FUNCTION SPACES ON P.C.F. SELF-SIMILAR SETS III: EMBEDDING AND INTERPOLATION THEOREMS

SHIPING CAO AND HUA QIU

ABSTRACT. We study the Sobolev spaces $H^p_\sigma(\Omega)$ and Besov spaces $B^{p,q}_\sigma(\Omega)$ with $\sigma\in\mathbb{R}$ and $1<p,q<\infty$, on products of p.c.f. self-similar sets in terms of the boundary behavior of functions. First, we establish a general embedding theorem which says that these function spaces are the restrictions of function spaces of the same type on a larger fractal domain without boundary. Towards this, we develop a throughout study on the relationship between various Sobolev and Besov type spaces, including $\tilde{H},\mathring{H}$ and $\tilde{B},\mathring{B}$. In contrast to the Euclidean case, one of the main differences comes from the appearance of many more critical orders of σ created by "derivatives" at boundary such that $\mathring{H}$ and $\mathring{B}$ present a critical phenomenon if σ is critical, and as a consequence $\tilde{H},\tilde{B}$ and $\mathring{H},\mathring{B}$ will be different spaces for any large order σ. Second, we provide a complete diagram of the interpolations of these function spaces in different situations. In particular, we allow spaces in the interpolation couple to involve the critical orders, and the resulted interpolation space, when it is of a critical order, will vary in different situations.

CONTENTS

2010 *Mathematics Subject Classification.* Primary 28A80.
Key words and phrases. p.c.f. self-similar sets, Sobolev space, Besov spaces, interpolation theorem, Laplacian.
The research of Qiu was supported by the NSFC grant 11471157.

1. Introduction

The theory of function spaces has been a longlasting topic in analysis, and has been playing a prominent role in the development of partial differential equations. In the general setting of metric measure spaces, there have been fruitful works [8, 9, 14, 16, 17, 18, 22] recently, involving various potential spaces and Lipschitz type spaces.

This work is devoted to the Sobolev spaces and Besov spaces on fractals, especially on products of fractals. It is well-known that on the post critically finite(p.c.f.) self-similar sets the analytic theory was developed by J. Kigami [24, 25], following several pioneering works on certain fractals by probabilistic methods [3, 4, 5, 15, 28, 29], which constructed Brownian motions, thus obtained the Laplacians indirectly as the generators. Since then, the theory of local self-similar Dirichlet forms on fractals has been widely studied and sub-Gaussian heat kernel estimates of the associate semigroups have been obtained [20, 27]. The initial study of function spaces on fractals in the general p.c.f. setting was launched by R.S. Strichartz [34], as well as the extension to products of fractals [35]. In these works, the p.c.f. self-similar sets are viewed as bounded domains with finite boundary points, while products of them are not p.c.f. but are more analogous to Euclidean spaces constructed as products of lines.

We are particularly interested in the function spaces on fractal domains **with boundary**. The boundaries of fractals or fractal boundaries bring many striking features for the boundary behavior of functions that never appear in Euclidean case. To reasonably establish the theory of function spaces on fractal domains, in particular, the Sobolev spaces and Besov spaces, the boundary condition of functions need to be dealt with in a more involved manner. In [35], Strichartz took an explorative study on the L^2 type of Sobolev spaces on products of p.c.f. self-similar sets. Because the boundary creates difficulties, the strategy is to work instead on a proper cover space of a product fractal that has no boundary, and then the study is transferred into the trace theorems or extension theorems of function spaces by embedding the product fractal into this cover space. In the p.c.f. setting, it is convenient to choose the cover space as the product of the "double covers" of fractals, by taking two copies of the fractals and identifying common boundary points. This pioneering work reveals the tip of the iceberg, and leaves many unknown aspects to uncover.

One of our goals is to fulfill the story of Sobolev spaces and additionally their real interpolations, the (heat) Besov spaces, on products of p.c.f. self-similar sets in the general L^p setting. We will use K to denote a p.c.f. self-similar set and $\tilde{K}$ its double cover. We will mainly deal with the Sobolev spaces $H^p_\sigma(\Omega)$ and Besov spaces $B^{p,q}_\sigma(\Omega)$ with $\sigma \in \mathbb{R}$, $p, q \in (1, \infty)$, and $\Omega = \tilde{K}^d$ or $\tilde{K}^d_+ := K \times \tilde{K}^{d-1}$, a product fractal or its half. Readers are suggested to refer to the monographs [30, 38] for the classical theorems of function spaces on domains with or without boundary in Euclidean case.

Firstly, we will study the trace theorem for function spaces on $\tilde{K}^d_+$ on the boundary, with $\sigma \geq 0$ (see Theorem 4.3) and prove an embedding theorem relating function spaces on $\tilde{K}^d$ and $\tilde{K}^d_+$. Mainly we devote to extend the results to real orders $\sigma \in \mathbb{R}$, especially the following embedding theorem (see Theorem 4.19):

Theorem 1. *Let $p, q \in (1, \infty)$, $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$ and $\sigma \in \mathbb{R}$. We have*

$$H^p_\sigma(\tilde{K}^d_+) = H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+} \quad \text{and} \quad B^{p,q}_\sigma(\tilde{K}^d_+) = B^{p,q}_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$$

if and only if $\sigma \notin \{\frac{d_S}{p'}, 2 - \frac{d_S}{p}\} - 2\mathbb{N}$, where d_S is spectral dimension of the Laplacian on K, and the restriction is in the sense of distribution.

In the above theorem, we define $H^p_\sigma(\tilde{K}^d_+)$ as the dual of $\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$, and $B^{p,q}_\sigma(\tilde{K}^d_+)$ as the dual of $\mathring{B}^{p',q'}_{-\sigma}(\tilde{K}^d_+)$ for negative orders σ, where the $\mathring{H}$ and $\mathring{B}$ spaces are the closures of $C^\infty_c(\tilde{K}^d_+)$ in corresponding spaces following J.L. Lions and E. Magenes [30]. In Euclidean case, for example, see book [38], the definition for spaces with negative orders σ may be alternately given directly by using this restriction. The theorem, analogous to the Euclidean case, shows that the two definitions agree, with countably many exceptional orders of σ, which comes from $\{\frac{d_S}{p}, 2 - \frac{d_S}{p'}\} + 2\mathbb{Z}_+$ by dual.

Although Theorem 1 is analogous to the Euclidean case, there exist major differences in the proof. In the fractal setting, as illustrated in [33], the tangents of functions at the boundary, defined in terms of multiharmonic functions in contrast to the classical Taylor approximation, contain much more information that do not take part in the matching conditions in extending functions from $\tilde{K}^d_+$ to $\tilde{K}^d$ across the boundary. This will bring many technical difficulties and involve many more critical orders of σ than $\{\frac{d_S}{p}, 2 - \frac{d_S}{p'}\} + 2\mathbb{Z}_+$. An intuitive explanation of the (countable infinitely many) critical orders is that they are the σ's such that higher order tangents may appear for functions at boundary in $H^p_{\sigma'}(\tilde{K}^d_+)$ with $\sigma' > \sigma$ compared to those in $H^p_\sigma(\tilde{K}^d_+) \setminus H^p_{\sigma'}(\tilde{K}^d_+)$, see (5.5) for the exact definition. However, in Euclidean case, the critical orders are exactly $\frac{1}{p} + \mathbb{Z}_+$ in extending functions from $\mathbb{R}^d_+$ to $\mathbb{R}^d$, and no more appears for the tangents.

The interpolation property of Sobolev spaces and Besov spaces on $\tilde{K}^d_+$ is the next main interest of this paper. We are particularly interested in the case that the interpolation couple involves critical orders of σ, which is also a difficult problem in Euclidean case. See Chapter 1, Section 18 in the monograph [30] by J.L. Lions and E. Magenes for the original problem. In order to include the consideration of interpolations between different p's (or evenly q's) in $(1, \infty)$, it is convenient to consider the critical set of couples $(\sigma, \frac{1}{p})$, denoted as $\mathcal{C}_{O_+} := \{(\sigma, \frac{1}{p}) : \sigma \text{ is critical in the } L^p \text{ setting}\}$.

