ONE-POINT INTERSECTION AND SAME-LEVEL SEPARATION IN PLANAR NESTED FRACTALS

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Abstract. The one-point intersection property and the same-level separation property have long been used as geometric assumptions in analysis on nested fractals. We prove that both properties hold for every planar nested fractal. Distinct cells of the same level meet in at most one point, and the distance between any two disjoint cells of the same level is bounded below by a fixed positive multiple of their diameter. We also prove both properties in any dimension when the fractal lies in the smallest affine subspace containing its boundary vertices.

1. Introduction

Nested fractals were introduced by Lindstrøm in his 1990 monograph [13], in which he constructed Brownian motion on these sets. With a finite boundary and reflection symmetry, they form an important class of self-similar sets in analysis on fractals, including the Sierpiński gasket and the Vicsek set. Following this construction, Fukushima [5] studied the associated Dirichlet forms, and Kumagai [11] and Fitzsimmons–Hambly–Kumagai [4] obtained estimates for transition densities. Subsequent work has studied function spaces and nonlinear energies on these sets; see, for example, [14, 6, 3, 10]. In this paper, we study two geometric properties underlying these studies.

Let us first introduce the notation and recall the definition of nested fractals. Let $L \in \mathbb{N}$, let $S \subset \mathbb{R}^L$ be a finite set with at least two points, and fix $0 < \rho < 1$. For each $q \in S$, let O_q be an orthogonal linear map of $\mathbb{R}^L$ and define

$$f_q(x) = q + \rho O_q(x - q), \qquad x \in \mathbb{R}^L. \tag{1.1}$$

Thus q is the fixed point of f_q, and all the similarities have contraction ratio ρ. Their *attractor* is the unique nonempty compact set K satisfying $K = \bigcup_{q\in S} f_q(K)$. A fixed point $p \in S$ is called *essential* if there are $r \in S$ and distinct $i, j \in S$ such that $f_i(p) = f_j(r)$. Let V_0 be the set of essential fixed points. We assume that $\#V_0 \geq 2$, where $\#$ denotes cardinality, and refer to V_0 as the *boundary*.

For $n \in \mathbb{N}$, let S^n be the set of words of length n with letters in S. If $w = w_1 \cdots w_n \in S^n$, put

$$|w| = n, \qquad f_w = f_{w_1} \circ \cdots \circ f_{w_n}, \qquad K_w = f_w(K), \qquad V_w = f_w(V_0).$$

We call K_w a *level-n cell* and V_w its boundary vertex set. Set $S^0 = \{\varnothing\}$, where $\varnothing$ is the empty word, and put $|\varnothing| = 0$, $f_\varnothing = \mathrm{id}$, $K_\varnothing = K$ and $V_\varnothing = V_0$. Here id denotes the identity map. For

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$p \in S$ and $k \geq 0$, the word p^k consists of k repetitions of p; in particular, $p^0 = \varnothing$. Products of maps denote composition, and f^k denotes the k-fold iterate of a map f, with $f^0 = \mathrm{id}$.

We use the following equivalent form of Barlow's definition [2, Definition 5.22].

Definition (nested fractal)**.** The attractor K of the system (1.1) is called a *nested fractal* if the following conditions hold.

(NF1) *Open set condition.* There is a nonempty bounded open set $\Omega \subset \mathbb{R}^L$ such that

$$f_q(\Omega) \subset \Omega \quad (q \in S), \qquad f_q(\Omega) \cap f_r(\Omega) = \varnothing \quad (q, r \in S,\ q \neq r).$$

(NF2) *Connectivity.* K is connected.

(NF3) *Symmetry.* For distinct $x, y \in V_0$, let g_{xy} be reflection in their perpendicular bisector. Then g_{xy} permutes $\{V_w : w \in S^n\}$ for every $n \geq 0$.

(NF4) *Nesting.* For every $n \geq 1$ and distinct $u, v \in S^n$,

$$K_u \cap K_v = V_u \cap V_v. \tag{1.2}$$

We call K *planar* if it is contained in an affine subspace of dimension at most two. For nonempty bounded sets $A, B \subset \mathbb{R}^L$, we write $\mathrm{dist}(A, B)$ for their Euclidean distance and $\operatorname{diam} A$ for the diameter of A.

By (NF4), distinct cells of the same level have only finitely many common points. In this paper, we focus on the following two properties.

One-point intersection property. For every $n \geq 1$ and distinct $u, v \in S^n$,

$$\#(K_u \cap K_v) \leq 1. \tag{1.3}$$

Same-level separation property. There is a constant $c > 0$ such that, for every $n \geq 1$ and $u, v \in S^n$,

$$K_u \cap K_v = \varnothing \quad \Longrightarrow \quad \mathrm{dist}(K_u, K_v) \geq c\rho^n. \tag{1.4}$$

Since $\operatorname{diam} K_w = \rho^{|w|} \operatorname{diam} K$, the estimate (1.4) compares the distance between disjoint cells with their diameter.

