{1\le i\le M} \mathcal{E}_p(\varphi_i)$. For each $K \in \mathscr{C}_{s\rho^2}$ with $K \cap U_0 \ne \emptyset$ take $w_K \in \widetilde{W}_*$ such that $F_{w_K} G = K$ and write A as the collection of such w_K, which satisfies $U_1 = \bigcup_{w_K \in A} F_{w_K} G$. Since $\psi_{x,s}$ is cell-preserving, each $w \in A$ admits $w' \in \widetilde{W}_*$ such that $\psi_{x,s}(F_w G) = F_{w'} G$, which satisfies $s\rho_{w'} \le C\rho_w$ for some constant $C \ge 1$. Write $A' := \{w' \in \widetilde{W}_* : w \in A\}$. Since A' is finite, there is $m \ge 1$ and $A'' \subset W_m$ such that

$$\bigcup_{w' \in A'} F_{w'} G = \overline{\bigcup_{w'' \in A''} F_{w''} G}.$$

Then we have

$$\begin{aligned}
\mathcal{E}_p(f_{x,s}) &= \sum_{K \in \mathscr{C}_{s\rho^2}} \rho_{w_K}^{-\theta_p} \mathcal{E}_p(f_{x,s} \circ F_{w_K}) = \sum_{w \in A} \rho_w^{-\theta_p} \mathcal{E}_p(f_{x,s} \circ F_w) \\
&= \sum_{w \in A} \rho_w^{-\theta_p} \mathcal{E}_p(\varphi_i \circ \psi_{x,s} \circ F_w) \\
&\le C \sum_{w' \in A'} (s\rho_{w'})^{-\theta_p} \mathcal{E}_p(\varphi_i \circ F_{w'}) \\
&= C s^{-\theta_p} \sum_{w'' \in A''} \rho_{w''}^{-\theta_p} \mathcal{E}_p(\varphi_i \circ F_{w''}) \\
&\le C s^{-\theta_p} \mathcal{E}_p(\varphi_i) \le C M_\varphi s^{-\theta_p},
\end{aligned}$$

where the first equation follows by (6.33), the last second and last line follow from (6.32). Thus we obtain the desired lower bound

$$R_{\mathcal{E}_p}(x, B_s^c(x)) \ge \frac{1}{\mathcal{E}_p(f_{x,s})} \ge c_3 s^{\theta_p}.$$

For each distinct $x, y \in G$, take $s := d(x, y)$ and then

$$R_{\mathcal{E}_p}(x, y) \geq R_{\mathcal{E}_p}(x, B_s^c(x)) \geq c_3 d(x, y)^{\theta_p},$$

which, together with the upper bound, proves (6.34).

Finally, suppose that $B_s^c(x) \neq \varnothing$. Since G is connected, there exists $y \in G$ with $d(x, y) = s$. Hence

$$R_{\mathcal{E}_p}(x, B_s^c(x)) \leq R_{\mathcal{E}_p}(x, y) \leq c_2 s^{\theta_p},$$

which proves (6.35). □

7 Eigenform and global uniqueness

This section proves uniqueness for a fixed pair (p, r) by comparing the largest and smallest ratios of two boundary eigenforms. Cellwise harmonicity from Lemma 7.1 handles the antisymmetric extremal case, while the remaining case is reduced by Theorem 7.2 to a one-vertex Perron–Frobenius asymptotic on the nested cells $F_0^n G$. Its proof follows the scheme of [18, Theorem 3.20], with the modifications needed for the countable level-one graph and the reflection fixing q_0, and is given in Appendix A. We then lift the uniqueness of the boundary eigenform, up to a positive scalar, to the self-similar form on G in Theorem 7.5.

Fix $p \in (1, \infty)$, $r \in (\rho^{p-1}, 1)$, and a symmetric p-resistance form $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ satisfying $\mathcal{R}_{p,r}\mathcal{D} = \lambda\mathcal{D}$. Let $(\mathcal{E}, \mathcal{F})$ be the global p-resistance form constructed from $\mathcal{D}$ in Section 6, and set $\kappa := \lambda^{1/(p-1)} \in (0, 1)$. The reflection in the vertical axis is denoted by σ. The following lemma does not require symmetry of $\mathcal{D}$.

Lemma 7.1. *Let $n \geq 1$. If $h \in \mathcal{H}_{\mathcal{E},\overline{V}_n}$, then $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$ for every $w \in W_n$. In particular, if $h \in \mathcal{H}_{\mathcal{E},V_0}$, then this conclusion holds for every $n \geq 1$ and $w \in W_n$. Moreover, for $h \in \mathcal{H}_{\mathcal{E},V_0}$ and $x \in V_1 \setminus \{q_0\}$,*

$$\sum_{\substack{w \in W_1 \\ x \in F_w V_0}} \gamma_w^{-1} \mathcal{D}\left(h \circ F_w|_{V_0}; \mathbf{1}^{V_0}_{F_w^{-1}x}\right) = 0, \tag{7.1}$$

and the sum is finite.

Proof. Let $h \in \mathcal{H}_{\mathcal{E},\overline{V}_n}$, and let g be obtained by replacing $h \circ F_w$ on each $w \in W_n$ by the $\mathcal{E}$-harmonic extension of its restriction to V_0. The pieces agree on their boundary intersections. The resistance inequality (2.2) on each cell and $\gamma_w \to 0$ outside finite subfamilies of W_n give continuity at the accumulation points, so $g \in \mathcal{F}$. Then $g = h$ on $\overline{V}_n$, and Theorem 6.9 gives

$$\mathcal{E}(h) \leq \mathcal{E}(g) = \sum_{w \in W_n} \gamma_w^{-1} \mathcal{D}(h \circ F_w|_{V_0}) \leq \sum_{w \in W_n} \gamma_w^{-1} \mathcal{E}(h \circ F_w) = \mathcal{E}(h).$$

Hence equality holds throughout, and therefore $\mathcal{E}(h \circ F_w) = \mathcal{D}(h \circ F_w|_{V_0})$ for every $w \in W_n$. Thus $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$ by Theorem 2.8. Since $\mathcal{H}_{\mathcal{E},V_0} \subset \mathcal{H}_{\mathcal{E},\overline{V}_n}$, the particular assertion follows.

Now let $h \in \mathcal{H}_{\mathcal{E},V_0}$ and fix $x \in V_1 \setminus \{q_0\}$. By Propositions 3.3 and 3.6, $\mathbf{1}_x^{V_1}$ extends to $\mathbf{1}_x^{\overline{V}_1} \in \mathcal{F}^{(1)}$. Set $v_x := h^{\mathcal{E}}_{\overline{V}_1}[\mathbf{1}_x^{\overline{V}_1}]$. Since $v_x|_{V_0} = 0$, harmonicity of h and (2.12) yield

$$\begin{aligned} 0 &= \mathcal{E}(h; v_x) = \mathcal{E}|_{\overline{V}_1}(h|_{\overline{V}_1}; \mathbf{1}_x^{\overline{V}_1}) = \mathcal{D}^{(1)}(h|_{\overline{V}_1}; \mathbf{1}_x^{\overline{V}_1}) \\ &= \sum_{\substack{w \in W_1 \\ x \in F_w V_0}} \gamma_w^{-1} \mathcal{D}\left(h \circ F_w|_{V_0}; \mathbf{1}^{V_0}_{F_w^{-1}x}\right). \end{aligned}$$

The last sum is finite by Proposition 3.3(1). □

Define

$$h_{\mathrm{s}} := h^{\mathcal{E}}_{V_0}[\mathbf{1}^{V_0}_{q_1} + \mathbf{1}^{V_0}_{q_2}], \qquad h_{\mathrm{a}} := h^{\mathcal{E}}_{V_0}[\mathbf{1}^{V_0}_{q_1} - \mathbf{1}^{V_0}_{q_2}]. \tag{7.2}$$

We call h_{s} and h_{a} the *symmetric* and *antisymmetric harmonic coordinates*, respectively. Thus $h_{\mathrm{s}} \circ \sigma = h_{\mathrm{s}}$ and $h_{\mathrm{a}} \circ \sigma = -h_{\mathrm{a}}$.

We will use the following one-vertex asymptotic. Its proof, together with the comparison results for the countable level-one graph, is given in Appendix A.