Since the function spaces of $\sigma < 0$ are defined as duals of $\mathring{H}$ and $\mathring{B}$ spaces, the critical set of the forthcoming interpolation theorem is reflected in a reasonable sense to be $\mathcal{C}_O := \{(\sigma, \frac{1}{p}) : (-\sigma, 1 - \frac{1}{p}) \in \mathcal{C}_{O_+}\}$, as illustrated in Figure 1, which consists of countably many parallel lines with slope $\frac{1}{d_S}$ between the two horizontal lines $\frac{1}{p} = 0$ and 1. For $p_0, p_1, p_\theta \in (1, \infty)$ and

$\theta \in (0,1)$ with $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, we need to consider the following three cases separately due to the critical set $\mathcal{C}_O$, as shown in Figure 1.

(**O1**). $(\sigma_\theta, \frac{1}{p_\theta}) \notin \mathcal{C}_O$;
(**O2**). $(\sigma_0, \frac{1}{p_0})$, $(\sigma_1, \frac{1}{p_1})$ and $(\sigma_\theta, \frac{1}{p_\theta})$ lie on a same critical line in $\mathcal{C}_O$;
(**O3**). otherwise.

Using $[\cdot,\cdot]_\theta$, $(\cdot,\cdot)_{\theta,q}$ to stand for the complex and real interpolations respectively, we will prove(see Theorem 5.20):

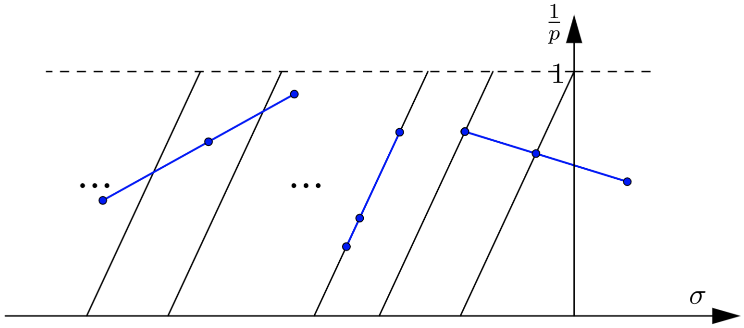

FIGURE 1. An illustration for $\mathcal{C}_O$ and the three cases of interpolations.

Theorem 2. *Let $\sigma_0, \sigma_1, \sigma \in \mathbb{R}$, $p_0, p_1, q_0, q_1, p, q \in (1,\infty)$, and put $\sigma_\theta = \theta\sigma_0 + (1-\theta)\sigma_1$, $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, and $\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$. We also write $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, $p'_\theta = \frac{p_\theta}{p_\theta - 1}$ and $q'_\theta = \frac{q_\theta}{q_\theta - 1}$. Then the interpolation results for $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ are given by the following table:*

	(O1)	***(O2)***	***(O3)***
$\big[H^{p_0}_{\sigma_0}(\tilde{K}^d_+), H^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta$	$H^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \big(\dot{H}^{p'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$	$H^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\big(\dot{H}^{p'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$
$\big(H^{p_0}_{\sigma}(\tilde{K}^d_+), H^{p_1}_{\sigma}(\tilde{K}^d_+)\big)_{\theta,p_\theta}$	$H^{p_\theta}_{\sigma}(\tilde{K}^d_+) = \big(\dot{H}^{p'_\theta}_{-\sigma}(\tilde{K}^d_+)\big)^*$	/	$\big(\dot{H}^{p'_\theta}_{-\sigma}(\tilde{K}^d_+)\big)^*$
$\big(H^{p}_{\sigma_0}(\tilde{K}^d_+), H^{p}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q}$	$B^{p,q}_{\sigma_\theta}(\tilde{K}^d_+) = \big(\dot{B}^{p',q'}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$	/	$\big(\dot{B}^{p',q'}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$,
$\big[B^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+), B^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \big(\dot{B}^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\big(\dot{B}^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$
$\big(B^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+), B^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q_\theta}$ *(if $p_\theta = q_\theta$)*	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \big(\dot{B}^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\big(\dot{B}^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde{K}^d_+)\big)^*$
$\big(B^{p,q_0}_{\sigma}(\tilde{K}^d_+), B^{p,q_1}_{\sigma}(\tilde{K}^d_+)\big)_{\theta,q_\theta}$	$B^{p,q_\theta}_{\sigma}(\tilde{K}^d_+) = \big(\dot{B}^{p',q'_\theta}_{-\sigma}(\tilde{K}^d_+)\big)^*$	$B^{p,q_\theta}_{\sigma}(\tilde{K}^d_+)$	/

where the $\dot{H}$ spaces with $\sigma \geq 0$ are contained in $\mathring{H}$ spaces and analogous to the Lions-Magenes spaces in Euclidean case, satisfying $\dot{H} = \mathring{H}$ if and only if $(\sigma, \frac{1}{p}) \notin \mathcal{C}_{O_+}$, and similarly for $\dot{B}$ spaces; the notation '$*$' means the dual space; also '/' means there exists only the trivial case.

We should mention that there is a large literature on the topic of function spaces on more general metric measure spaces from other different points of view, see [8, 9, 14, 16, 17, 18, 22] and the references therein. See also [1, 2, 10, 11, 21] and the references therein for recent works on function spaces on fractals.

In addition, the techniques and results from the books [7, 13, 19], the paper on pseudo differential operators [23] and the paper on smooth bump functions [31] are important for our developments.

Now we briefly introduce the organization of this paper. In the remaining of this introduction, we will briefly review some basic concepts, including the p.c.f. self-similar sets, the construction of Dirichlet forms and Laplacians, and the Sobolev spaces $H^p_\sigma(\tilde{K}^d)$ and Besov spaces $B^{p,q}_\sigma(\tilde{K}^d)$ defined on $\tilde{K}^d$.

In Section 2, we will develop the main tool of this paper, the relations of various function sequence spaces. In fact, we will extract the boundary information of a function into several sequences of rescaling functions on cells approaching to the boundary. We will only deal with a simple case, and leave a further discussion to Appendix A.

In Section 3, we deal with Sobolev spaces $H^p_\sigma(\Omega)$ for $\sigma\geq 0$ on general fractal domains $\Omega := K^l\times\tilde{K}^{d-l}$ with $0\leq l\leq d$ which include $\tilde{K}^d$, $\tilde{K}^d_+$ as special cases. We will prove an embedding result(Theorem 3.5) that $H^p_\sigma(\Omega)=H^p_\sigma(\tilde{K}^d)|_\Omega$, as well as a decomposition(Theorem 3.17) of a Sobolev space $H^p_\sigma(\Omega)$ into the union of a "kernel" part and a "sequence" part, based on the sequence spaces we studied in Section 2.

We will focus on $\Omega=\tilde{K}^d_+$ in the next two sections for simplicity, though quite a large portion of the results can be extended to more general Ω's with boundary.

In Section 4, first we present a trace theorem for values and normal derivatives (of Laplacians) of functions in $H^p_\sigma(\tilde{K}^d_+)$ or $B^{p,q}_\sigma(\tilde{K}^d_+)$ at boundary(Theorem 4.3). Then we introduce two classes of Sobolev type spaces $\tilde{H}^p_\sigma(\tilde{K}^d_+),\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and two classes of Besov type spaces $\tilde{B}^{p,q}_\sigma(\tilde{K}^d_+),\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$ contained in the H and B spaces, and provide an exact characterization of these spaces in terms of the boundary behavior of functions(traces for $\tilde{H}$, $\tilde{B}$ spaces, and tangents for $\mathring{H}$, $\mathring{B}$ spaces, see Theorem 4.6 and 4.10). At last, we will use the above results to prove Theorem 1(or 4.19), the first main result in this paper.

Finally in Section 5, we will devote to prove the second main result in this paper, Theorem 2(or 5.20). We will first derive an interpolation theorem for $\mathring{H}$ and $\mathring{B}$ spaces with $\sigma\geq 0$, then extends this to the real order case based on a dual approach. The main difficulty arises in this fractal setting due to the appearance of tangents that do not take part in the matching conditions of functions at boundary. We will provide a long proof to overcome it.

In addition to the main story, we will present three appendixes. Appendix A is a supplement to Section 2, and will be crucial in Section 4 and 5. Appendix B includes a short discussion about the definition of Sobolev spaces in the sense of distributions. Appendix C collects several useful facts and important concepts which will be used throughout the paper.

Throughout this paper, we always write $f\lesssim g$ to mean that $f\leq Cg$ for some constant $C>0$, and write $f\asymp g$ if both $f\lesssim g$ and $g\lesssim f$ hold.