These two properties are often assumed in analysis on nested fractals. One-point intersection describes how cells are joined. Fitzsimmons–Hambly–Kumagai stated this property for affine nested fractals in 1994 [4, Proposition 2.2(4)]. In 1997, Kumagai treated it as an additional assumption, noting that the earlier proof required a further hypothesis [12, p. 227, (A5) and the following remark]. In 1998, Barlow listed as an open problem whether one-point intersection follows from the definition of a nested fractal [2, Remark 5.25(2)(c)]. More recently, Alonso Ruiz–Baudoin assumed this property in their work on oscillations of measures of bounded variation [1, Section 2.1]. The one-point intersection property is also assumed by the second author and Wang in their very recent work on connective constants for self-avoiding walks on nested fractal graphs [15, p. 4, (2)]. The need for this assumption in that work motivated the present paper.

Same-level separation controls the distance between disjoint cells. It is also closely related to one-point intersection. A related finite-chain condition already appears in Kumagai's 1993 work on heat-kernel estimates [11, Assumption 2.2]. Related geometric conditions are also used by Hu–Wang in their effective resistance estimates [7, Condition (C1)] and by Radchenko–Zähle in their study of stochastic heat equations [16, Assumption 1]. Later, related assumptions were used by Pietruska-Pałuba–Stós in their study of Poincaré inequalities and Sobolev-space

comparisons [14], and by Gu–Lau in their study of Besov norms [6]. The estimate (1.4) also appears explicitly in Kigami's work on nonlinear energies [10, Assumption 4.41]. More recently, Kajino–Shimizu used a related metric condition in their theorem on Ahlfors regular conformal dimension [8, Assumption B.6 and Theorem B.8].

For same-level separation, several special cases are known. Pietruska-Pałuba–Stós prove their Property (P) when all the defining similarities have the same orthogonal part [14, Remark 1 and Appendix]. Gu–Lau prove (1.4) for homogeneous *post-critically finite* (p.c.f.) fractals defined by homotheties [6, Proposition 2.5]. Kigami proves (1.4) when $O_p = I$ for every $p \in V_0$, where I is the identity linear map [10, Appendix 6.3].

Although these geometric assumptions have been used for many years, it remains open whether both properties follow from the definition of a nested fractal in general. We prove that they do for every planar nested fractal. We also prove them in higher dimensions when the boundary affinely spans the attractor.

For a nonempty set $A \subset \mathbb{R}^L$, let $\mathrm{aff}(A)$ denote its affine hull, the smallest affine subspace containing A, and put $E = \mathrm{aff}(K)$. Our main results are as follows.

Theorem 1.1. *Let K be a nested fractal in $\mathbb{R}^L$. If K has the same-level separation property, then it has the one-point intersection property.*

Theorem 1.2. *Let K be a nested fractal. If K is planar or $\mathrm{aff}(V_0) = E$, then K has the same-level separation property and the one-point intersection property.*

We prove Theorem 1.1 by contradiction. Suppose that two distinct cells of the same level have two common points. By uniqueness of boundary addresses and compactness of the orthogonal group, we find pairs of disjoint cells whose distance divided by their common diameter tends to zero. This contradicts same-level separation.

For Theorem 1.2, we use the sufficient condition in Proposition 3.2: the two similarities contracting toward each common boundary point agree after the same number of iterations. In the plane, we verify this condition using invariant Jordan arcs and their slope parameters, following Tetenov–Yudin–Kadirova [18, Section 5]. The case $\mathrm{aff}(V_0) = E$ uses reflection symmetry instead. Write $E_0 = E - E$ for the direction space of E. Since $f_q(E) = E$, each O_q preserves E_0. We say that an orthogonal map has *finite order* if some positive power of it is the identity. We use reflections to show that $O_p|_{E_0}$ has finite order for every $p \in V_0$, which gives the sufficient condition.

The paper is organized as follows. In Section 2, we give a criterion for same-level separation and prove Theorem 1.1. In Section 3, we prove Theorem 1.2. In Section 4, we discuss earlier results and some open questions. The proof of Lemma 3.4 is given in the appendix.

2. Same-level separation implies one-point intersection

We prove Theorem 1.1 by rescaling cells near a common boundary point. We first recall the uniqueness of boundary addresses.

Set $V_n = \bigcup_{w \in S^n} V_w$ for $n \geq 0$ and $V_* = \bigcup_{n \geq 0} V_n$. An *address* of $x \in K$ is an infinite word $\omega = \omega_1 \omega_2 \cdots$, with letters in S, such that $x \in K_{\omega_1 \cdots \omega_n}$ for every $n \geq 1$. Every point of K has an address, and each infinite word determines a unique point, since the diameters of these cells tend to zero. For $p \in S$, write $p^\infty = ppp \cdots$.

Lemma 2.1. *For every $p \in V_0$ and $n \geq 1$,*

$$\{w \in S^n : p \in K_w\} = \{p^n\}. \tag{2.1}$$

Moreover, $V_0 \cap K_w \subset V_w$ for every finite word w.