Theorem 7.2. *For $h \in \mathcal{H}_{\mathcal{E},V_0}$, set*

$$c_0(h) := -\operatorname{sgn} \mathcal{D}(h|_{V_0}; \mathbf{1}^{V_0}_{q_0}) \left| \frac{\mathcal{D}(h|_{V_0}; \mathbf{1}^{V_0}_{q_0})}{\mathcal{D}(h_{\mathrm{s}}|_{V_0}; \mathbf{1}^{V_0}_{q_0})} \right|^{1/(p-1)}. \tag{7.3}$$

Then

$$\kappa^{-n}\big(h \circ F_0^n - h(q_0)\mathbf{1}_G\big) \longrightarrow c_0(h) h_{\mathrm{s}} \tag{7.4}$$

uniformly on G and in the seminorm $\mathcal{E}^{1/p}$.

We now use the convergence in (7.4) to compare two eigenforms.

Let $\widetilde{\mathcal{D}} \in \mathcal{M}_p(V_0)$ satisfy $\mathcal{R}_{p,r}\widetilde{\mathcal{D}} = \lambda\widetilde{\mathcal{D}}$, and let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be the global p-resistance form constructed from $\widetilde{\mathcal{D}}$ in Section 6. Define $\Theta_+ := \max_{u \notin \mathbb{R}\mathbf{1}_{V_0}} \frac{\mathcal{D}(u)}{\widetilde{\mathcal{D}}(u)}$ and $\Theta_- := \min_{u \notin \mathbb{R}\mathbf{1}_{V_0}} \frac{\mathcal{D}(u)}{\widetilde{\mathcal{D}}(u)}$, and

$$\mathcal{A}_\pm := \{u \in \mathbb{R}^{V_0} : \mathcal{D}(u) = \Theta_\pm \widetilde{\mathcal{D}}(u)\}.$$

We call $\mathcal{A}_+$ and $\mathcal{A}_-$ the *extremal cones* which clearly contain all constant functions. They are closed by continuity of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. The extrema are attained by nonconstant functions and $0 < \Theta_- \le \Theta_+ < \infty$. Since $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ have the same eigenvalue, the finite-level constructions use the same weights γ_w. The definition of $\Theta_\pm$ therefore gives, for every $m \ge 0$,

$$\mathcal{F}^{(m)} = \widetilde{\mathcal{F}}^{(m)}, \qquad \Theta_- \widetilde{\mathcal{D}}^{(m)}(u) \le \mathcal{D}^{(m)}(u) \le \Theta_+ \widetilde{\mathcal{D}}^{(m)}(u), \quad u \in \mathcal{F}^{(m)}.$$

Passing to the limits in (6.20) and (6.21) gives

$$\mathcal{F} = \widetilde{\mathcal{F}}, \qquad \Theta_- \widetilde{\mathcal{E}}(u) \le \mathcal{E}(u) \le \Theta_+ \widetilde{\mathcal{E}}(u), \qquad u \in \mathcal{F}. \tag{7.5}$$

Lemma 7.3. (1) *For $u \in \mathbb{R}^{V_0}$, $\mathcal{D}(u; \mathbf{1}^{V_0}_{q_0}) = 0 \iff u \in \mathbb{R}\mathbf{1}_{V_0} + \mathbb{R}\, h_{\mathrm{a}}|_{V_0}$.*

(2) *If $u \in \mathcal{A}_\pm$, then its $\mathcal{E}$- and $\widetilde{\mathcal{E}}$-harmonic extensions coincide; writing the common extension as H_u, one has $H_u \circ F_w|_{V_0} \in \mathcal{A}_\pm$, $\quad w \in W_1$.*

Proof. For the implication from right to left in (1), translation invariance and (2.7) reduce the claim to $u = h_{\mathrm{a}}|_{V_0}$. Reflection invariance, $\sigma(q_0) = q_0$, and $h_{\mathrm{a}} \circ \sigma = -h_{\mathrm{a}}$ then give

$$\mathcal{D}(h_{\mathrm{a}}|_{V_0}; \mathbf{1}^{V_0}_{q_0}) = \mathcal{D}(-h_{\mathrm{a}}|_{V_0}; \mathbf{1}^{V_0}_{q_0}) = -\mathcal{D}(h_{\mathrm{a}}|_{V_0}; \mathbf{1}^{V_0}_{q_0}),$$

so $\mathcal{D}(h_{\mathrm{a}}|_{V_0}; \mathbf{1}^{V_0}_{q_0}) = 0$. Conversely, suppose that $\mathcal{D}(u; \mathbf{1}^{V_0}_{q_0}) = 0$. If $u(q_1) \ne u(q_2)$, then, after subtracting a constant, we can write $u = a(h_{\mathrm{a}}|_{V_0} + t\mathbf{1}^{V_0}_{q_0})$ with $a \ne 0$. By homogeneity, it is enough to consider $h_{\mathrm{a}}|_{V_0} + t\mathbf{1}^{V_0}_{q_0}$. The quantity $\mathcal{D}(h_{\mathrm{a}}|_{V_0} + t\mathbf{1}^{V_0}_{q_0}; \mathbf{1}^{V_0}_{q_0})$ is strictly monotone in t by (2.6), and it vanishes at $t = 0$ by the reflection argument above. Hence $t = 0$, and $u \in \mathbb{R}\mathbf{1}_{V_0} + \mathbb{R}h_{\mathrm{a}}|_{V_0}$. If $u(q_1) = u(q_2)$, subtracting a constant leaves a multiple of $\mathbf{1}^{V_0}_{q_0}$. By homogeneity, $\mathcal{D}(u; \mathbf{1}^{V_0}_{q_0}) = 0$ can then hold only when that multiple is zero. Thus u is constant.

For (2), let $\widetilde{H}_u$ be the $\widetilde{\mathcal{E}}$-harmonic extension of u and suppose first that $u \in \mathcal{A}_+$. By trace minimality and (7.5),

$$\mathcal{D}(u) \le \mathcal{E}(\widetilde{H}_u) \le \Theta_+ \widetilde{\mathcal{E}}(\widetilde{H}_u) = \Theta_+ \widetilde{\mathcal{D}}(u) = \mathcal{D}(u).$$

Hence $\widetilde{H}_u$ is also the $\mathcal{E}$-harmonic extension of u; write the common extension as H_u. Applying the self-similarity identity from Theorem 6.9 and Lemma 7.1 to both forms gives

$$\mathcal{D}(u) = \mathcal{E}(H_u) = \sum_{w\in W_1} \gamma_w^{-1}\mathcal{D}(H_u \circ F_w|_{V_0}),$$
$$\Theta_+\widetilde{\mathcal{D}}(u) = \Theta_+\widetilde{\mathcal{E}}(H_u) = \Theta_+ \sum_{w\in W_1} \gamma_w^{-1}\widetilde{\mathcal{D}}(H_u \circ F_w|_{V_0}).$$

The two left-hand sides are equal, while each summand in the first series is at most the corresponding summand in the second. Since all weights are positive, equality holds termwise. Therefore $H_u \circ F_w|_{V_0} \in \mathcal{A}_+$ for every $w \in W_1$. The case $u \in \mathcal{A}_-$ follows by interchanging the two forms. □

Theorem 7.4. *For every $p \in (1,\infty)$ and $r \in (\rho^{p-1}, 1)$, there exists a p-resistance form $\mathcal{D}_{p,r} \in \mathcal{M}_p(V_0)$, unique up to multiplication by a positive constant, such that*

$$\mathcal{R}_{p,r}\mathcal{D}_{p,r} = \lambda_p(r)\mathcal{D}_{p,r}.$$

Every renormalization eigenform belongs to $\mathcal{M}_p^{\mathrm{sym}}(V_0)$.

Proof. Existence of a symmetric eigenform $\mathcal{D}$ follows from Theorem 5.7. Let $\widetilde{\mathcal{D}}$ be any other eigenform and use the notation introduced above. By Proposition 5.6, it has the same eigenvalue $\lambda_p(r)$. Suppose that it is not a positive multiple of $\mathcal{D}$, so that $\Theta_- < \Theta_+$, and set $\mathcal{L}_{\mathrm{a}} := \mathbb{R}\mathbf{1}_{V_0} + \mathbb{R}h_{\mathrm{a}}|_{V_0}$. We distinguish two cases.