1.1. **The p.c.f. self-similar sets.** The main objects we study in this paper are the p.c.f. self-similar sets. Let $\{F_i\}_{i=1}^N$ be an *iterated function system (i.f.s.)*, a finite collection of contractions, on a complete metric space $(\mathcal{M},d)$. The associated self-similar set is the unique compact set $K\subset\mathcal{M}$ satisfying $K=\bigcup_{i=1}^N F_iK$. For $m\geq 1$, we define $W_m=\{1,\cdots,N\}^m$ the collection of *words* of length m, and for each $w=w_1w_2\cdots w_m\in W_m$, denote

$$F_w=F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_m}.$$

For uniformity, we set $W_0=\{\emptyset\}$, with $F_\emptyset$ being the identity map. For convenience, let $W_*=\bigcup_{m=0}^\infty W_m$ be the collection of all finite words.

Let $\Sigma=\{1,2,\cdots,N\}^{\mathbb{N}}$ be the shift space endowed with the natural product topology. There is a continuous surjection $\pi:\Sigma\to K$ defined by

$$\pi(\omega)=\bigcap_{m\geq 1}F_{[\omega]_m}K,$$

where for $\omega=\omega_1\omega_2\cdots$ in Σ we write $[\omega]_m=\omega_1\omega_2\cdots\omega_m\in W_m$ for each $m\geq 1$. Let

$$C_K=\bigcup_{i\neq j}F_iK\cap F_jK,\quad \mathcal{C}=\pi^{-1}(C_K),\quad \mathcal{P}=\bigcup_{m\geq 1}\sigma^m\mathcal{C},$$

where σ is the shift map define as $\sigma(\omega_1\omega_2\cdots)=\omega_2\omega_3\cdots$, $\mathcal{P}$ is called the *post critical set*. Call K a *p.c.f. self-similar set* if $\#\mathcal{P}<\infty$. In what follows, we always assume that K is a connected p.c.f. self-similar set.

1.2. **Dirichlet forms.** Let $V_0=\pi(\mathcal{P})$ and call it the *boundary* of K. For $m\geq 1$, we always have $F_wK\cap F_{w'}K\subset F_wV_0\cap F_{w'}V_0$ for any $w\neq w'\in W_m$. For $m\geq 1$, denote $V_m=\bigcup_{w\in W_m}F_wV_0$ and let $l(V_m)=\{f: f \text{ maps } V_m \text{ into } \mathbb{C}\}$. Write $V_*=\bigcup_{m\geq 0}V_m$.

Let $H=(H_{xy})_{x,y\in V_0}$ be a symmetric linear operator(matrix) on $l(V_0)$. H is called a *(discrete) Laplacian* on V_0 if H is non-positive definite; $Hu=0$ if and only if u is constant on V_0; and $H_{xy}\geq 0$ for any $x\neq y\in V_0$. Given a Laplacian H on V_0 and a vector $\boldsymbol{r}=\{r_i\}_{i=1}^N$ with $r_i>0$, $1\leq i\leq N$, define the *(discrete) Dirichlet form* $\big(\mathcal{E}_0,l(V_0)\big)$ on V_0 by

$$\mathcal{E}_0(f,g)=-(f,Hg),$$

for $f,g\in l(V_0)$, and inductively $\big(\mathcal{E}_m,l(V_m)\big)$ on V_m by

$$\mathcal{E}_m(f,g)=\sum_{i=1}^N r_i^{-1}\mathcal{E}_{m-1}(f\circ F_i,g\circ F_i),\quad m\geq 1,$$

for $f,g\in l(V_m)$. Write $\mathcal{E}_m(f):=\mathcal{E}_m(f,f)$ for short.

Say $(H,\boldsymbol{r})$ is a *harmonic structure* if for any $f\in l(V_0)$,

$$\mathcal{E}_0(f)=\min\{\mathcal{E}_1(g):g\in l(V_1),g|_{V_0}=f\}.$$

In addition, call $(H,\boldsymbol{r})$ a *regular harmonic structure* if $0<r_i<1$, $\forall 1\leq i\leq N$. In this paper, we will always assume that there exists a regular harmonic structure associated with K.

Now for each $f\in C(K)$, the sequence $\{\mathcal{E}_m(f)\}_{m\geq 0}$ is nondecreasing, so the following definition makes sense. Let $dom\mathcal{E}=\{f\in C(K):\lim\limits_{m\to\infty}\mathcal{E}_m(f)<\infty\}$, and

$$\mathcal{E}(f,g)=\lim_{m\to\infty}\mathcal{E}_m(f,g)\text{ for } f,g\in dom\mathcal{E}.$$

We write $\mathcal{E}(f):=\mathcal{E}(f,f)$ for short, and call $\mathcal{E}(f)$ the *energy* of f. Note that the form $(\mathcal{E},dom\mathcal{E})$ satisfies the *self-similar property*,

$$\mathcal{E}(f,g)=\sum_{w\in W_m}r_w^{-1}\mathcal{E}(f\circ F_i,g\circ F_i),\quad \forall m\geq 1,f,g\in dom\mathcal{E},$$

with $r_w:=r_{w_1}r_{w_2}\cdots r_{w_m}$. It is known that $(\mathcal{E},dom\mathcal{E})$ turns out to be a *local regular Dirichlet form* on $L^2(K,\mu)$ for any Radon measure μ on K.

There is a natural metric on K related with the energy form $(\mathcal{E},dom\mathcal{E})$, called the *effective resistance metric*, which is defined as

$$R(x,y)=\big(\min\{\mathcal{E}(f):f\in dom\mathcal{E}\text{ and } f(x)=1,f(y)=0\}\big)^{-1},\quad \forall x\neq y\in K.$$

It is easy to see that the Hausdorff dimension of K with respect to $R(\cdot,\cdot)$ is the unique positive number d_H satisfying $\sum_{i=1}^N r_i^{d_H}=1$. In this paper, we always choose a proper self-similar measure μ on K that matches with $R(\cdot,\cdot)$. To be more precise, we fix a weight vector $\{\mu_i\}_{i=1}^N$ such that $\mu_i=r_i^{d_H}$, and let μ be the unique probability measure supported on K such that

$$\mu(A)=\sum_{i=1}^N \mu_i\mu(F_i^{-1}A),\quad \forall A\subset K.$$

One can easily check that $\mu(F_wK)=\mu_w:=\mu_{w_1}\cdots\mu_{w_m}$, for each $w\in W_m$.

For $f\in dom\mathcal{E}$, say $\Delta f=u$ if

$$\mathcal{E}(f,\psi)=-\int_K u\psi d\mu$$

holds for any $\psi\in dom_0\mathcal{E}$, with $dom_0\mathcal{E}:=\{\psi\in dom\mathcal{E}:\psi|_{V_0}=0\}$. In particular, we define $dom_{L^p(K)}\Delta=\big\{f\in dom\mathcal{E}:\Delta f\in L^p(K)\big\}$ for $1<p<\infty$, where and from now on, we abbreviate $L^p(K,\mu)$ to $L^p(K)$.

We are also interested in the double cover $\tilde{K}$, which consists of two copies of K identified at all boundary points. For convenience, we sometimes denote the two copies K_+ and K_-.

One can simply define an energy form $(\tilde{\mathcal{E}},dom\tilde{\mathcal{E}})$ on $\tilde{K}$ by

$$\tilde{\mathcal{E}}(f,g)=\mathcal{E}_+(f|_{K_+},g|_{K_+})+\mathcal{E}_-(f|_{K_-},g|_{K_-}),$$

with $dom\tilde{\mathcal{E}}=\{f\in C(\tilde{K}):f|_{K_+}\in dom\mathcal{E}_+ \text{ and } f|_{K_-}\in dom\mathcal{E}_-\}$, where $(\mathcal{E}_+,dom\mathcal{E}_+)$ and $(\mathcal{E}_-,dom\mathcal{E}_-)$ are the natural energy forms on K_+,K_- respectively. Furthermore, we take $\tilde{\mu}$ to be the measure on $\tilde{K}$ which coincides with μ on each copy $K_\pm$, and define the Laplacian $\tilde{\Delta}$ on $\tilde{K}$ as before (there is no boundary in this case).

In fact, for the double cover $\tilde{K}$, we can alternately define $\tilde{\Delta}$ with the Bessel potential $(1-\tilde{\Delta})^{-1}=\int_0^\infty e^{-t}P_tdt$, where $\{P_t\}_{t\geq 0}$ is the associated heat semigroup on $\tilde{K}$. This definition is shown to be consistent with the former definition in [23], and $\mathcal{D}(\tilde{\Delta})=dom_{L^p(\tilde{K})}\tilde{\Delta}$ in the L^p setting.