Proof. By [2, Proposition 5.24], K is p.c.f. For $p \in V_0$, we have $f_p(p) = p$, so [9, Lemma 1.3.14] shows that p^∞ is the unique address of p. This proves (2.1). If $p \in V_0 \cap K_w$, then $w = p^{|w|}$ and $p = f_w(p) \in V_w$. □

By (NF4) and Lemma 2.1, every address of a first-level intersection point has the form ip^∞, with $i \in S$ and $p \in V_0$.

We now reduce the same-level separation estimate to cells approaching a common boundary point. Define the finite set

$$\mathcal{J} = \{(i, p; j, r) : i, j \in S,\ i \neq j,\ p, r \in V_0,\ f_i(p) = f_j(r)\}.$$

For $J = (i, p; j, r) \in \mathcal{J}$ and $k \geq 0$, put

$$H_{J,k} = (f_j f_r^k)^{-1} f_i f_p^k. \tag{2.2}$$

Since $f_i f_p^k$ and $f_j f_r^k$ have the same contraction ratio, $H_{J,k}$ is an isometry. By (1.1), we have

$$H_{J,k}(x) = r + O_r^{-k} O_j^{-1} O_i O_p^k (x - p), \qquad x \in \mathbb{R}^L. \tag{2.3}$$

For $a, b \in S$, it follows that

$$\operatorname{dist}(K_{ip^k a}, K_{jr^k b}) = \rho^{k+1} \operatorname{dist}\big(H_{J,k}(K_a), K_b\big). \tag{2.4}$$

The two cells on the left have level $k + 2$.

The following proposition expresses same-level separation in terms of these rescaled cells.

Proposition 2.2. *The same-level separation property holds if and only if there is $\varepsilon > 0$ such that, for all $J \in \mathcal{J}$, $k \geq 0$ and $a, b \in S$,*

$$H_{J,k}(K_a) \cap K_b = \varnothing \quad \Longrightarrow \quad \operatorname{dist}\big(H_{J,k}(K_a), K_b\big) \geq \varepsilon. \tag{2.5}$$

Proof. If (1.4) holds with constant c, then (2.4) gives (2.5) with $\varepsilon = c\rho$.

Conversely, suppose (2.5) holds. Let δ be the minimum distance between disjoint first-level cells, and set $\delta = 1$ if there are no such pairs. Thus $\delta > 0$. Fix disjoint level-n cells K_u, K_v, where $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$. For $0 \leq l \leq n$, write $u|_l = u_1 \cdots u_l$ and $v|_l = v_1 \cdots v_l$, with both prefixes empty when $l = 0$. Let

$$t = \min\{1 \leq l \leq n : K_{u|_l} \cap K_{v|_l} = \varnothing\}.$$

Thus t is the first level at which the two cells are disjoint. If $u|_{t-1} = v|_{t-1}$, the level-t cells are images of disjoint first-level cells under the same similarity. Hence

$$\operatorname{dist}(K_{u|_t}, K_{v|_t}) \geq \delta \rho^{t-1}.$$

Otherwise, let $s < t - 1$ be the length of the longest common prefix w of u and v. Choose $x \in K_{u|_{t-1}} \cap K_{v|_{t-1}}$. Put $i = u_{s+1}$ and $j = v_{s+1}$, so $i \neq j$. By (NF4), $f_w^{-1}(x) = f_i(p) = f_j(r)$ for some $p, r \in V_0$. By (2.1), we have $u_{s+2} \cdots u_{t-1} = p^k$ and $v_{s+2} \cdots v_{t-1} = r^k$, where $k = t - s - 2 \geq 0$. Setting $a = u_t$ and $b = v_t$, we obtain

$$K_{u|_t} = f_w(K_{ip^k a}), \qquad K_{v|_t} = f_w(K_{jr^k b}).$$

These sets are disjoint. Equations (2.4) and (2.5) give

$$\operatorname{dist}(K_{u|_t}, K_{v|_t}) \geq \varepsilon\rho^{s+k+1} = \varepsilon\rho^{t-1}.$$

In either case, since the original cells are subsets of these level-t cells,

$$\operatorname{dist}(K_u, K_v) \geq \min\{\delta, \varepsilon\}\rho^{t-1} \geq \min\{\delta, \varepsilon\}\rho^{n-1}.$$

This proves (1.4). □

We now use this criterion to prove Theorem 1.1. We write $|x|$ for the Euclidean norm of $x \in \mathbb{R}^L$.

Proof of Theorem 1.1. To verify (1.3), it suffices to consider first-level cells. Indeed, removing the longest common prefix and applying the corresponding inverse similarity reduces any pair of distinct same-level cells to subsets of distinct first-level cells.

Suppose that K has same-level separation but not one-point intersection. Then two distinct first-level cells K_i, K_j have two distinct common points. Write these points as

$$x = f_i(p) = f_j(r), \qquad y = f_i(q) = f_j(s),$$

where $p, q, r, s \in V_0$, $p \neq q$ and $r \neq s$. Fix $J = (i, p; j, r)$ and write $H_k = H_{J,k}$ for this proof. Then $H_0(q) = s$.