Case 1: one extremal cone contains only antisymmetric functions modulo constants. Interchanging $+$ and $-$ if necessary, assume $\mathcal{A}_+ \subset \mathcal{L}_{\mathrm{a}}$. Since $\mathcal{A}_+$ contains a nonconstant element, translation invariance and p-homogeneity imply $h_{\mathrm{a}}|_{V_0} \in \mathcal{A}_+$. By Lemma 7.3(2),

$$h_{\mathrm{a}} \circ F_w|_{V_0} \in \mathcal{L}_{\mathrm{a}}, \qquad w \in W_1.$$

Set $x_n := F_1^n q_0$. For $n \geq 1$, the two level-one cells containing x_n are $F_{1^{n-1}0}G$ and $F_{1^n0}G$. Hence, for suitable $a_n, b_n, t_n, s_n \in \mathbb{R}$,

$$h_{\mathrm{a}} \circ F_{1^{n-1}0}|_{V_0} = a_n\mathbf{1}_{V_0} + t_n h_{\mathrm{a}}|_{V_0}, \qquad h_{\mathrm{a}} \circ F_{1^n0}|_{V_0} = b_n\mathbf{1}_{V_0} + s_n h_{\mathrm{a}}|_{V_0}.$$

By Lemma 7.3(1),

$$\mathcal{D}(h_{\mathrm{a}}|_{V_0}; \mathbf{1}_{q_0}^{V_0}) = 0.$$

Together with reflection antisymmetry and $\mathcal{D}(h_{\mathrm{a}}|_{V_0}; h_{\mathrm{a}}|_{V_0}) = \mathcal{D}(h_{\mathrm{a}}|_{V_0})$, this gives

$$\mathcal{D}(h_{\mathrm{a}}|_{V_0}; \mathbf{1}_{q_1}^{V_0}) = \frac{1}{2}\mathcal{D}(h_{\mathrm{a}}|_{V_0}) > 0. \tag{7.6}$$

At x_n, the cells $F_{1^{n-1}0}G$ and $F_{1^n0}G$ correspond to the local vertices q_1 and q_0, respectively. Thus the vertex equation (7.1) gives

$$0 = \gamma_{1^{n-1}0}^{-1}\mathcal{D}\big(a_n\mathbf{1}_{V_0} + t_n h_{\mathrm{a}}|_{V_0}; \mathbf{1}_{q_1}^{V_0}\big) + \gamma_{1^n0}^{-1}\mathcal{D}\big(b_n\mathbf{1}_{V_0} + s_n h_{\mathrm{a}}|_{V_0}; \mathbf{1}_{q_0}^{V_0}\big).$$

The second term vanishes by Lemma 7.3(1), while homogeneity in (2.7) and (7.6) show that the first term can vanish only if $t_n = 0$. Consequently, $h_{\mathrm{a}}(x_n) = h_{\mathrm{a}}(x_{n-1})$ for every $n \geq 1$. Since $x_n \to q_1$, continuity gives $h_{\mathrm{a}}(q_1) = h_{\mathrm{a}}(q_0) = 0$, contrary to the boundary value $h_{\mathrm{a}}(q_1) = 1$.

Case 2: both extremal cones contain functions that are not antisymmetric modulo constants. Choose $u_\pm \in \mathcal{A}_\pm \setminus \mathcal{L}_{\mathrm{a}}$. By Lemma 7.3(1), $\mathcal{D}(u_\pm; \mathbf{1}_{q_0}^{V_0}) \neq 0$. Let $H_\pm$ be their common harmonic extensions. By Lemma 7.3(2), the boundary values of

$$\kappa^{-n}\big(H_\pm \circ F_0^n - H_\pm(q_0)\mathbf{1}_G\big)$$

remain in $\mathcal{A}_\pm$. By Theorem 7.2 and the formula (7.3), their limits are nonzero multiples of $h_{\mathrm{s}}|_{V_0}$. Since the extremal cones are closed and invariant under addition of constants and scalar multiplication, $h_{\mathrm{s}}|_{V_0} \in \mathcal{A}_+ \cap \mathcal{A}_-$. As $h_{\mathrm{s}}|_{V_0}$ is nonconstant, this implies $\Theta_+ = \Theta_-$, again a contradiction.

Both cases are impossible. Hence every eigenform is a positive multiple of $\mathcal{D}$, and is therefore symmetric. □

We now lift the uniqueness of the eigenform on V_0 to the associated global form.

Theorem 7.5. *Fix $p \in (1,\infty)$ and $r \in (\rho^{p-1}, 1)$, set $\lambda = \lambda_p(r)$, $\gamma_0 = \lambda$, $\gamma_1 = \gamma_2 = r$, and $\gamma_w = \prod_{j=1}^{|w|} \gamma_{w_j}$. Up to a positive scalar, there is a unique p-resistance form $(\mathcal{E}, \mathcal{F})$ on G satisfying the following properties:*

(a) $$\mathcal{F} = \left\{ f \in C(G) : f \circ F_w \in \mathcal{F} \quad \forall\, w \in W_1,\ \sum_{w \in W_1} \gamma_w^{-1} \mathcal{E}(f \circ F_w) < \infty \right\}.$$

(b) *For every $f \in \mathcal{F}$, $\mathcal{E}(f) = \sum_{w \in W_1} \gamma_w^{-1} \mathcal{E}(f \circ F_w)$.*

Proof. By Theorem 6.9, the p-resistance form $(\mathcal{E}, \mathcal{F})$ constructed in Section 6 satisfies (a)–(b). It remains to prove uniqueness.

Let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be another p-resistance form satisfying (a)–(b), and denote the traces of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ to V_0 by $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively. By (6.1) and (6.22), $\mathcal{R}_{p,r}\mathcal{D} = \lambda \mathcal{D}$. On the other hand, (a)–(b) and the final assertion of Theorem 2.8 give $\mathcal{R}_{p,r}\widetilde{\mathcal{D}} = \lambda \widetilde{\mathcal{D}}$. Hence Theorem 7.4 yields $\widetilde{\mathcal{D}} = c\mathcal{D}$ for some $c > 0$.

Following Definition 6.1, for $m \geq 0$, let $\widetilde{\mathcal{D}}^{(m)}$ denote the compatible level-m energy obtained from $\widetilde{\mathcal{D}}$ using the same weights γ_w. We show that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}$ are recovered from these finite-level energies as follows:

$$\widetilde{\mathcal{F}} = \left\{ f \in C(G) : \lim_{m \to \infty} \widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}) < \infty \right\}, \tag{7.7}$$

and, for every $f \in \widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}(f) = \lim_{m \to \infty} \widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}). \tag{7.8}$$

We first identify $\widetilde{\mathcal{D}}^{(m)}$ with the trace of $\widetilde{\mathcal{E}}$ on $\overline{V}_m$. Let $f \in C(G)$ with $\widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}) < \infty$, and define $\widetilde{H}_m f$ cellwise by requiring, for every $w \in W_m$, that $\widetilde{H}_m f \circ F_w$ be the $\widetilde{\mathcal{E}}$-harmonic extension of $f \circ F_w|_{V_0}$ from V_0. The boundary values agree on cell intersections. Writing $L_m := \widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m})$, the resistance inequality (2.2) gives

$$\operatorname{osc}_{F_w G}(\widetilde{H}_m f)^p \leq C_1 \widetilde{\mathcal{D}}(f \circ F_w|_{V_0}) \leq C_1 \gamma_w L_m, \qquad w \in W_m.$$

Since $\gamma_w \to 0$ outside finite subfamilies of W_m, the cell oscillations tend uniformly to zero there, and the pieces glue to a continuous function on G. Iterating (a)–(b) then shows that $\widetilde{H}_m f \in \widetilde{\mathcal{F}}$ and

$$\widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}) = \sum_{w \in W_m} \gamma_w^{-1} \widetilde{\mathcal{D}}(\widetilde{H}_m f \circ F_w|_{V_0}) = \sum_{w \in W_m} \gamma_w^{-1} \widetilde{\mathcal{E}}(\widetilde{H}_m f \circ F_w) = \widetilde{\mathcal{E}}(\widetilde{H}_m f). \tag{7.9}$$

Conversely, if $g \in \widetilde{\mathcal{F}}$ agrees with f on $\overline{V}_m$, then the trace inequality in Theorem 2.8 on each m-cell and (b) give $\widetilde{\mathcal{E}}(g) \geq \widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m})$. Thus $\widetilde{\mathcal{D}}^{(m)}$ is the trace of $\widetilde{\mathcal{E}}$ on $\overline{V}_m$, and $\widetilde{H}_m f = h^{\widetilde{\mathcal{E}}}_{\overline{V}_m}[f|_{\overline{V}_m}]$.

If $f \in \widetilde{\mathcal{F}}$, the trace inequality in Theorem 2.8 gives $\widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}) \leq \widetilde{\mathcal{E}}(f)$ for every m. Since $\overline{V}_m \subset \overline{V}_{m+1}$, the final assertion of Theorem 2.8 shows that these trace energies are nondecreasing in m; their limit is therefore finite.