1.3. **Function spaces on fractal domains.** We will study Sobolev spaces and Besov spaces on products of fractals $\Omega=K^l\times\tilde{K}^{d-l}$, where $d\in\mathbb{N}$ and $0\leq l\leq d$. There is a natural product measure, still denoted by μ by a bit abuse of notation, and we can define $\Delta=\Delta^{(1)}+\cdots+\Delta^{(d)}$ as the Laplacian on $\tilde{K}^d$, where $\Delta^{(i)}$ is the Laplacian on the i-th "direction". (We will not distinguish between Δ and $\tilde{\Delta}$ for convenience.) Detailed discussions on the definition of Laplacians will be given in Section 3.

For a special case $\tilde{K}^d$, using theorems about the Calderón-Zygmund operators on fractals [23], we can see that Laplacian Δ defined above is the generator of the heat semigroup $\{U_t\}_{t\geq 0}$ on $L^p(\tilde{K}^d)$, where U_t is the product of corresponding heat operators in all directions. In particular, the Bessel potential $(1-\Delta)^{-\sigma/2}=\Gamma(\sigma/2)^{-1}\int_0^\infty t^{\sigma/2}e^{-t}U_tdt$ is well-defined. We define Sobolev spaces as follows.

Definition 1.1. *For $p\in(1,\infty)$, $\sigma\geq 0$, define the* Sobolev space

$$H^p_\sigma(\tilde{K}^d)=(1-\Delta)^{-\sigma/2}L^p(\tilde{K}^d),$$

with norm $\|f\|_{H^p_\sigma(\tilde{K}^d)}=\big\|(1-\Delta)^{\sigma/2}f\big\|_{L^p(\tilde{K}^d)}$.

Using the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$, we define the Besov spaces $B^{p,q}_\sigma(\tilde{K}^d)$ as follows.

Definition 1.2. *For $p,q\in(1,\infty)$ and $\sigma>0$, define the* heat Besov space

$$B^{p,q}_\sigma(\tilde{K}^d)=\Big\{f\in L^p(\tilde{K}^d):\Big(\int_0^\infty\big(t^{-\sigma/2}\|(t\Delta)^k e^{t\Delta}f\|_{L^p(\tilde{K}^d)}\big)^q dt/t\Big)^{1/q}<\infty\Big\},$$

with $k\in\mathbb{N}\cap(\sigma/2,\infty)$, and norm $\|f\|_{B^{p,q}_\sigma(\tilde{K}^d)}=\|f\|_{L^p(\tilde{K}^d)}+\big(\int_0^\infty(t^{-\sigma/2}\|(t\Delta)^k e^{t\Delta}(f)\|_{L^p(\tilde{K}^d)})^q dt/t\big)^{1/q}$.

Note that in the above definition, different choices of k will provide equivalent norms $\|\cdot\|_{B^{p,q}_\sigma(\tilde{K}^d)}$, see [8, 9, 19, 22]. In addition, the Besov spaces $B^{p,q}_\sigma(\tilde{K}^d)$ are real interpolations of Sobolev spaces $H^p_\sigma(\tilde{K}^d)$, see book [19].

Lemma 1.3. *For $p,q\in(1,\infty)$, $\sigma>0$ and $\theta\in(0,1)$, we have*

$$\big(H^p_0(\tilde{K}^d),H^p_\sigma(\tilde{K}^d)\big)_{\theta,q}=B^{p,q}_{\theta\sigma}(\tilde{K}^d).$$

For subdomains $\Omega=K^l\times\tilde{K}^{d-l}\subset\tilde{K}^d$, we will provide a definition of Sobolev spaces and Besov spaces for integer orders first, then extend to positive real orders using interpolation (Definition 3.4 and 4.1). In addition, we will show that these function spaces on Ω are just the restrictions of corresponding type spaces on $\tilde{K}^d$ to Ω (Theorem 3.5, Proposition 4.2).

Moreover, in Section 4, we will extend the definitions of these spaces to negative orders.

2. Basic structures of sequence spaces

In this section, we study sequence spaces with values in a Banach space X. These sequence spaces will play essential roles in reflecting the boundary behavior of functions in Sobolev or Besov spaces on p.c.f. self-similar sets. ~~See Section 3 for a decomposition theorem of Sobolev spaces, and Section 4-6 for various applications of the results in this section.~~

We begin with a simple case.

Definition 2.1. *Let $1<p<\infty$, $\alpha>0$ and $\gamma\in\mathbb{C}\setminus\{0\}$.*

(a). Define $l^p_\alpha(X)=\big\{\boldsymbol{s}=\{s_n\}_{n\geq 0}:\{\alpha^{-n}\|s_n\|_X\}_{n\geq0}\in l^p\big\}$, with norm $\|\boldsymbol{s}\|_{l^p_\alpha(X)}=\big\|\alpha^{-n}\|s_n\|_X\big\|_{l^p}$.

(b). Define $l^p_\alpha(X,\gamma)=\big\{\boldsymbol{s}=\{s_n\}_{n\geq0}:\{s_{n+1}-\gamma s_n\}_{n\geq0}\in l^p_\alpha(X)\big\}$, with norm $\|\boldsymbol{s}\|_{l^p_\alpha(X,\gamma)}=|\gamma|^{-1}\big\|\{s_{n+1}-\gamma s_n\}_{n\geq0}\big\|_{l^p_\alpha(X)}+\|s_0\|_X$.

(c). Define $\overline{l^p_\alpha(X)}^\gamma$ the closure of $l^p_\alpha(X)$ in $l^p_\alpha(X,\gamma)$.

We will compare the spaces defined above. It is convenient to introduce the following operators.

Definition 2.2. *Let $\alpha>0$ and $\gamma\in\mathbb{C}\setminus\{0\}$.*

(a). For each $s\in X$, define $\vec{1}(s)=\{s,s,\cdots\}$, which is a sequence of constant value.

(b). Define $\Lambda(\gamma):X^{\mathbb{Z}_+}\to X^{\mathbb{Z}_+}$ such that $\Lambda(\gamma)\big(\{s_n\}_{n\geq0}\big)=\{\gamma^n s_n\}_{n\geq0}$.

Throughout this paper, we write $X=\oplus_{k=1}^m X_k$ for Banach spaces X and $X_k, 1\leq k\leq m$, if

1. $X_k\subset X$ and $\|\cdot\|_{X_k}\asymp\|\cdot\|_X$, for each $1\leq k\leq m$;
2. For each $x\in X$, there is a unique representation $x=\sum_{k=1}^m x_k$, with $x_k\in X_k, 1\leq k\leq m$.

Lemma 2.3. *Let $\alpha>0$ and $\gamma\in\mathbb{C}\setminus\{0\}$.*

(a). For $1<p<\infty$, we have that $\Lambda(\gamma^{-1})$ is an isometry from $l^p_\alpha(X,\gamma)$ to $l^p_{\alpha|\gamma|^{-1}}(X,1)$.

(b). For $1<p<\infty$, we have

$$l^p_\alpha(X,\gamma)=\begin{cases}\overline{l^p_\alpha(X)}^\gamma, & \text{if } \alpha\geq|\gamma|,\\ \overline{l^p_\alpha(X)}^\gamma\oplus\Lambda(\gamma)\vec{1}(X), & \text{if } \alpha<|\gamma|.\end{cases}\tag{2.1}$$

In addition,

$$\overline{l^p_\alpha(X)}^\gamma=l^p_\alpha(X) \text{ if and only if } \alpha\neq|\gamma|.\tag{2.2}$$

Proof. (a) is easy. For (b), it is enough to consider the case $\gamma=1$ by (a).

Let $\boldsymbol{s}=\{s_n\}_{n\geq0}\in l^p_\alpha(X,1)$. Define $\boldsymbol{t}=\{t_n\}_{n\geq0}$ with $t_0=s_0$ and $t_n=s_n-s_{n-1}$ for $n\geq1$. We discuss three cases separately as follows.