For $k \geq 1$, we first note that (2.1) implies $K_{p^k q} \cap V_0 = \varnothing$. Indeed, if $b \in V_0 \cap K_{p^k q}$, then $p^k q = b^{k+1}$, contrary to $p \neq q$. Since $K_i \cap K_j \subset f_i(V_0)$, we have

$$K_{ip^k q} \cap K_j \subset f_i(K_{p^k q} \cap V_0) = \varnothing.$$

In particular, $H_k(K_q) \cap K_s = \varnothing$ for every $k \geq 1$.

Next, we use compactness of the group $O(L)$ of orthogonal maps of $\mathbb{R}^L$. Choose strictly increasing integers n_l such that $(O_p^{n_l}, O_r^{n_l})$ converges in $O(L) \times O(L)$. By passing to a subsequence, we may also require $k_l = n_{l+1} - n_l \to \infty$. Continuity of inversion and multiplication gives

$$O_p^{k_l} \longrightarrow I, \qquad O_r^{k_l} \longrightarrow I.$$

Equation (2.3) now yields $H_{k_l}(q) \to H_0(q) = s$. Since $q \in K_q$ and $s \in K_s$,

$$0 < \operatorname{dist}\big(H_{k_l}(K_q), K_s\big) \leq |H_{k_l}(q) - s| \longrightarrow 0.$$

This contradicts Proposition 2.2. □

3. Sufficient conditions for the two properties

We now prove Theorem 1.2. After reducing the ambient space to the affine hull of K, we derive a sufficient condition from Proposition 2.2 and verify it in the two cases.

Recall that $E = \operatorname{aff}(K)$. The following lemma allows us to take E as the ambient space.

Lemma 3.1. *The attractor K is a nested fractal in E for the system $\{f_q|_E : q \in S\}$, with the same cells, boundary sets and Euclidean distances.*

Proof. Since $f_q(K) \subset K$, we have $f_q(E) \subset E$. The two affine spaces have the same dimension, so $f_q(E) = E$. Connectivity and nesting are unchanged. To check symmetry, note first that every g_{xy} preserves V_n by (NF3). For any $z \in K$, a level-n cell containing z has a boundary vertex at distance at most $\rho^n \operatorname{diam} K$ from z. Hence V_* is dense in K. It follows

that $g_{xy}(K) = K$ and $g_{xy}(E) = E$. The restriction of g_{xy} to E is reflection in the perpendicular bisector of x, y within E, and still permutes the boundary vertex sets at each level.

Finally, the equivalence of the open set condition and the *strong open set condition* [17, Theorem 2.2] allows us to choose Ω in (NF1) with $\Omega \cap K \neq \varnothing$. The set $\Omega \cap E$ is then nonempty and relatively open in E, and its images are pairwise disjoint and contained in $\Omega \cap E$. This proves the open set condition on E. □

By Lemma 3.1, we may work on E. We identify E isometrically with $\mathbb{R}^d$, where $d = \dim E \geq 1$. We keep the notation f_q and O_q for the restricted similarities and their orthogonal parts.

3.1. **A sufficient condition.** Fix $J = (i, p; j, r) \in \mathcal{J}$, and put $z_0 = f_i(p) = f_j(r)$. The similarities

$$T_i = f_i f_p f_i^{-1}, \qquad T_j = f_j f_r f_j^{-1} \tag{3.1}$$

both fix z_0 and have contraction ratio ρ. They describe the contractions toward z_0 within K_i and K_j, respectively.

Proposition 3.2. *Suppose that for every $J \in \mathcal{J}$ there is an integer $m = m(J) \geq 1$ such that*

$$T_i^m = T_j^m. \tag{3.2}$$

Then K has the same-level separation property and the one-point intersection property.

Proof. Fix $J \in \mathcal{J}$ and choose m as in (3.2). For $k \geq 0$, we have $f_i f_p^{k+m} = T_i^m f_i f_p^k$ and $f_j f_r^{k+m} = T_j^m f_j f_r^k$. Therefore (2.2) and (3.2) give

$$H_{J,k+m} = (T_j^m f_j f_r^k)^{-1} (T_i^m f_i f_p^k) = H_{J,k}.$$

Thus only finitely many maps $H_{J,k}$ occur for each J. Since $\mathcal{J}$ and S are finite and the cells are compact, the distances $\operatorname{dist}(H_{J,k}(K_a), K_b)$ between disjoint pairs have a uniform positive lower bound. Hence Proposition 2.2 gives same-level separation, and Theorem 1.1 gives one-point intersection. □

Recall that an orthogonal map has finite order if some positive power of it is the identity.