Conversely, let $f \in C(G)$ and assume $L := \lim_{m\to\infty} \widetilde{\mathcal{D}}^{(m)}(f|_{\overline{V}_m}) < \infty$. Set $E_m := \widetilde{\mathcal{E}}(\widetilde{H}_m f)$. By (7.9), $E_m \uparrow L$. If $n \geq m$, then $(\widetilde{H}_n f + \widetilde{H}_m f)/2$ agrees with f on $\overline{V}_m$, and the minimizing property of $\widetilde{H}_m f$ gives

$$\widetilde{\mathcal{E}}\left(\frac{\widetilde{H}_n f + \widetilde{H}_m f}{2}\right) \geq E_m. \tag{7.10}$$

For $1 < p \leq 2$, applying (2.3) to the half-sum and half-difference yields

$$\widetilde{\mathcal{E}}\left(\frac{\widetilde{H}_n f - \widetilde{H}_m f}{2}\right)^{1/(p-1)} \leq \left(\frac{E_n + E_m}{2}\right)^{1/(p-1)} - E_m^{1/(p-1)} \longrightarrow 0.$$

For $p \geq 2$, applying (2.4) and (7.10) gives

$$\widetilde{\mathcal{E}}(\widetilde{H}_n f - \widetilde{H}_m f) \leq 2\left(E_n^{1/(p-1)} + E_m^{1/(p-1)}\right)^{p-1} - 2^p E_m \longrightarrow 0.$$

Thus $\{\widetilde{H}_m f\}_m$ is $\widetilde{\mathcal{E}}^{1/p}$-Cauchy. Since all its members have the same value at q_0, completeness modulo constants in Definition 2.1 $(\mathrm{RF2})_p$ gives $h \in \widetilde{\mathcal{F}}$ such that $\widetilde{\mathcal{E}}(\widetilde{H}_m f - h) \to 0$ and $h(q_0) = f(q_0)$. The resistance inequality (2.2) implies $\widetilde{H}_m f \to h$ pointwise, whereas on V_* the sequence is eventually equal to f. Hence $h = f$ on the dense set V_* and therefore on G, since both h and f are continuous. Thus $f \in \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}(\widetilde{H}_m f - f) \to 0$. The triangle inequality for $\widetilde{\mathcal{E}}^{1/p}$ now gives $\widetilde{\mathcal{E}}(f) = \lim_m E_m = L$. This proves (7.7)–(7.8).

Finally, $\widetilde{\mathcal{D}} = c\mathcal{D}$ gives $\widetilde{\mathcal{D}}^{(m)} = c\mathcal{D}^{(m)}$ for every $m \geq 0$. Comparing (7.7)–(7.8) with the construction of $(\mathcal{E}, \mathcal{F})$ in Section 6, we obtain $\widetilde{\mathcal{F}} = \mathcal{F}$ and $\widetilde{\mathcal{E}} = c\mathcal{E}$. □

We are now in a position to complete the proofs of Theorems 1 and 2.

Proof of Theorem 1. For part (a), the existence of a boundary eigenform follows from Theorem 5.7, while Proposition 5.6 shows that all eigenforms have the same eigenvalue. The uniqueness and symmetry of the eigenform are given by Theorem 7.4.

For part (b), the compatible finite-level forms associated with the boundary eigenpair have a global limit by Theorem 6.7. The resulting form is self-similar, regular, and strongly local by Theorem 6.9, and its uniqueness up to a positive scalar follows from Theorem 7.5. □

Proof of Theorem 2. The existence and uniqueness of r_p satisfying $\lambda_p(r_p) = r_p^2$ follow from Theorem 5.10. Let $(\mathcal{E}_p, \mathcal{F}_p)$ be the global form associated with the corresponding boundary eigenpair. For $w \in W_1$, the definition of the weights and the relation $1 - \rho = \rho^2$ give

$$\gamma_w = r_p^{|w|+1} = \rho_w^{\theta_p}, \qquad \theta_p = \frac{\log r_p}{\log \rho}.$$

Finally, the two-sided resistance estimate is precisely Proposition 6.12. □

8 Dependence on p

This section proves the monotonicity and continuity of the standard renormalization factor r_p with respect to p. The standard global forms $(\mathcal{E}_p, \mathcal{F}_p)$ used below are characterized by Corollary 6.10, and their resistance metrics $R_{\mathcal{E}_p}$ satisfy the two-sided estimate in Proposition 6.12.

8.1 Monotonicity

We adapt the argument of [16, Theorem 7.9], proved in the p.c.f. setting, to the present countable-cell structure. For each $p \in (1,\infty)$, let $r_p \in (\rho^{p-1}, 1)$ satisfy $\lambda_p(r_p) = r_p^2$, put $\lambda_p = r_p^2$, and fix a corresponding eigenform $\mathcal{D}_p \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ and its associated global form $(\mathcal{E}_p, \mathcal{F}_p)$. For $n \geq 1$ and $\tau \in W_n$, write $\gamma_{p,\tau}$ for the standard cell weight. Since $W_n = W_1^n$, this weight is $\gamma_{p,\tau} = r_p^{|\tau|+n}$. In particular, $\gamma_{p,w} = r_p^{|w|+1}$ for $w \in W_1$.

Theorem 8.1. *The map $p \longmapsto r_p^{1/(p-1)}$ is non-increasing on $(1,\infty)$. Equivalently, for $1 < p_1 \leq p_2 < \infty$,*

$$r_{p_2}^{1/(p_2-1)} \leq r_{p_1}^{1/(p_1-1)}. \tag{8.1}$$

Consequently, both $p \mapsto r_p$ and $p \mapsto \lambda_p = r_p^2$ are strictly decreasing on $(1,\infty)$.

Proof. Fix $1 < p_1 < p_2 < \infty$ and write $r_i := r_{p_i}$, $s_i := r_i^{1/(p_i-1)}$, $\mathcal{D}_i := \mathcal{D}_{p_i}$, and $(\mathcal{E}_i, \mathcal{F}_i) := (\mathcal{E}_{p_i}, \mathcal{F}_{p_i})$. Let $B = \{q_1, q_2\}$ and let h_1 be the p_1-harmonic function on $G \setminus B$ with $h_1(q_1) = 0$ and $h_1(q_2) = 1$. Every word in W_1 ends in 0, so $F_w G \cap B = \varnothing$ for every $w \in W_1$. If $h_1 \circ F_w$ were constant for every $w \in W_1$, the connected cell-incidence graph in Proposition 3.3 (1) would force the same constant on all level-one cells; continuity at q_1 and q_2 would then give $h_1(q_1) = h_1(q_2)$, a contradiction. We may therefore choose $w \in W_1$ such that $h_{1,w} := h_1 \circ F_w$ is nonconstant. Regularity and the generalized contraction property in Theorem 6.7 give $\chi_w \in \mathcal{F}_1$ such that $\chi_w = 1$ on $F_w G$ and $\chi_w = 0$ on B. Hence, by (2.13),

$$C_1 := \inf_{z \in F_w G} R_{\mathcal{E}_1}(z, B) \geq \mathcal{E}_1(\chi_w)^{-1} > 0.$$

For $x, y \in V_0$ and $\tau \in W_n$, apply the harmonic oscillation estimate Proposition 2.10 to h_1 at the points $F_w F_\tau x, F_w F_\tau y$. Since these points lie in $F_w G$, (6.31) together with (6.30) and the lower bound $C_1 > 0$ give a constant $C_2 > 0$ such that

$$R_{\mathcal{E}_1}(F_w F_\tau x, F_w F_\tau y) \leq C_2 \gamma_{1,w} \gamma_{1,\tau},$$

The harmonic oscillation estimate consequently gives a constant $C_3 = C_3(w) > 0$, independent of n, τ, x, and y, such that

$$\operatorname{osc}_{V_0}(h_{1,w} \circ F_\tau) \leq C_3 \gamma_{1,\tau}^{1/(p_1-1)}, \qquad \tau \in W_n. \tag{8.2}$$

By homogeneity and compactness of the unit-oscillation sphere in $\mathbb{R}^{V_0}/\mathbb{R}\mathbf{1}_{V_0}$, there exists $C_4 > 0$ such that

$$\mathcal{D}_2(u) \leq C_4 \operatorname{osc}_{V_0}(u)^{p_2-p_1} \mathcal{D}_1(u), \qquad u \in \mathbb{R}^{V_0}.$$

Using (8.2) in the level-n energy therefore gives a constant $C_5 = C_4 C_3(w)^{p_2-p_1} > 0$ such that

$$\mathcal{D}_2^{(n)}(h_{1,w}|_{\overline{V}_n}) \leq C_5 \sum_{\tau \in W_n} \gamma_{2,\tau}^{-1} \gamma_{1,\tau}^{(p_2-1)/(p_1-1)} \gamma_{1,\tau}^{-1} \mathcal{D}_1(h_{1,w} \circ F_\tau|_{V_0}). \tag{8.3}$$