Case 1: $\alpha<1$. In this case, we have $\boldsymbol{t}\in l^p_\alpha(X)$ so that $s_\infty=\lim_{n\to\infty}s_n=\sum_{n=0}^\infty t_n$ is well defined. In addition, by the Minkowski inequality, we have the estimate

$$\begin{aligned}\big\|\boldsymbol{s}-\vec{1}(s_\infty)\big\|_{l^p_\alpha(X)}&=\big\|\alpha^{-n}\|s_n-s_\infty\|_X\big\|_{l^p}=\big\|\alpha^{-n}\|\sum_{m=n+1}^\infty t_m\|_X\big\|_{l^p}\\&\leq\big\|\alpha^{-n}\sum_{m=n+1}^\infty\|t_m\|_X\big\|_{l^p}=\big\|\sum_{m=1}^\infty\alpha^m\alpha^{-n-m}\|t_{m+n}\|_X\big\|_{l^p}\\&\leq(\sum_{m=1}^\infty\alpha^m)\|\boldsymbol{t}\|_{l^p_\alpha(X)}\lesssim\|\boldsymbol{s}\|_{l^p_\alpha(X,1)}.\end{aligned}$$

As a consequence, we have $\big(\boldsymbol{s}-\vec{1}(s_\infty)\big)+\vec{1}(s_\infty)\in l^p_\alpha(X)\oplus\vec{1}(X)$. Thus, $l^p_\alpha(X,1)\subset l^p_\alpha(X)\oplus\vec{1}(X)$. The other direction is easy. So both (2.1) and (2.2) follows in this case.

Case 2: $\alpha=1$. Let's first show that $\vec{1}(X)\subset\overline{l^p_\alpha(X)}^1$. For any $s\in X$ and $m\geq1$, we define a sequence $\boldsymbol{v}_m=\{v_{mn}s\}_{n\geq0}$ as follows,

$$v_{mn}=\begin{cases}\frac{m-n}{m}, & \text{if } n<m,\\ 0, & \text{if } n\geq m.\end{cases}$$

It is easy to see that $\|\boldsymbol{v}_m-\vec{1}(s)\|_{l^p_\alpha(X,1)}=m^{-1+1/p}\|s\|_X$. So we have $\vec{1}(X)\subset\overline{l^p_\alpha(X)}^1$.

Next, fix $\alpha'<1$, clearly we have $\vec{1}(X)\subset\overline{l^p_\alpha(X)}^1$ and $l^p_{\alpha'}(X)\subset\overline{l^p_\alpha(X)}^1$. As a consequence, we have $l^p_{\alpha'}(X,1)\subset\overline{l^p_\alpha(X)}^1$ by Case 1. Then (2.1) follows noticing that $l^p_{\alpha'}(X,1)$ is dense in $l^p_\alpha(X,1)$. In addition, we clearly have $\vec{1}(X)\subset\overline{l^p_\alpha(X)}^1\setminus l^p_\alpha(X)$, and so (2.2) follows.

Case 3: $\alpha>1$. In this case, we have the estimate that

$$\begin{aligned}\|\boldsymbol{s}\|_{l^p_\alpha(X)}&=\big\|\alpha^{-n}\|\sum_{m=0}^n t_m\|_X\big\|_{l^p}\leq\big\|\alpha^{-n}\sum_{m=0}^n\|t_{n-m}\|_X\big\|_{l^p}\\&=\big\|\sum_{m=0}^n\alpha^{-m}\alpha^{m-n}\|t_{n-m}\|_X\big\|_{l^p}\leq(\sum_{m=0}^\infty\alpha^{-m})\|\boldsymbol{t}\|_{l^p_\alpha(X)}\lesssim\|\boldsymbol{s}\|_{l^p_\alpha(X,1)}.\end{aligned}$$

As a consequence, we have $l^p_\alpha(X,1)=l^p_\alpha(X)$, and (2.1), (2.2) follows. □

2.1. **The spaces $l^p_{\alpha,\beta}(X_0,X_1)$ and $l^p_{\alpha,\beta}(X_0,X_1,\gamma)$.**

Definition 2.4. *Let (X_0,X_1) be an interpolation couple of Banach spaces and $\alpha,\beta>0$.*

(a). For $1<p<\infty$, define

$$l^p_{\alpha,\beta}(X_0,X_1)=l^p_{\alpha\beta}(X_0)\cap l^p_\beta(X_1),$$

with norm $\|\boldsymbol{s}\|_{l^p_{\alpha,\beta}(X_0,X_1)}=\|\boldsymbol{s}\|_{l^p_{\alpha\beta}(X_0)}+\|\boldsymbol{s}\|_{l^p_\beta(X_1)}$.

(b). For $1<p<\infty$ and $\gamma\in\mathbb{C}\setminus\{0\}$, define

$$l^p_{\alpha,\beta}(X_0,X_1,\gamma)=l^p_{\alpha\beta}(X_0,\gamma)\cap l^p_\beta(X_1,\gamma),$$

with norm $\|\boldsymbol{s}\|_{l^p_{\alpha,\beta}(X_0,X_1,\gamma)}=\|\boldsymbol{s}\|_{l^p_{\alpha\beta}(X_0,\gamma)}+\|\boldsymbol{s}\|_{l^p_\beta(X_1,\gamma)}$.

(c). Define $\overline{l^p_{\alpha,\beta}(X_0,X_1)}^\gamma$ the closure of $l^p_{\alpha,\beta}(X_0,X_1)$ in $l^p_{\alpha,\beta}(X_0,X_1,\gamma)$.

In this paper, we are most interested in the coefficients

$$\alpha\in(0,1),\quad \beta\in(1,\infty),\quad |\gamma|\in(0,1]. \tag{2.3}$$

Also, in our applications, we will always have $X_1\subset X_0$. The following lemma, same as Lemma 2.3 (a), provides some convenience by reducing the coefficient γ to be 1.

Lemma 2.5. *Let $\alpha,\beta>0$, $\gamma\in\mathbb{C}\setminus\{0\}$ and $1<p<\infty$, we have that $\Lambda(\gamma^{-1})$ is an isometry from $l^p_{\alpha,\beta}(X_0,X_1,\gamma)$ to $l^p_{\alpha,\beta|\gamma|^{-1}}(X_0,X_1,1)$.*

Our aim is to recover a decomposition of $l^p_{\alpha,\beta}(X_0,X_1,\gamma)$ as $l^p_\alpha(X,\gamma)$ in Lemma 2.3 (b). We will deal with this in a more concrete setting in the rest subsections. In this subsection, let's first see what is the limit of a sequence in $l^p_{\alpha,\beta}(X_0,X_1,\gamma)$. This is easy for readers who are familiar with the real interpolation of function spaces, but we still provide a short proof below.

For convenience, let's briefly review the J-method of real interpolation.

J-method. Let $\overline{X}=(X_0,X_1)$ be an interpolation couple. Define $\Sigma(\overline{X})=X_0+X_1$ and $\Delta(\overline{X})=X_0\cap X_1$. We have a family of equivalent norms $J(t,s)=\max\big\{\|s\|_{X_0},t\|s\|_{X_1}\big\}$ on $\Delta(\overline{X})$ with $t>0$. It is easy to see that $J(t,s)$ is increasing, continuous, and convex of t.

For a measurable positive function φ on $\mathbb{R}_+$ and $\theta\in(0,1)$, $p\in[1,\infty]$, we write $\Phi_{\theta,p}\big(\varphi(t)\big)=\Big(\int_0^\infty\big(t^{-\theta}\varphi(t)\big)^pdt/t\Big)^{1/p}$ with usual modification when $p=\infty$.

Then for $\theta\in(0,1)$, $p\in[1,\infty]$, we have

$$\begin{aligned}(X_0,X_1)_{\theta,p}=\ &\big\{s\in\Sigma(\overline{X}):s=\textstyle\int_0^\infty u(t)dt/t \text{ for some measurable function}\\ &\qquad u:\mathbb{R}_+\to\Delta(\overline{X})\text{ and }\Phi_{\theta,p}\big(J(t,u(t)\big)<\infty\big\}\end{aligned}$$

with norm $\|s\|_{\overline{X}_{\theta,p}}=\inf_u\Phi_{\theta,p}\big(J(t,u(t))\big)$. In the above identity, $u(t)$ is strongly measurable in $\Delta(\overline{X})$, and the integral is taken in $\Sigma(\overline{X})$. See the book [7] for unexplained details.

Lemma 2.6. *Assume $X_1\subset X_0$ continuously, and assume (2.3). Then, for $\alpha\beta<|\gamma|$, the following operator $\Gamma_\gamma:l^p_{\alpha,\beta}(X_0,X_1,\gamma)\to X_0$ is well-defined:*

$$\Gamma_\gamma\big(\{s_n\}_{n\geq0}\big)=\lim_{n\to\infty}\gamma^{-n}s_n.$$

Moreover, $\Gamma_\gamma:l^p_{\alpha,\beta}(X_0,X_1,\gamma)\to(X_0,X_1)_{\theta,p}$ is bounded and surjective for $\theta=1+\frac{\log\beta|\gamma|^{-1}}{\log\alpha}$.