Corollary 3.3. *If O_p has finite order for every $p \in V_0$, then K has the same-level separation property and the one-point intersection property.*

Proof. For $J = (i, p; j, r) \in \mathcal{J}$, choose a positive common multiple m of the orders of O_p and O_r. Then $O_p^m = O_r^m = I$, and

$$T_i^m(x) = z_0 + \rho^m (x - z_0) = T_j^m(x), \qquad x \in E,$$

so Proposition 3.2 applies. □

3.2. **The case $\dim E \leq 2$.** In the planar case, we verify (3.2) using invariant arcs. A *Jordan arc* is a homeomorphic image of the closed interval $[0, 1]$. An arc γ ending at a fixed point of a contracting similarity T is called *invariant* for T if $T^m(\gamma) \subset \gamma$ for some integer $m \geq 1$.

Identify the plane with $\mathbb{C}$. For an arc γ with endpoint z_0, choose a continuous argument $\alpha(z)$ of $z - z_0$ on $\gamma \setminus \{z_0\}$. The arc has *slope parameter* $\lambda \in \mathbb{R}$ at z_0 (we use an equivalent form of [18, Definition 5.4]) if there is $C > 0$ such that

$$\big|\alpha(z) - \lambda \log |z - z_0|\big| \leq C \tag{3.3}$$

for $z \in \gamma \setminus \{z_0\}$ sufficiently close to z_0. This definition does not depend on the choice of α, since two continuous arguments differ by a constant multiple of 2π.

Lemma 3.4 (cf. [18])**.** *Let K be a planar nested fractal.*

(a) *For every $p \in V_0$, there is a nonconstant Jordan arc in K ending at p which is invariant for f_p.*
(b) *If $z \in K_i \cap K_j$ for distinct $i, j \in S$, then every Jordan arc in K ending at z has the same slope parameter there.*

Parts (a) and (b) are special cases of [18, Proposition 5.2(iii) and Theorem 5.6], respectively. For completeness, we provide a direct proof of Lemma 3.4 in the appendix. The next lemma gives existence of the slope parameter and its relation to the rotation angle.

Lemma 3.5. *Let $z_0 \in \mathbb{C}$, $0 < \tau < 1$ and $\theta \in \mathbb{R}$, and let*

$$U(z) = z_0 + \tau e^{\mathrm{i}\theta}(z - z_0).$$

Suppose that γ is a nonconstant Jordan arc ending at z_0 and $U(\gamma) \subset \gamma$. Then γ has a unique slope parameter λ at z_0, and

$$\theta \equiv \lambda \log \tau \pmod{2\pi}. \tag{3.4}$$

Proof. Choose a continuous argument $\alpha(z)$ of $z - z_0$ on $\gamma \setminus \{z_0\}$. The function $\alpha(U(z)) - \alpha(z)$ is continuous on the connected set $\gamma \setminus \{z_0\}$ and takes values in $\theta + 2\pi\mathbb{Z}$, so it is constant, say β. Put $\lambda = \beta/\log\tau$ and $h(z) = \alpha(z) - \lambda \log|z - z_0|$. Since $|U(z) - z_0| = \tau|z - z_0|$, we have $h(U(z)) = h(z)$.

To show that h is bounded, let a be the other endpoint of γ. Since $U(\gamma) \subset \gamma$, the image $U(\gamma)$ is the subarc from z_0 to $U(a)$. Let δ be the subarc from $U(a)$ to a. It is compact and does not contain z_0, so h is bounded on δ. Moreover,

$$\gamma \setminus \{z_0\} = \bigcup_{n \geq 0} U^n(\delta),$$

since $\gamma = \delta \cup U(\gamma)$ and the nested subarcs $U^n(\gamma)$ shrink to $\{z_0\}$. The identity $h \circ U = h$ now shows that h is bounded on $\gamma \setminus \{z_0\}$. Thus λ is a slope parameter. It is unique by (3.3), since $\log|z - z_0| \to -\infty$ as $z \to z_0$. Finally, (3.4) follows from $\lambda \log\tau = \beta \in \theta + 2\pi\mathbb{Z}$. □

We now apply the preceding two lemmas to verify (3.2) in the planar case.

Proof of Theorem 1.2 in the planar case. If $d = 1$, then every orthogonal part is I or $-I$, and Corollary 3.3 applies. We therefore assume that $d = 2$.

Fix $J = (i, p; j, r) \in \mathcal{J}$. Let z_0, T_i, T_j be as in (3.1). By Lemma 3.4(a), there are nonconstant Jordan arcs $\gamma_p, \gamma_r \subset K$ ending at p, r, respectively, and integers $m_p, m_r \geq 1$ such that

$$f_p^{m_p}(\gamma_p) \subset \gamma_p, \qquad f_r^{m_r}(\gamma_r) \subset \gamma_r.$$

Set $\Gamma_i = f_i(\gamma_p)$ and $\Gamma_j = f_j(\gamma_r)$. These arcs lie in K_i and K_j, respectively, and both end at z_0. Choose a positive even integer m divisible by m_p and m_r. Then T_i^m and T_j^m preserve orientation, and

$$T_i^m(\Gamma_i) \subset \Gamma_i, \qquad T_j^m(\Gamma_j) \subset \Gamma_j.$$

Since $z_0 \in K_i \cap K_j$, Lemma 3.4(b) gives a common slope parameter λ for Γ_i and Γ_j at z_0.