If $s_2 > s_1$, set $q = (s_1/s_2)^{p_2-1} < 1$. For $\tau \in W_n$, put $\ell_n(\tau) := |\tau| + n$. Since $\tau \in W_n$, the definition of the standard weights gives $\gamma_{i,\tau} = r_i^{|\tau|+n} = r_i^{\ell_n(\tau)}$, and $\ell_n(\tau) \geq n$. Hence

$$\gamma_{2,\tau}^{-1} \gamma_{1,\tau}^{(p_2-1)/(p_1-1)} = q^{\ell_n(\tau)} \leq q^n.$$

Thus (8.3) yields a constant $C_6 > 0$ satisfying

$$\mathcal{D}_2^{(n)}(h_{1,w}|_{\overline{V}_n}) \leq C_6 q^n \mathcal{E}_1(h_{1,w}) \longrightarrow 0.$$

This contradicts (6.22), since $h_{1,w}|_{V_0}$ is nonconstant and

$$0 < \mathcal{D}_2(h_{1,w}|_{V_0}) \le \mathcal{D}_2^{(n)}(h_{1,w}|_{\overline{V}_n}).$$

Therefore $s_2 \le s_1$, proving (8.1).

Finally, $0 < s_2 \le s_1 < 1$ and $p_2 > p_1$ give

$$r_{p_2} = s_2^{p_2-1} \le s_1^{p_2-1} < s_1^{p_1-1} = r_{p_1}.$$

Hence both r_p and $\lambda_p = r_p^2$ are strictly decreasing. □

8.2 Continuity

To prove continuity, we establish compactness of normalized p_n-resistance forms on V_0 as $p_n \to p$ and combine it with the continuity theorem for varying exponents in Theorem 4.4. It is convenient to use the parameter $s = r^{1/(p-1)} \in (\rho, 1)$, so that $r = s^{p-1}$.

Throughout this subsection, we identify $\mathcal{L}(V_0)$ with $\mathbb{R}^3$ as in Definition 4.1 and interpret convergence of resistance forms with varying exponent p as local uniform convergence on $\mathcal{L}(V_0)$. Recall from Section 5 that conductance is defined by $C_{\mathcal{D}}(x,y) = R_{\mathcal{D}}(x,y)^{-1}$. Thus, whenever $\mathcal{D}_n \in \mathcal{M}_{p_n}(V_0)$, we use

$$C_{\mathcal{D}_n}(q_0,q_1) = R_{\mathcal{D}_n}(q_0,q_1)^{-1},$$

where $R_{\mathcal{D}_n}$ is the effective p_n-resistance associated with $\mathcal{D}_n$.

Lemma 8.2. *Let $p_n \to p \in (1,\infty)$ and let $\mathcal{D}_n \in \mathcal{M}_{p_n}^{\mathrm{sym}}(V_0)$ satisfy*

$$\mathcal{D}_n(u_*) = 1, \qquad \inf_n C_{\mathcal{D}_n}(q_0,q_1) > 0,$$

where $u_* = (1/2, 0, 1)$ is as in equation (5.1). Then there exist a subsequence $\{\mathcal{D}_{n_k}\}$ and $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$ such that $\mathcal{D}_{n_k} \to \mathcal{D}$ locally uniformly on $\mathcal{L}(V_0)$ and $\mathcal{D}(u_*) = 1$.

Proof. Set $\mathcal{N}_n := \mathcal{D}_n^{1/p_n}$ and $C_1 := \inf_n C_{\mathcal{D}_n}(q_0,q_1) > 0$. By Lemma 5.4,

$$\frac{1}{2} C_1^{1/p_n} \operatorname{osc}_{V_0}(v) \le \mathcal{N}_n(v) \le 2 \operatorname{osc}_{V_0}(v), \qquad v \in \mathcal{L}(V_0).$$

Since $p_n \to p$, there exists $C_2 > 0$ such that

$$C_2 \operatorname{osc}_{V_0}(v) \le \mathcal{N}_n(v) \le 2 \operatorname{osc}_{V_0}(v) \tag{8.4}$$

for all sufficiently large n. Moreover,

$$|\mathcal{N}_n(u) - \mathcal{N}_n(v)| \le \mathcal{N}_n(u-v) \le 2 \operatorname{osc}_{V_0}(u-v).$$

Thus $\{\mathcal{N}_n\}$ is locally uniformly bounded and equicontinuous on $\mathcal{L}(V_0) \simeq \mathbb{R}^3$. By the local Arzelà–Ascoli theorem, there are a subsequence $\{\mathcal{N}_{n_k}\}$ and a continuous seminorm $\mathcal{N}$ such that $\mathcal{N}_{n_k} \to \mathcal{N}$ locally uniformly on $\mathcal{L}(V_0)$. Relabeling this subsequence as $\{\mathcal{N}_n\}$, set $\mathcal{D} := \mathcal{N}^p$. Since $p_n \to p$ along the relabeled subsequence, $\mathcal{D}_n = \mathcal{N}_n^{p_n} \to \mathcal{D}$ locally uniformly on $\mathcal{L}(V_0)$. The reflection symmetry, and the normalization pass to the limit. Passing to the limit in (8.4) gives $C_2 \operatorname{osc}_{V_0}(v) \le \mathcal{N}(v)$, so $\mathcal{D}$ is nondegenerate.

We next verify the generalized contraction property $(\mathrm{RF5})_p$, and then complete the verification of $(\mathrm{RF1})_p$. Let $q \in (0,p]$, $t \in [p,\infty]$, and let $T = (T_1,\dots,T_k) : \mathbb{R}^m \to \mathbb{R}^k$ satisfy $T(0) = 0$ and

$$\|T(x) - T(y)\|_{\ell^t} \le \|x - y\|_{\ell^q}, \qquad x, y \in \mathbb{R}^m.$$

Set $q_n := q \wedge p_n$ and $t_n := t \vee p_n$. Using the standard monotonicity of finite-dimensional ℓ^a quasi-norms, $\|z\|_{\ell^b} \le \|z\|_{\ell^a}$ for $0 < a \le b \le \infty$,

$$\|T(x) - T(y)\|_{\ell^{t_n}} \le \|T(x) - T(y)\|_{\ell^t} \le \|x - y\|_{\ell^q} \le \|x - y\|_{\ell^{q_n}}.$$

Thus T satisfies the contraction hypothesis for $(\mathrm{RF5})_{p_n}$. For $u = (u_1, \dots, u_m) \in \mathcal{L}(V_0)^m$, define $T_j(u)$ pointwise by

$$T_j(u)(x) := T_j(u_1(x), \dots, u_m(x)), \qquad x \in V_0, \quad 1 \le j \le k.$$

Then

$$\left\| \left(\mathcal{N}_n(T_j(u))\right)_{j=1}^k \right\|_{\ell^{t_n}} \le \left\| \left(\mathcal{N}_n(u_i)\right)_{i=1}^m \right\|_{\ell^{q_n}}.$$

Passing to the limit, using $q_n \to q$, $t_n \to t$, and $\mathcal{N}_n(v) \to \mathcal{N}(v)$ for every fixed $v \in \mathcal{L}(V_0)$, together with continuity of finite-dimensional ℓ^a quasi-norms gives

$$\left\| \left(\mathcal{D}(T_j(u))^{1/p}\right)_{j=1}^k \right\|_{\ell^t} \le \left\| \left(\mathcal{D}(u_i)^{1/p}\right)_{i=1}^m \right\|_{\ell^q}.$$

Hence $\mathcal{D}$ satisfies $(\mathrm{RF5})_p$. Since $\mathcal{N}$ is a seminorm and the nondegeneracy of $\mathcal{D}$ was established above, $\mathcal{D}$ also satisfies $(\mathrm{RF1})_p$. By Remark 2.2, $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$. □

Theorem 8.3. *The maps $p \mapsto r_p^{1/(p-1)}$, $p \mapsto r_p$, and $p \mapsto \lambda_p = r_p^2$ are continuous on $(1, \infty)$.*

Proof. Set $s_p := r_p^{1/(p-1)} \in (\rho, 1)$. Let $p_n \to p \in (1, \infty)$ and write $s_n := s_{p_n}$ and $r_n := r_{p_n} = s_n^{p_n - 1}$. Choose $1 < p_- < p < p_+ < \infty$. For all sufficiently large n, $p_- < p_n < p_+$, and the monotonicity in Theorem 8.1 gives $s_{p_+} \le s_n \le s_{p_-}$. Thus $\{s_n\}$ is contained in a compact subinterval of $(\rho, 1)$.