Proof. By Lemma 2.5, we only need consider the case $\gamma = 1$, noticing that $\Gamma_\gamma = \Gamma_1\Lambda(\gamma^{-1})$.

In this case, $\alpha\beta < 1$, and we have $l^p_{\alpha,\beta}(X_0, X_1, 1) = l^p_{\alpha\beta}(X_0, 1) \cap l^p_{\beta}(X_1, 1) \subset l^p_{\alpha\beta}(X_0, 1)$. Then, by the discussion in Lemma 2.3 (b), we know the limit exists of each sequence in $l^p_{\alpha,\beta}(X_0, X_1, 1)$, and so Γ_1 is well-defined.

Next, we need to show that $\Gamma_1 : l^p_{\alpha,\beta}(X_0, X_1, 1) \to (X_0, X_1)_{\theta,p}$ is bounded and surjective. Let $s = \int_0^\infty u(t)dt/t \in \Sigma(\overline{X}) = X_0$ with a strongly measurable function $u : \mathbb{R}_+ \to \Delta(\overline{X}) = X_1$. For $n \in \mathbb{Z}$, define $u_n = \int_{\alpha^n}^{\alpha^{n-1}} u(t)dt/t$. Then we have $s = \sum_{n=-\infty}^{\infty} u_n$ and

$$\Big(\sum_{n=-\infty}^{\infty}\big(\alpha^{-n\theta}J(\alpha^n, u_n)\big)^p\Big)^{1/p} \lesssim \Phi_{\theta,p}\big(J(t, u(t))\big). \tag{2.4}$$

For $n \le 0$, it is clear that $\alpha^{-n\theta}J(\alpha^n, u_n) = \alpha^{-n\theta}\max\big\{\|u_n\|_{X_0}, \alpha^n\|u_n\|_{X_1}\big\} \asymp \alpha^{(1-\theta)n}\|u_n\|_{X_1}$. Let $s_0 = \sum_{n=-\infty}^{0} u_n$, we then have

$$J(1, s_0) \asymp \|s_0\|_{X_1} \le \sum_{n=-\infty}^{0}\|u_n\|_{X_1} \lesssim \big(\sum_{n=-\infty}^{0}\alpha^{-(1-\theta)np'}\big)^{\frac{1}{p'}}\Big(\sum_{n=-\infty}^{0}\big(\alpha^{-n\theta}J(\alpha^n, u_n)\big)^p\Big)^{\frac{1}{p}}, \tag{2.5}$$

where $p' = \frac{p}{p-1}$. For $n \ge 1$, denote $s_n = u_n$, and define $\bar{u} = (-\log\alpha)^{-1}\sum_{n=0}^{\infty} 1_{(\alpha^n, \alpha^{n-1}]}s_n$. Then by using estimates (2.4) and (2.5), we have

$$\Phi_{\theta,p}\big(J(t, \bar{u}(t))\big) \lesssim \Big(\sum_{n=0}^{\infty}\big(\alpha^{-n\theta}J(\alpha^n, s_n)\big)^p\Big)^{1/p} \lesssim \Phi_{\theta,p}\big(J(t, u(t))\big).$$

On the other hand, since $\alpha^\theta = \alpha\beta$ and $\alpha^{\theta-1} = \beta$, we can easily check

$$\Big(\sum_{n=0}^{\infty}\big(\alpha^{-n\theta}J(\alpha^n, s_n)\big)^p\Big)^{1/p} \asymp \big\|\{s_n\}_{n\ge0}\big\|_{l^p_{\alpha,\beta}(X_0,X_1)}.$$

It follows by J-method that $s \in (X_0, X_1)_{\theta,p}$ if and only if there is $\boldsymbol{s} = \{s_n\}_{n\ge0} \in l^p_{\alpha,\beta}(X_0, X_1)$ such that $s = \sum_{n=0}^{\infty} s_n$. In addition, $\|s\|_{\overline{X}_{\theta,p}} \asymp \inf\big\{\|\{s_n\}_{n\ge0}\|_{l^p_{\alpha,\beta}(X_0,X_1)} : \sum_{n=0}^{\infty} s_n = s\big\}$.

Obviously, this is equivalent to say that $s \in (X_0, X_1)_{\theta,p}$ if and only if $s = \Gamma_1(\boldsymbol{s})$ for some $\boldsymbol{s} \in l^p_{\alpha,\beta}(X_0, X_1, 1)$, and $\|s\|_{\overline{X}_{\theta,p}} \asymp \inf\big\{\|\boldsymbol{s}\|_{l^p_{\alpha,\beta}(X_0,X_1,1)} : s = \Gamma_1(\boldsymbol{s})\big\}$. □

2.2. **The interpolation couple $\overline{\mathcal{D}(L^\sigma)} = \big(X, \mathcal{D}(L^\sigma)\big)$.** From now on, we restrict our consideration to a more concrete setting. In particular, in this subsection, we will derive a decomposition result analogous to Lemma 2.3 (b).

In the rest of this section, we will assume that X is a Banach space with a sectorial operator L. In addition, we may assume the following.

(**L1**). *L is sectorial of angle $\omega \in [0, \frac{\pi}{2})$.*
(**L2**). *$\{(1 + L)^{it}\}_{t\in\mathbb{R}}$ is a C_0-group.*

Readers can find a systematic discussion on sectorial operators in [], see also Appendix C. In particular, there is a one to one correspondence between (single valued) sectorial operators L of angle $\omega \in [0, \frac{\pi}{2})$ and bounded (injective) holomorphic semigroups $\{e^{-tL}\}_{t\ge0}$.

For $\sigma \geq 0$, we denote $\mathcal{D}(L^\sigma)$ the domain of L^σ with norm $\|s\|_{\mathcal{D}(L^\sigma)} = \|s\|_X + \|L^\sigma s\|_X$. For short, we will write $\overline{\mathcal{D}(L^\sigma)} := \big(X, \mathcal{D}(L^\sigma)\big)$. Our interest is the following sequence spaces,

$$\begin{aligned} l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) &:= l^p_{\alpha^\sigma,\beta}\big(X, \mathcal{D}(L^\sigma)\big), \\ l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}, \gamma\big) &:= l^p_{\alpha^\sigma,\beta}\big(X, \mathcal{D}(L^\sigma), \gamma\big), \\ \overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^\gamma &:= \overline{l^p_{\alpha^\sigma,\beta}\big(X, \mathcal{D}(L^\sigma)\big)}^\gamma, \end{aligned}$$

where $\sigma \geq 0$ and $\alpha \in (0,1), \beta \in (1,\infty), \gamma \in (0,1]$ as in (2.3). Here, we denote $\overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^\gamma$ the closure of $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ in $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}, \gamma\big)$.

We need the following proposition from the book [19] (Chapter 6, Section 6.2.3).

Proposition 2.7 ([19])**.** *Assume (**L1**). For $\sigma > 0, 1 \leq p \leq \infty$ and any fixed $\sigma' > \sigma$, let*

$$X_{\sigma,p} = \big\{s \in X : \big(\int_0^\infty t^{-\sigma p}\|(tL)^{\sigma'} e^{-tL}(s)\|_X^p dt/t\big)^{1/p} < \infty\big\}.$$

Then $X_{\sigma,p} = \big(X, \mathcal{D}(L^{\sigma'})\big)_{\theta,p}$ with $0 < \theta = \sigma/\sigma' < 1$, and norm $\|s\|_{X_{\sigma,p}}$ equivalent to

$$\|s\|_X + \big(\int_0^\infty t^{-\sigma p}\|(tL)^{\sigma'} e^{-tL}(s)\|_X^p dt/t\big)^{1/p}.$$

As a consequence of the above proposition, we can derive the following lemma.