We now compare the rotation angles. Write

$$T_i^m(z) = z_0 + \rho^m e^{\mathrm{i}\theta_i}(z - z_0), \qquad T_j^m(z) = z_0 + \rho^m e^{\mathrm{i}\theta_j}(z - z_0),$$

with $\theta_i, \theta_j \in \mathbb{R}$. By Lemma 3.5, both angles are congruent to $\lambda \log \rho^m$ modulo 2π. Since the two maps have the same fixed point and contraction ratio, $T_i^m = T_j^m$. The result follows from Proposition 3.2. □

3.3. **The case** $\mathrm{aff}(V_0) = E$**.** We now assume $\mathrm{aff}(V_0) = E$ and prove that O_p has finite order for every $p \in V_0$. We first give a lemma about the geometry of V_0. Translate the average of the points in V_0 to the origin, so that E becomes a linear subspace. For vectors $u, v \in E$, write $\langle u, v\rangle$ for their inner product, and $p^\perp = \{v \in E : \langle p, v\rangle = 0\}$ for the orthogonal complement of a nonzero vector p.

Let $\ell = \min\{|x - y| : x, y \in V_0,\ x \neq y\}$. For $p \in V_0$, define

$$B_p = \{q \in V_0 : |q - p| = \ell\}.$$

Note that the graph with vertex set V_0 and edges between points at distance ℓ is connected; see [9, Lemma 3.8.6].

Lemma 3.6. *Suppose* $\dim E \geq 2$*. Then for every* $p \in V_0$*,*

$$\mathrm{span}\{x - y : x, y \in B_p\} = p^\perp. \tag{3.5}$$

Proof. Since each g_{xy} permutes V_0, it fixes the origin. It also interchanges x and y, so $|x| = |y|$. Thus all points of V_0 have the same positive norm.

For $p \in V_0$, put $W_p = \mathrm{span}\{q - p : q \in B_p\}$. We first show that $W_p = E$ for every $p \in V_0$. The connectedness of the graph implies $B_p \neq \varnothing$. If $q \in B_p$, then g_{pq} maps B_p onto B_q, and hence $g_{pq}(W_p) = W_q$. On the other hand, g_{pq} is now a linear reflection, and $g_{pq}(v) - v$ is a multiple of $q - p$ for every $v \in E$. Since $q - p \in W_p$, this gives $g_{pq}(W_p) = W_p$. Thus $W_q = W_p$ along each edge. By connectedness, all these spaces are equal. Every difference of two boundary vertices is a sum of edge vectors along a path, so it belongs to this common space. Since V_0 affinely spans E, we obtain $W_p = E$.

We now fix $p \in V_0$ and put $N_p = \mathrm{span}\{x - y : x, y \in B_p\}$. For $x, y \in B_p$, the equalities $|x| = |y|$ and $|x - p| = |y - p|$ imply $\langle p, x - y\rangle = 0$. Hence $N_p \subset p^\perp$. Fix $q_0 \in B_p$. Writing $q - p = (q - q_0) + (q_0 - p)$ for $q \in B_p$ gives

$$E = W_p = N_p + \mathbb{R}(q_0 - p).$$

It follows that $\dim N_p \geq \dim E - 1$. Since $p \neq 0$ and $N_p \subset p^\perp$, equality (3.5) follows. □

Proof of Theorem 1.2 when $\mathrm{aff}(V_0) = E$*.* If $\dim E = 1$, every orthogonal part is I or $-I$, so Corollary 3.3 applies. We therefore assume $\dim E \geq 2$ and fix $p \in V_0$.

For distinct $x, y \in B_p$, let $h = g_{xy}$. Then $h(p) = p$ because $|p - x| = |p - y|$. By (NF3), h permutes the level-k boundary vertex sets. By (2.1), V_{p^k} is the only such set containing p. Hence

$$h(V_{p^k}) = V_{p^k}, \qquad k \geq 0.$$

Thus the isometry $C_k = f_p^{-k} h f_p^k$ permutes V_0. Since $\mathrm{aff}(V_0) = E$, each C_k is determined by its values on V_0. There are only finitely many such permutations, so we may choose integers $0 \leq a < b$ with $C_a = C_b$. This gives

$$f_p^{b-a} h = h f_p^{b-a}.$$

Comparing linear parts, we obtain $O_p^{b-a}h = hO_p^{b-a}$, since h is linear.

Since there are only finitely many pairs of distinct points $x, y \in B_p$, we may choose a positive integer m divisible by all the exponents obtained above. Put $A = O_p^m$. Then A commutes with every g_{xy} for distinct $x, y \in B_p$. For such a pair, put $v = x - y$. The (-1)-eigenspace of g_{xy} is $\mathbb{R}v$. Since A commutes with g_{xy}, we have $Av \in \mathbb{R}v$. As A is orthogonal, $Av = \pm v$, and hence $A^2 v = v$.