It is enough to identify the limit of an arbitrary convergent subsequence. After relabeling, suppose that $s_n \to s \in (\rho, 1)$ and hence $r_n \to r := s^{p-1}$.

For each n, choose a normalized standard eigenform $\mathcal{D}_n \in \mathcal{M}_{p_n}^{\mathrm{sym}}(V_0)$ such that

$$\mathcal{D}_n(u_*) = 1, \qquad \mathcal{R}_{p_n, r_n} \mathcal{D}_n = r_n^2 \mathcal{D}_n.$$

Recall that $\widetilde{\mathcal{R}}_{p,r}$ denotes the normalized renormalization operator defined in (5.2). Since $\mathcal{D}_n(u_*) = 1$, the preceding relation is equivalent to $\mathcal{D}_n = \widetilde{\mathcal{R}}_{p_n, r_n} \mathcal{D}_n$. Hence Proposition 5.3 gives $C_{\mathcal{D}_n}(q_0, q_1) \ge c_{p_n, r_n}$.

With $r_n = s_n^{p_n - 1}$, the explicit bound in that proposition is

$$c_{p_n, r_n} = \frac{1}{b_{p_n}} (1 - s_n)^{p_n - 1} \frac{s_n^{p_n - 1} - \rho^{p_n - 1}}{s_n^{p_n - 1} \rho^{p_n - 1}}.$$

Since $b_p = 2^p c_1^p c_2$, this expression is continuous and strictly positive on compact subsets of $(1, \infty) \times (\rho, 1)$. Consequently, $\inf_n C_{\mathcal{D}_n}(q_0, q_1) > 0$.

By Lemma 8.2, after passing to a further subsequence, $\mathcal{D}_n \to \mathcal{D}$ uniformly on every compact subset of $\mathbb{R}^3$ for some normalized $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$.

Applying Theorem 4.4 and passing to the limit in the standard eigenvalue equation, we obtain $\mathcal{R}_{p,r} \mathcal{D} = r^2 \mathcal{D}$. Thus r is a standard renormalization factor for exponent p. By the uniqueness in Theorem 5.10, $r = r_p$, and therefore $s = r^{1/(p-1)} = r_p^{1/(p-1)} = s_p$.

Every convergent subsequence of $\{s_n\}$ has the same limit s_p; hence $s_n \to s_p$. Finally, $r_{p_n} = s_n^{p_n - 1} \to s_p^{p-1} = r_p$, and the continuity of $p \mapsto \lambda_p = r_p^2$ follows immediately. □

We now combine the results of Sections 8.1 and 8.2 to complete the proof of Theorem 3.

Proof of Theorem 3. The non-increasing property of $p \mapsto r_p^{1/(p-1)}$, together with the strict decrease of $p \mapsto r_p$ and $p \mapsto \lambda_p = r_p^2$, is exactly Theorem 8.1. The continuity of all three maps is given by Theorem 8.3. These two results prove the theorem. □

Appendix A One-vertex Perron–Frobenius asymptotics

This appendix proves Theorem 7.2 by modifying Kajino and Shimizu's Perron–Frobenius argument in [18, Theorem 3.20] to accommodate the countable level-one graph and the reflection fixing q_0. We first establish the localized and strict comparison results and then derive the top-cell estimates.

Throughout the first part of this appendix, fix $p \in (1,\infty)$, $r \in (\rho^{p-1},1)$, and a p-resistance form $\mathcal{D} \in \mathcal{M}_p(V_0)$ satisfying $\mathcal{R}_{p,r}\mathcal{D} = \lambda\mathcal{D}$. Let $(\mathcal{E},\mathcal{F})$ be the global p-resistance form on G constructed from this eigenpair in Section 6. Recall that $\mathcal{D}^{(m)}$ denotes the compatible level-m form on $\overline{V}_m$ defined in (6.4), obtained by summing the weighted copies of $\mathcal{D}$ over the cells in W_m. Thus $\mathcal{F} = \{f \in C(G) : \lim_{m\to\infty} \mathcal{D}^{(m)}(f|_{\overline{V}_m}) < \infty\}$ and $\mathcal{E}(f) = \lim_{m\to\infty} \mathcal{D}^{(m)}(f|_{\overline{V}_m})$ for $f \in \mathcal{F}$. The localized restriction and comparison results through Proposition A.3 require no symmetry of $\mathcal{D}$.

Lemma A.1. *Let $U \subsetneq G$ be open and let $f \in \mathcal{F}$ satisfy $f = 0$ on $\partial_G U$. Then $f\mathbf{1}_U \in \mathcal{F}$.*

Proof. For $n \geq 1$, define $\varphi_n(t) := t - \big(n^{-1} \wedge (t \vee (-n^{-1}))\big)$ for $t \in \mathbb{R}$. Then φ_n is 1-Lipschitz, $\varphi_n(t) \to t$, and Proposition 2.13 gives

$$\mathcal{E}\big(f - \varphi_n(f)\big) \longrightarrow 0. \tag{A.1}$$

Because $\varphi_n(f)$ vanishes where $|f| \leq n^{-1}$, continuity of f and the condition $f = 0$ on $\partial_G U$ show that $K_n := \overline{U} \cap \operatorname{supp} \varphi_n(f)$ is a compact subset of U. By regularity and the generalized contraction property in Theorem 6.7 , there is $\eta_n \in \mathcal{F}$ such that

$$0 \leq \eta_n \leq 1, \qquad \eta_n = 1 \text{ on } K_n, \qquad \eta_n = 0 \text{ on } G \setminus U.$$

Since Proposition 2.4 (2) shows that $\mathcal{F}$ is an algebra, $g_n := \varphi_n(f)\eta_n = \varphi_n(f)\mathbf{1}_U \in \mathcal{F}$.

For $m,n \geq 1$, set $d_{m,n} := \varphi_n(f) - \varphi_m(f)$, $a_{m,n} := g_n - g_m$, and $b_{m,n} := d_{m,n} - a_{m,n}$. Since $g_k = \varphi_k(f)\mathbf{1}_U$ for every k, one has $a_{m,n} = d_{m,n}\mathbf{1}_U$ and $b_{m,n} = d_{m,n}\mathbf{1}_{G\setminus U}$, and hence $a_{m,n}b_{m,n} = 0$ on G. Applying the strong-locality identity (6.27) with $f = a_{m,n}$, $g = b_{m,n}$, and $h = 0$ gives

$$\mathcal{E}(d_{m,n}) = \mathcal{E}(a_{m,n} + b_{m,n}) = \mathcal{E}(a_{m,n}) + \mathcal{E}(b_{m,n}).$$

Therefore

$$\mathcal{E}(g_n - g_m) = \mathcal{E}(a_{m,n}) \leq \mathcal{E}(d_{m,n}).$$

Consequently,

$$\mathcal{E}(g_n - g_m)^{1/p} \leq \mathcal{E}\big(f - \varphi_n(f)\big)^{1/p} + \mathcal{E}\big(f - \varphi_m(f)\big)^{1/p} \longrightarrow 0$$

by (A.1). Thus $\{g_n\}$ is Cauchy in the energy seminorm.

Fix $x_0 \in G \setminus U$. Since $g_n(x_0) = 0$, completeness in Definition 2.1 $(\mathrm{RF}2)_p$ and (6.23) give a function $g \in \mathcal{F}$ such that $g_n \to g$ pointwise on G. On the other hand, $\varphi_n(f) \to f$, so $g_n \to f\mathbf{1}_U$ pointwise on G. Hence $g = f\mathbf{1}_U$, as required. □

Proposition A.2. *Let $U \subsetneq G$ be open and let $f,g \in \mathcal{H}_{\mathcal{E},G\setminus U}$. If $f \leq g$ on $\partial_G U$, then $f \leq g$ on U.*

Proof. Set $w := (f-g)^+$. By the generalized contraction property in Theorem 6.7 , $w \in \mathcal{F}$. The boundary assumption gives $w = 0$ on $\partial_G U$, so Lemma A.1 implies $w\mathbf{1}_U \in \mathcal{F}$. Therefore $\psi := w\mathbf{1}_{G\setminus U} = w - w\mathbf{1}_U \in \mathcal{F}$. Define $\widehat{f} := f - \psi$ and $\widehat{g} := g + \psi$. These functions agree with f and g, respectively, on U. We first check that they remain harmonic on U. Let $\xi \in \mathcal{F}$ satisfy $\xi|_{G\setminus U} = 0$. Since $\psi\xi = 0$, the strong-locality identity (6.27) gives, for every $t \in \mathbb{R}$,

$$\mathcal{E}(f - \psi + t\xi) + \mathcal{E}(f) = \mathcal{E}(f - \psi) + \mathcal{E}(f + t\xi).$$

Subtracting the equality at $t = 0$, dividing by pt, and then letting $t \to 0$, (2.5) and the harmonicity of f yield

$$\mathcal{E}(\widehat{f};\xi) = \mathcal{E}(f;\xi) = 0.$$

The same argument with $g + \psi$ gives $\mathcal{E}(\widehat{g};\xi) = 0$. Hence $\widehat{f}, \widehat{g} \in \mathcal{H}_{\mathcal{E},G\setminus U}$.