Lemma 2.8. *Assume (**L1**). Let $k \in \mathbb{N}$, $\varphi \in C_c(0,1)$ satisfying $\int_0^1 \varphi(t)dt = 1$ and*

$$\int_0^1 t^j \varphi(t) dt = 0 \text{ for } j = 1, 2, \cdots, k-1.$$

For $0 < \alpha < 1$, $s \in X$, we define $S^{L,\varphi}_\alpha(s) = \{S^{L,\varphi}_\alpha(s)_n\}_{n\geq 0}$ by

$$S^{L,\varphi}_\alpha(s)_n = \alpha^{-n}\int_0^{\alpha^n} \varphi(\alpha^{-n}t) e^{-tL}(s)dt, \quad \forall n \geq 0.$$

Then, for $\theta > 0$ and $0 < \sigma < k$, we have $S^{L,\varphi}_\alpha : X_{\sigma,p} \to l^p_{\alpha^{\sigma+\theta},\alpha^{-\theta}}\big(\overline{\mathcal{D}(L^{\sigma+\theta})}, 1\big)$ and $\Gamma_1\big(S^{L,\varphi}_\alpha(s)\big) = s, \forall s \in X_{\sigma,p}$.

Proof. Let $s \in X_{\sigma,p}$. First, we immediately have

$$\big\|L^{\sigma+\theta} S^{L,\varphi}_\alpha(s)_n\big\|_X \lesssim \alpha^{-n\theta}\big(\int_{c\alpha^n}^{\alpha^n} t^{-\sigma p}\|(tL)^{\sigma+\theta} e^{-tL}(s)\|_X^p dt/t\big)^{1/p}, \tag{2.6}$$

where we assume φ supports on $[c,1]$ with $c > 0$.

Next, using the assumption on φ, we have

$$S^{L,\varphi}_\alpha(s)_{n+1} - S^{L,\varphi}_\alpha(s)_n = \alpha^{-n}(-1)^k \int_0^{\alpha^n} \alpha^{nk}\varPhi(\alpha^{-n}t)\frac{d^k}{dt^k} e^{-tL}(s)dt,$$

where $\Phi(t)=\int_{0\leq t_1\leq t_2\leq\cdots\leq t_k\leq t}\big(\alpha^{-1}\varphi(\alpha^{-1}t_1)-\varphi(t_1)\big)dt_k\cdots dt_1$. In fact, Φ is characterized as the unique function supported in $[c\alpha,1]$ such that $\Phi^{(k)}(t)=\alpha^{-1}\varphi(\alpha^{-1}t)-\varphi(t)$. So we have

$$\begin{aligned}\big\|S^{L,\varphi}_\alpha(s)_{n+1}-S^{L,\varphi}_\alpha(s)_n\big\|_X&\lesssim\alpha^{n\sigma}\Big(\int_{c\alpha^{n+1}}^{\alpha^n}t^{-\sigma p}\|t^k\frac{d^k}{dt^k}e^{-tL}(s)\|_X^p dt/t\Big)^{1/p}\\&=\alpha^{n\sigma}\Big(\int_{c\alpha^{n+1}}^{\alpha^n}t^{-\sigma p}\|(tL)^k e^{-tL}(s)\|_X^p dt/t\Big)^{1/p}.\end{aligned}\tag{2.7}$$

Combining estimates (2.6), (2.7), and using Proposition 2.7, we then have

$$S^{L,\varphi}_\alpha(s)\in l^p_{\alpha^\sigma}(X,1)\cap l^p_{\alpha^{-\theta}}\big(\mathcal{D}(L^{\sigma+\theta})\big).$$

Noticing that $\alpha^{-\theta}>1$, we have $S^{L,\varphi}_\alpha(s)\in l^p_{\alpha^{\sigma+\theta},\alpha^{-\theta}}\big(\overline{\mathcal{D}(L^{\sigma+\theta})},1\big)$ by Lemma 2.3. Lastly, since $\int_0^1\varphi(t)dt=1$, we have $\lim_{n\to\infty}S^{L,\varphi}_\alpha(s)_n=s$ in X, and thus $\Gamma_1\big(S^{L,\varphi}_\alpha(s)\big)=s$ in $X_{\sigma,p}$. □

Combining Lemma 2.6 and 2.8, we are able to derive the following decomposition of the spaces $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$.

Proposition 2.9. *Assume (**L1**) and (2.3). Let $1<p<\infty$, $k\geq 1$, and define φ and $S^{L,\varphi}_\alpha$ as in Lemma 2.8. Then for $0<\sigma<k-\frac{\log\beta\gamma^{-1}}{\log\alpha}$ we have,*

$$l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)=\begin{cases}\overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}, & \text{if } \alpha^\sigma\beta\geq\gamma,\\ \overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}\oplus\Lambda(\gamma)S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},p}), & \text{if } \alpha^\sigma\beta<\gamma.\end{cases}\tag{2.8}$$

In particular, we have

$$\overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}=l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)\ \textit{if and only if}\ \alpha^\sigma\beta\neq\gamma.\tag{2.9}$$

Proof. We consider three cases separately.

Case 1: $\alpha^\sigma\beta>\gamma$. In this case, we can see that $l^p_{\alpha^\sigma\beta}(X)=l^p_{\alpha^\sigma\beta}(X,\gamma)$ and $l^p_\beta\big(\mathcal{D}(L^\sigma)\big)=l^p_\beta\big(\mathcal{D}(L^\sigma),\gamma\big)$ by Lemma 2.3. Both (2.8) and (2.9) follows.

Case 2: $\alpha^\sigma\beta=\gamma$. Using Lemma 2.3, we can see that

$$\Lambda(\gamma)\vec{1}\big(\mathcal{D}(L^\sigma)\big)\subset l^p_{\alpha^\sigma\beta}\big(\mathcal{D}(L^\sigma),\gamma\big)=\overline{l^p_{\alpha^\sigma\beta}\big(\mathcal{D}(L^\sigma)\big)}^{\gamma}\subset\overline{l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}.$$

Observe that for any $s\neq 0$ in $\mathcal{D}(L^\sigma)$, $\Lambda(\gamma)\vec{1}(s)\notin l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$. We have (2.9) proved.

In addition, we see that

$$\big\{\{\gamma^n s\}_{n\geq 0}:s\in\mathcal{D}(L^\sigma)\big\}\cup\big\{\{\delta_{nm}s\}_{n\geq 0}:s\in\mathcal{D}(L^\sigma)\text{ and }m\geq 0\big\}$$

spans a dense subspace of $l^p_{\alpha^\sigma,\beta}(\overline{\mathcal{D}(L^\sigma)},\gamma)$. So (2.8) follows.

Case 3: $\alpha^\sigma\beta<\gamma$. First, we have $S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},p})\subset l^p_{\alpha^\sigma,\beta\gamma^{-1}}\big(\overline{\mathcal{D}(L^\sigma)},1\big)$ by applying Lemma 2.8. So we have

$$l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)\oplus\Lambda(\gamma)S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},p})\subset l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big),$$

applying Lemma 2.5.

On the other hand, let $\boldsymbol{s} \in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$. We have $s_\infty := \Gamma_\gamma(\boldsymbol{s}) \in X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},p}$ by Lemma 2.6. Define $\boldsymbol{s}' = \boldsymbol{s} - \Lambda(\gamma)S^{L,\varphi}_\alpha(s_\infty)$. Then we have

$$\boldsymbol{s}' \in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) = l^p_{\alpha^\sigma\beta}(X) \cap l^p_\beta\big(\mathcal{D}(L^\sigma)\big)$$

by Lemma 2.3, noticing that $s'_\infty = \lim_{n\to\infty}\gamma^{-n}s'_n = 0$ in X. Thus, we have

$$l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big) \subset l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) \oplus \Lambda(\gamma)S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},p}).$$

So (2.8) and (2.9) follow immediately. □

2.3. **The spaces $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ and $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$.** In this last subsection, we will develop some interpolation properties of the sequence spaces. The following result from the book [], Chapter 6 is useful.

Proposition 2.10 ([19]). *Assume (**L2**). Then for $\sigma > 0$, $\theta \in (0,1)$, we have*

$$\mathcal{D}(L^{\theta\sigma}) = [X, \mathcal{D}(L^\sigma)]_\theta,$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation space.