By Lemma 3.6, the vectors $x - y$, with $x, y \in B_p$, span $p^\perp$. Thus A preserves $p^\perp$ and $A^2 = I$ there. Since A is orthogonal, it also preserves $\mathbb{R}p$. This space is one-dimensional, so $A^2 = I$ there as well. Therefore $O_p^{2m} = A^2 = I$ on E. Since p is arbitrary, O_p has finite order for every $p \in V_0$. The same-level separation and one-point intersection properties of K therefore follow from Corollary 3.3. □

4. Remarks

We conclude with a few remarks on connections with earlier work and some open questions.

Remark 4.1. Theorem 1.2 removes the need to assume one-point intersection separately in [1, 12, 15], when K is planar or $\mathrm{aff}(V_0) = \mathrm{aff}(K)$. The same-level separation estimate (1.4) is precisely Condition (H) in [6] and Assumption 4.41 in [10]. It also gives Condition (C1) in [7] when all partition weights are equal to ρ. It implies Property (P) in [14], the finite-chain condition in [11, 16], and the metric condition in [8]. These implications use the following observation: if $c > 0$ is a constant in (1.4) and $|x - y| < c\rho^n$ for $x, y \in K$, then any level-n cells containing x and y intersect.

Remark 4.2. Proposition 3.2 also applies to the two cases treated in earlier work. If all the orthogonal parts O_q are equal to a fixed orthogonal map O, as in Pietruska-Pałuba–Stós [14], then T_i and T_j in (3.1) have the same fixed point, contraction ratio and orthogonal part O, so $T_i = T_j$. If $O_p = I$ for every $p \in V_0$, as in Kigami [10], then Corollary 3.3 applies directly.

Remark 4.3. Let d_E be the Euclidean metric on K, and write $\dim_{\mathrm{ARC}}(K, d_E)$ for its *Ahlfors regular conformal dimension*; see [8, Section B.2]. Combining Theorem 1.2 with the result of Kajino–Shimizu [8, Theorem B.8], we obtain

$$\dim_{\mathrm{ARC}}(K, d_E) = 1, \tag{4.1}$$

when K is planar or $\mathrm{aff}(V_0) = \mathrm{aff}(K)$. Here one-point intersection allows us to pass from the symmetry of boundary vertex sets in (NF3) to the cell symmetry required in their theorem, by [9, Proposition 3.8.19]; see also [8, Remark 8.48]. Same-level separation gives the metric condition [8, Assumption B.6] for $d_E/\operatorname{diam} K$, as explained in Remark 4.1.

Remark 4.4. The two proofs in Section 3 use different arguments. In the plane, winding of arcs at a common endpoint gives (3.2), without proving finite order of the individual orthogonal parts. In contrast, when $\mathrm{aff}(V_0) = E$, the fact that an isometry of E is determined by its values on V_0 allows us to prove finite order. If $\mathrm{aff}(V_0) \subsetneq E$, the values on V_0 no longer determine an isometry of E.

Remark 4.5. For general nested fractals, the following questions remain open.

(1) Does every nested fractal have the same-level separation property? If not, does every nested fractal still have the one-point intersection property?

(2) Does the one-point intersection property imply the same-level separation property?

We also ask whether the sufficient condition in Proposition 3.2 is necessary for same-level separation.

Appendix. Proof of Lemma 3.4

Let $K \subset \mathbb{C}$ be a planar nested fractal. We first construct an invariant arc at each point of V_0, and then compare the slope parameters of arcs with a common endpoint.

Lemma A.1. *For every $p \in V_0$, there are a nonconstant Jordan arc $\gamma \subset K$ ending at p and an integer $m \geq 1$ such that $f_p^m(\gamma) \subset \gamma$.*

Proof. By [9, Theorem 1.6.2 and Proposition 1.6.4], K is locally connected and arcwise connected. In particular, any two distinct points of K can be joined by a Jordan arc in K.

Fix $p \in V_0$, write $f = f_p$, and put $D = V_0 \setminus \{p\}$. By Lemma 2.1, $D \cap f(K) = \varnothing$, and p belongs to no other first-level cell. Thus $f(K)$ is a neighbourhood of p in K. For each $y \in D$, choose a Jordan arc in K from y to p. Let b_y be its first point in $f(K)$ when travelling from y, and let δ_y be the subarc from y to b_y. Since $f(K)$ is a neighbourhood of p, we have $b_y \neq p$. Before reaching b_y, the arc lies in the union of the other first-level cells. This union is closed, so (NF4) gives

$$b_y \in f(K) \cap \bigcup_{q \in S \setminus \{p\}} K_q \subset f(V_0).$$

Thus the chosen arcs define a map $\varphi : D \to D$ by $b_y = f(\varphi(y))$, and