For $x \in G\setminus U$, if $f(x) \le g(x)$ then $\psi(x) = 0$, while if $f(x) > g(x)$ then $\psi(x) = f(x) - g(x)$. In either case, $\widehat{f}(x) \le \widehat{g}(x)$. Applying the weak comparison principle Proposition 2.9 with boundary set $G \setminus U$ gives $\widehat{f} \le \widehat{g}$ on G. Since $\widehat{f} = f$ and $\widehat{g} = g$ on U, the desired inequality follows. □

Proposition A.3. *Let $f, g \in \mathcal{H}_{\mathcal{E},V_0}$ satisfy $f|_{V_0} \le g|_{V_0}$ and $f|_{V_0} \ne g|_{V_0}$. Then*

$$f(x) < g(x), \qquad x \in V_1 \setminus \{q_0\}.$$

Proof. By the weak comparison principle in Proposition 2.9, we have $f \le g$ on G. Suppose that equality holds at some $x \in V_1 \setminus \{q_0\}$. Choose $w \in W_1 \setminus \{0\}$ such that $x \in F_w V_0$, which is possible by Proposition 3.3 (1) .

We first prove a local equality statement. Set

$$U := f \circ F_w|_{V_0}, \qquad V := g \circ F_w|_{V_0}, \qquad B := \{y \in V_0 : U(y) < V(y)\}.$$

Since $U = V$ at $F_w^{-1}x$, the set B is proper. Suppose that $B \ne \varnothing$, and let $a := \min_{y\in B}\big(V(y) - U(y)\big) > 0$ and $u := U + a\mathbf{1}_B^{V_0}$.

Fix $y \in V_0 \setminus B$ and set $z := F_w y$. For every $\tau \in W_1$ such that $z \in F_\tau V_0$, Proposition 2.6 gives

$$\mathcal{D}\big(f \circ F_\tau|_{V_0}; \mathbf{1}^{V_0}_{F_\tau^{-1}z}\big) \ge \mathcal{D}\big(g \circ F_\tau|_{V_0}; \mathbf{1}^{V_0}_{F_\tau^{-1}z}\big).$$

Applying (7.1) to f and g shows that the weighted sum of these nonnegative differences is zero; hence each difference vanishes. In particular,

$$\mathcal{D}(U; \mathbf{1}_y^{V_0}) = \mathcal{D}(V; \mathbf{1}_y^{V_0}).$$

Since $U \le u \le V$ and $U(y) = u(y) = V(y)$, another application of Proposition 2.6 yields

$$\mathcal{D}(U; \mathbf{1}_y^{V_0}) = \mathcal{D}(u; \mathbf{1}_y^{V_0}) = \mathcal{D}(V; \mathbf{1}_y^{V_0}), \qquad y \in V_0 \setminus B.$$

Summing over $V_0 \setminus B$ and using $\mathcal{D}(U; \mathbf{1}_{V_0}) = \mathcal{D}(u; \mathbf{1}_{V_0}) = 0$ gives

$$\mathcal{D}(U; \mathbf{1}_B^{V_0}) = \mathcal{D}(u; \mathbf{1}_B^{V_0}).$$

On the other hand, since $\mathbf{1}_B^{V_0} \notin \mathbb{R}\mathbf{1}_{V_0}$ and $a > 0$, the strict monotonicity in Theorem 2.5 gives

$$\mathcal{D}(U; \mathbf{1}_B^{V_0}) < \mathcal{D}(u; \mathbf{1}_B^{V_0}),$$

a contradiction. Thus $B = \varnothing$, and equality holds at every vertex of $F_w V_0$.

Starting with $F_w V_0$ and repeating this local argument along finite cell-edge paths in the connected graph from Proposition 3.3 (1) that avoid q_0 gives $f = g$ at every vertex of $V_1 \setminus \{q_0\}$. In particular, equality holds at the two lower vertices of the top cell $F_0 V_0$. If equality did not hold at q_0, the same local argument on $F_0 V_0$, using the vertex equations at its two lower vertices, would force equality at all three vertices. Hence equality also holds at q_0, and $f = g$ on V_1. Since $V_0 \subset \overline{V_1}$ and f, g are continuous on G, it follows that $f = g$ on V_0, contradicting $f|_{V_0} \ne g|_{V_0}$.

Thus $f(x) < g(x)$ for every $x \in V_1 \setminus \{q_0\}$. □

From now on, assume in addition that $\mathcal{D} \in \mathcal{M}_p^{\mathrm{sym}}(V_0)$. We use the functions h_{s} and h_{a} defined in (7.2).

Proposition A.4. *There exists $\eta \in \mathbb{R}$ such that*

$$h_{\mathrm{s}} \circ F_0 = \lambda^{1/(p-1)} h_{\mathrm{s}} = \kappa h_{\mathrm{s}}, \qquad h_{\mathrm{a}} \circ F_0 = \eta h_{\mathrm{a}}, \qquad |\eta| < \kappa. \tag{A.2}$$

Proof. Lemma 7.1 and uniqueness of harmonic extensions in Theorem 2.8 , together with reflection symmetry, give $h_{\mathrm{s}} \circ F_0 = c\, h_{\mathrm{s}}$ and $h_{\mathrm{a}} \circ F_0 = \eta h_{\mathrm{a}}$ for some $c, \eta \in \mathbb{R}$. Since q_0 belongs only to $F_0^n G$ at level n, the final assertion of Theorem 2.8 gives, for every $h \in \mathcal{H}_{\mathcal{E},V_0}$ and $n \geq 1$,

$$\mathcal{D}(h|_{V_0}; \mathbf{1}_{q_0}^{V_0}) = \lambda^{-n} \mathcal{D}(h \circ F_0^n|_{V_0}; \mathbf{1}_{q_0}^{V_0}). \tag{A.3}$$

First, since $h_{\mathrm{s}}|_{V_0} + \mathbf{1}_{q_0}^{V_0} = \mathbf{1}_{V_0}$, the strict monotonicity in (2.6) gives

$$\mathcal{D}(h_{\mathrm{s}}|_{V_0}; \mathbf{1}_{q_0}^{V_0}) < \mathcal{D}(\mathbf{1}_{V_0}; \mathbf{1}_{q_0}^{V_0}) = 0.$$

Applying (A.3) to h_{s} with $n = 1$ and using the homogeneity in (2.7), we may cancel this nonzero quantity and obtain

$$\operatorname{sgn}(c)|c|^{p-1} = \lambda.$$

As $\lambda > 0$, we have $c > 0$, and hence

$$c = \lambda^{1/(p-1)} = \kappa.$$

On V_0, $-h_{\mathrm{s}} \leq h_{\mathrm{a}} \leq h_{\mathrm{s}}$, with equality in neither comparison, so the strong comparison principle Proposition A.3 at $F_0 q_1$ gives $-\kappa < \eta < \kappa$. □

Proposition A.5. *For every $h \in \mathcal{H}_{\mathcal{E},V_0}$,*

$$\operatorname{osc}_G(h \circ F_0) \leq \kappa \operatorname{osc}_G(h). \tag{A.4}$$

Equality holds when $h(q_1) = h(q_2)$, and the inequality is strict otherwise.

Proof. Subtracting $h(q_0)\mathbf{1}_G$, we may assume that $h(q_0) = 0$. If $h(q_1) = h(q_2)$, then h is a multiple of h_{s}, and the equality follows from the first identity in Proposition A.4, namely (A.2). Otherwise, after changing the sign if necessary, write $h(q_1) = a < b = h(q_2)$. The weak comparison principle Proposition 2.9 and the strong comparison principle Proposition A.3 give

$$\kappa a < h(F_0 q_i) < \kappa b, \qquad i = 1, 2.$$

Together with $h(F_0 q_0) = h(q_0) = 0$ and the boundary maximum principle from Proposition 2.9 for $h \circ F_0$, these strict inequalities yield $\operatorname{osc}_G(h \circ F_0) < \kappa \operatorname{osc}_G(h)$. Together with the symmetric case, this proves (A.4). □

Proof of Theorem 7.2. After subtracting $h(q_0)\mathbf{1}_G$, assume $h(q_0) = 0$ and set

$$h_n := \kappa^{-n} h \circ F_0^n, \qquad a_n := h_n(q_1), \qquad b_n := h_n(q_2).$$

After changing the sign of h, assume $a_0 \leq b_0$. Suppose that $a_n \leq b_n$. Comparing $a_n h_{\mathrm{s}}$, h_n, and $b_n h_{\mathrm{s}}$ on V_0 and applying Proposition 2.9 gives

$$a_n \leq a_{n+1}, \qquad b_{n+1} \leq b_n.$$

Set $U := \{x \in G \setminus V_0 : |x - q_1| > |x - q_2|\}$, the open half of G adjacent to q_2. Reflection symmetry and $a_n \leq b_n$ give $h_n \circ \sigma \leq h_n$ on $\partial_G U$. Applying the localized weak comparison principle Proposition A.2 on U and evaluating at $F_0 q_2$ gives $a_{n+1} \leq b_{n+1}$. Hence, by induction, for some $a, b \in \mathbb{R}$,

$$a_n \uparrow a, \qquad b_n \downarrow b, \qquad a \leq b.$$

Let $h_\infty := h^{\mathcal{E}}_{V_0}[(0,a,b)]$. The weak comparison principle Proposition 2.9 gives the L^∞-stability estimate

$$\left\|h^{\mathcal{E}}_{V_0}[u] - h^{\mathcal{E}}_{V_0}[v]\right\|_{\infty,G} \le \|u-v\|_{\infty,V_0}, \qquad u,v \in \mathbb{R}^{V_0}.$$

Since the boundary values $(0,a_n,b_n)$ converge to $(0,a,b)$, this estimate gives $h_n \to h_\infty$ uniformly on G. Passing to the limit in $h_{n+1} = \kappa^{-1} h_n \circ F_0$ gives

$$h_\infty = \kappa^{-1} h_\infty \circ F_0.$$

The strict part of Proposition A.5 therefore forces $a = b$, so $h_\infty = a h_{\mathrm{s}}$.

By (A.3) and $\kappa^{p-1} = \lambda$,

$$\mathcal{D}(h_n|_{V_0}; \mathbf{1}^{V_0}_{q_0}) = \mathcal{D}(h|_{V_0}; \mathbf{1}^{V_0}_{q_0}).$$

Passing to the limit in the preceding identity and using $h_\infty = a h_{\mathrm{s}}$ together with the homogeneity relation in (2.7), we obtain $a = c_0(h)$ by (7.3). Finally, Lemma 7.1 shows that each $h_n \in \mathcal{H}_{\mathcal{E},V_0}$, so (2.10) gives

$$\mathcal{E}(h_n) = \mathcal{D}(h_n|_{V_0}) \longrightarrow \mathcal{D}(h_\infty|_{V_0}) = \mathcal{E}(h_\infty).$$

Since $h_n \to h_\infty$ uniformly on G, Proposition 2.12(2) then yields $\mathcal{E}(h_n - h_\infty) \to 0$. □

References

[1] C. Bandt and H. Rao, *Topology and separation of self-similar fractals in the plane*, Nonlinearity **20** (2007), no. 6, 1463–1474.

[2] M. T. Barlow and R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet*, Ann. Inst. H. Poincaré Probab. Statist. **25** (1989), no. 3, 225–257.

[3] M. T. Barlow and R. F. Bass, *Brownian motion and harmonic analysis on Sierpiński carpets*, Canad. J. Math. **51** (1999), no. 4, 673–744.

[4] M. T. Barlow, R. F. Bass, T. Kumagai and A. Teplyaev, *Uniqueness of Brownian motion on Sierpiński carpets*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 3, 655–701.

[5] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields **79** (1988), no. 4, 543–623.

[6] A. Björn and J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics, vol. 17, European Mathematical Society, Zürich, 2011.

[7] S. Cao, Q. Gu and H. Qiu, *p-energies on p.c.f. self-similar sets*, Adv. Math. **405** (2022), Paper No. 108517.

[8] S. Cao and H. Qiu, *Resistance forms on self-similar sets with finite ramification of finite type*, Potential Anal. **54** (2021), 581–606.

[9] S. Cao and H. Qiu, *Brownian motion on the golden ratio Sierpiński gasket*, Proc. Roy. Soc. Edinburgh Sect. A **154** (2024), no. 3, 699–726.

[10] S. Cao and H. Qiu, *Dirichlet forms on unconstrained Sierpiński carpets*, Probab. Theory Related Fields **189** (2024), no. 1–2, 613–657.

[11] S. Cao and H. Qiu, *Uniqueness and convergence of resistance forms on unconstrained Sierpiński carpets*, Trans. Amer. Math. Soc., published electronically February 2026.

[12] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd rev. and extended ed., De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter, Berlin, 2010.

[13] S. Goldstein, *Random walks and diffusions on fractals*, in Percolation Theory and Ergodic Theory of Infinite Particle Systems (Minneapolis, Minn., 1984–85), H. Kesten (ed.), IMA Vol. Math. Appl. 8, Springer, New York– Berlin, 1987, 121–129.

[14] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993.

[15] P. E. Herman, R. Peirone and R. S. Strichartz, *p-energy and p-harmonic functions on Sierpiński gasket type fractals*, Potential Anal. **20** (2004), no. 2, 125–148.

[16] N. Kajino and R. Shimizu, *Contraction properties and differentiability of p-energy forms with applications to nonlinear potential theory on self-similar sets*, 2024, arXiv:2404.13668v4 [math.FA].

[17] N. Kajino and R. Shimizu, *Korevaar–Schoen p-energy forms and associated p-energy measures on fractals*, in Facets of Contemporary Analysis, Geometry and Non-Euclidean Statistics (T. Shioya, S. Shimizu and W. Tuschmann, eds.), Tohoku Ser. Math. Sci. 1, Springer, Singapore, 2026, 113–182.

[18] N. Kajino and R. Shimizu, *p-energy forms on fractals: Recent progress*, in From Classical Analysis to Analysis on Fractals: A Tribute to Robert Strichartz, vol. 2, Appl. Numer. Harmon. Anal., Springer, 2026, to appear, arXiv:2501.09002.

[19] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. **6** (1989), no. 2, 259–290.

[20] J. Kigami, *Analysis on Fractals*, Cambridge University Press, Cambridge, 2001.

[21] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, Mem. Amer. Math. Soc. **216** (2012), no. 1015, vi+132 pp.

[22] J. Kigami, *Conductive homogeneity of compact metric spaces and construction of p-energy*, Mem. Eur. Math. Soc. **5**, European Mathematical Society, Berlin, 2023.

[23] N. J. Korevaar and R. M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), no. 4, 561–659.

[24] S. Kusuoka, *A diffusion process on a fractal*, in Probabilistic Methods in Mathematical Physics (Katata and Kyoto, 1985), K. Itô and N. Ikeda (eds.), Academic Press, Boston, MA, 1987, 251–274.

[25] S. Kusuoka and X. Y. Zhou, *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields **93** (1992), no. 2, 169–196.

[26] K.-S. Lau and S.-M. Ngai, *A generalized finite type condition for iterated function systems*, Adv. Math. **208** (2007), no. 2, 647–671.

[27] T. Lindstrøm, *Brownian Motion on Nested Fractals*, Mem. Amer. Math. Soc. **83** (1990), no. 420, 128 pp.

[28] M. Moran, *Hausdorff measure of infinitely generated self-similar sets*, Monatsh. Math. **122** (1996), no. 4, 387–399.

[29] M. Murugan and R. Shimizu, *First-order Sobolev spaces, self-similar energies and energy measures on the Sierpiński carpet*, Comm. Pure Appl. Math. **78** (2025), no. 9, 1523–1608.

[30] S.-M. Ngai and Y. Wang, *Hausdorff dimension of self-similar sets with overlaps*, J. London Math. Soc. (2) **63** (2001), no. 3, 655–672.

[31] H. Rao and Z.-Y. Wen, *A class of self-similar fractals with overlap structure*, Adv. in Appl. Math. **20** (1998), no. 1, 50–72.

[32] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoam. **16** (2000), no. 2, 243–279.

[33] R. Shimizu, *Construction of p-energy and associated energy measures on Sierpiński carpets*, Trans. Amer. Math. Soc. **377** (2024), no. 2, 951–1032.

[34] R. S. Strichartz and C. Wong, *The p-Laplacian on the Sierpiński gasket*, Nonlinearity **17** (2004), no. 2, 595–616.

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