$$\delta_y \setminus \{f(\varphi(y))\} \subset K \setminus f(K). \tag{A.1}$$

To construct the invariant arc, choose a periodic point y_0 of φ, which exists because D is finite and nonempty. Let $m \geq 1$ be its period and put $y_k = \varphi^k(y_0)$ for $k \geq 0$. The arc $f^k(\delta_{y_k})$ joins $f^k(y_k)$ to $f^{k+1}(y_{k+1})$. By (A.1), it lies in $f^k(K) \setminus f^{k+1}(K)$ except for its last endpoint. Since $D \cap f(K) = \varnothing$, consecutive arcs meet only at their common endpoint, and nonconsecutive arcs are disjoint. As $f^k(K)$ shrinks to $\{p\}$,

$$\gamma = \{p\} \cup \bigcup_{k \geq 0} f^k(\delta_{y_k})$$

is a Jordan arc from y_0 to p. Since $y_{k+m} = y_k$, we have $f^m(\gamma) \subset \gamma$. □

The following lemma compares two arcs with a common endpoint.

Lemma A.2. *Let $\gamma, \eta \subset \mathbb{C}$ be Jordan arcs with common endpoint z_0, and suppose that $\gamma \cap \eta = \{z_0\}$. If γ has slope parameter λ at z_0, then so does η.*

Intuitively, two arcs approaching the same endpoint cannot wind around it at different rates without crossing.

Proof. Choose $r > 0$ sufficiently small. Replace each arc by its subarc from its last point on $|z - z_0| = r$ to z_0, keeping the notation γ and η. Except for their endpoints, both arcs lie in $0 < |z - z_0| < r$, and we may assume that (3.3) holds throughout $\gamma \setminus \{z_0\}$.

Put $a = \log r$ and $H = \{w \in \mathbb{C} : \operatorname{Re} w < a\}$. Under the covering map $w \mapsto z_0 + e^w$, choose a lift G of $\gamma \setminus \{z_0\}$. It is a simple curve with one endpoint on the boundary line ∂H, and

$\operatorname{Re} w \to -\infty$ along its other end. Its translates $G_k = G + 2\pi k\mathrm{i}$, $k \in \mathbb{Z}$, are pairwise disjoint. By (3.3), there is $C > 0$ such that

$$|\operatorname{Im} w - \lambda \operatorname{Re} w| \le C, \qquad w \in G.$$

Choose a lift G' of $\eta \setminus \{z_0\}$. Its endpoint on ∂H lies strictly between the endpoints of G_k and G_{k+1} for some $k \in \mathbb{Z}$. The curves G_k and G_{k+1}, together with the segment joining their endpoints on ∂H, form a Jordan curve after adjoining the point at infinity. Let D be the complementary domain contained in H. The curve G' starts on that segment and otherwise lies in H. Since it meets neither G_k nor G_{k+1}, the Jordan curve theorem gives $G' \setminus \partial H \subset D$.

The curves G_k and G_{k+1}, together with the segment joining their endpoints, lie in the strip

$$-C + 2\pi k \le \operatorname{Im} w - \lambda \operatorname{Re} w \le C + 2\pi(k+1). \tag{A.2}$$

Since any point of H outside this strip can be joined to ∂H along a line on which $\operatorname{Im} w - \lambda \operatorname{Re} w$ is constant, without meeting the Jordan curve, D also lies in this strip.

Thus $\operatorname{Im} w - \lambda \operatorname{Re} w$ is bounded on G'. In the original coordinates, $\operatorname{Re} w = \log|z - z_0|$ and $\operatorname{Im} w$ gives a continuous argument of $z - z_0$ along $\eta \setminus \{z_0\}$. This proves (3.3) for η. □

Proof of Lemma 3.4. Part (a) follows from Lemma A.1. To prove part (b), fix $z \in K_i \cap K_j$ with $i \ne j$. By (NF4), write $z = f_i(p)$ with $p \in V_0$. Choose an invariant arc γ at p as in Lemma A.1, and put $\Gamma = f_i(\gamma)$. Choose an even integer $m \ge 1$ such that $f_p^m(\gamma) \subset \gamma$. Then $f_i f_p^m f_i^{-1}$ preserves orientation and maps Γ into itself. By Lemma 3.5, Γ has a slope parameter λ at z.

Next, choose a nonconstant Jordan arc $\eta \subset K_j$ ending at z, using the arcwise connectedness of K_j [9, Theorem 1.6.2]. Since $K_i \cap K_j$ is finite, we may shorten Γ and η near z so that $\Gamma \cap \eta = \{z\}$. By Lemma A.2, η also has slope parameter λ at z.

Finally, let $\xi \subset K$ be any other Jordan arc ending at z. Shorten it so that $\xi \cap V_1 = \{z\}$. By (NF4), the sets $K_q \setminus V_1$, $q \in S$, are pairwise disjoint and both open and closed in $K \setminus V_1$. Since $\xi \setminus \{z\}$ is connected, it lies in one of them, and hence $\xi \subset K_q$ for some $q \in S$. If $q = i$, then $\xi \cap \eta = \{z\}$; otherwise, $\xi \cap \Gamma = \{z\}$. In either case, Lemma A.2 gives slope parameter λ for ξ at z. □

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