Lemma 2.11. *Assume (**L2**) and (2.3). Then for $p \in (1,\infty)$, $\sigma_1,\sigma_2 \geq 0$ and $\theta \in (0,1)$, we have*

$$\big[l^p_{\alpha^{\sigma_1},\beta}\big(\overline{\mathcal{D}(L^{\sigma_1})}\big), l^p_{\alpha^{\sigma_2},\beta}\big(\overline{\mathcal{D}(L^{\sigma_2})}\big)\big]_\theta = l^p_{\alpha^{\sigma_\theta},\beta}\big(\overline{\mathcal{D}(L^{\sigma_\theta})}\big), \text{ with } \sigma_\theta = (1-\theta)\sigma_1 + \theta\sigma_2.$$

Proof. First, we consider the interpolation couple $\big(l^p(X), l^p_{\alpha^\sigma,\alpha^{-\sigma}}\big(\overline{\mathcal{D}(L^\sigma)}\big)\big)$. By a little abuse of the notations, we write $\Lambda(\alpha)L : l^p(X) \to l^p(X)$ as

$$\Lambda(\alpha)L\big(\{s_n\}_{n\geq0}\big) = \big\{\alpha^n L(s_n)\big\}_{n\geq0}.$$

One can show that $\Lambda(\alpha)L$ is sectorial. In fact, for each $\lambda > 0$, we have

$$\lambda\big(\lambda + \Lambda(\alpha)L\big)^{-1}\big(\{s_n\}_{n\geq0}\big) = \big\{\lambda(\lambda+\alpha^n L)^{-1}(s_n)\big\}_{n\geq0}.$$

Thus $\lambda\big(\lambda+\Lambda(\alpha)L\big)^{-1}$ is uniformly bounded by $\sup_{\lambda>0,\varepsilon>0}\big\|\lambda(\lambda+\varepsilon L)^{-1}\big\|$. The fact that $\Lambda(\alpha)L$ is sectorial follows from Proposition 2.1.1 (a) and (f) in book [].

Next, we can check that $\big\{\big(1+\Lambda(\alpha)L\big)^{it}\big\}_{t\in\mathbb{R}}$ is a C_0-group. First, we have $\big(1+\Lambda(\alpha)L\big)^{it} \in \mathcal{L}\big(l^p(X)\big)$. In fact, $\big(1+\Lambda(\alpha)L\big)^{it}\big(\{s_n\}_{n\geq0}\big) = \big\{(1+\alpha^n L)^{it}(s_n)\big\}_{n\geq0}$, and we have the following estimate for each term

$$\begin{aligned}\big\|(1+\alpha^n L)^{it}\big\| &= \big\|\big(\alpha^n + (1-\alpha^n)(1+L)^{-1}\big)^{it}(1+L)^{it}\big\| \\ &\leq \big\|\big(\frac{\alpha^n}{1-\alpha^n} + (1+L)^{-1}\big)^{it}\big\| \cdot \big\|(1+L)^{it}\big\| \leq C\big\|(1+L)^{it}\big\|,\end{aligned}$$

where C is clearly independent of n by Proposition 3.5.5 (c) in []. Second, we can see that $\mathcal{D}\big(1+\Lambda(\alpha)L\big) \cap \mathcal{R}\big(1+\Lambda(\alpha)L\big)$ is dense in $l^p(X)$, since $\mathcal{D}(L)$ is dense in X and $\{\delta_{nm}s\}_{n\geq0} \in \mathcal{D}\big(1+\Lambda(\alpha)L\big) \cap \mathcal{R}\big(1+\Lambda(\alpha)L\big)$ for any $s \in \mathcal{D}(L)$ and $m \geq 0$, where $\mathcal{D}\big(1+\Lambda(\alpha)L\big)$ and $\mathcal{R}\big(1+\Lambda(\alpha)L\big)$ are the domain and range of the operator $1+\Lambda(\alpha)L$ respectively. Combining the above two claims, by Corollary 3.5.7 in [], we have $\big\{\big(1+\Lambda(\alpha)L\big)^{it}\big\}_{t\in\mathbb{R}}$ is a C_0-group.

Thus, we can apply Proposition 2.10 to conclude that

$$\big[l^p(X), l^p_{\alpha^\sigma,\alpha^{-\sigma}}\big(\overline{\mathcal{D}(L^\sigma)}\big)\big]_\theta = \big[l^p(X), \mathcal{D}\big((\Lambda(\alpha)L)^\sigma\big)\big]_\theta = \mathcal{D}\big((\Lambda(\alpha)L)^{\theta\sigma}\big) = l^p_{\alpha^{\theta\sigma},\alpha^{-\theta\sigma}}\big(\overline{D(L^{\theta\sigma})}\big).$$

By a standard argument of complex interpolation, we then conclude that

$$\big[l^p_\beta(X), l^p_{\alpha^\sigma,\beta}\big(\overline{D(L^\sigma)}\big)\big]_\theta = l^p_{\alpha^{\theta\sigma},\beta}\big(\overline{\mathcal{D}(L^{\theta\sigma})}\big).$$

The lemma follows by using the reiteration theorem. □

Now we have a class of sequence spaces that are stable under complex interpolation, by the reiteration theorem of real interpolations.

Definition 2.12. *Assume (**L2**) and (2.3), and let $p,q\in(1,\infty)$. Define*

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) = \big(l^p_\beta(X), l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})}\big)\big)_{\frac{\sigma}{\sigma'},q},$$

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big) = \big(l^p_\beta(X,\gamma), l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})},\gamma\big)\big)_{\frac{\sigma}{\sigma'},q},$$

for some $\sigma'>\sigma>0$.

We have the following decomposition concerning the spaces.

Proposition 2.13. *Assume (2.3), (**L1**) and (**L2**). Let $p,q\in(1,\infty)$, $k\geq 1$, φ and $S^{L,\varphi}_\alpha$ as in Lemma 2.8. Then for $0<\sigma<k-\frac{\log\beta\gamma^{-1}}{\log\alpha}$, we have*

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big) = \begin{cases} \overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}, & \text{if } \alpha^\sigma\beta\geq\gamma,\\ \overline{l^{p,q}_{\alpha^\sigma,\beta,}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}\oplus\Lambda(\gamma)S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},q}), & \text{if } \alpha^\sigma\beta<\gamma.\end{cases} \tag{2.10}$$

Here, we denote $\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}$ the closure of $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ in $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$. In particular, we have

$$\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma} = l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) \text{ if and only if } \alpha^\sigma\beta\neq\gamma. \tag{2.11}$$

Proof. We consider three cases separately.

Case 1: $\alpha^\sigma\beta>\gamma$. In this case, we choose σ' such that $-\frac{\log\beta\gamma^{-1}}{\log\alpha}>\sigma'>\sigma$, then

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) = \big(l^p_\beta(X), l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})}\big)\big)_{\frac{\sigma}{\sigma'},q} = \big(l^p_\beta(X,\gamma), l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})},\gamma\big)\big)_{\frac{\sigma}{\sigma'},q} = l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$$

by Lemma 2.3 and Proposition 2.9. Both (2.10) and (2.11) hold in this case.

Case 2: $\alpha^\sigma\beta=\gamma$. In this case, we take $\sigma'>\sigma$, and notice that

$$\begin{cases} l^p_\beta\big(\mathcal{D}(L^{\sigma'})\big)\subset l^p_\beta(X),\\ l^p_{\alpha^{\sigma'}\beta}\big(\mathcal{D}(L^{\sigma'})\big)\subset l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})}\big),\end{cases} \quad\text{and}\quad \begin{cases} l^p_\beta\big(\mathcal{D}(L^{\sigma'}),\gamma\big)\subset l^p_\beta(X,\gamma),\\ l^p_{\alpha^{\sigma'}\beta}\big(\mathcal{D}(L^{\sigma'}),\gamma\big)\subset l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})},\gamma\big).\end{cases}$$

As a consequence, we have by using real interpolation (see Theorem 5.6.1 in the book []),

$$l^q_{\alpha^\sigma\beta}\big(\mathcal{D}(L^{\sigma'})\big)\subset l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big),\qquad l^q_{\alpha^\sigma\beta}\big(\mathcal{D}(L^{\sigma'}),\gamma\big)\subset l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big). \tag{2.12}$$

Then, we can see that

$$\{\gamma^n s\}_{n\geq0}\in\overline{l^q_{\alpha^\sigma\beta}\big(\mathcal{D}(L^{\sigma'})\big)}^{\gamma}\subset\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma} \text{ for any } s\in\mathcal{D}(L^{\sigma'}) \tag{2.13}$$

by using Lemma 2.3. Now, using (2.13), we have $l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})},\gamma\big)\subset\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}$. Since $l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})},\gamma\big)$ is dense in $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$ by the property of real interpolation, we see that $l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)\subset\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^{\gamma}$, and thus (2.10) follows immediately.

It remains to show (2.11), i.e. $\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)}^\gamma \neq l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ in this case. Obviously, we have $l^p_{\alpha^{\sigma'},\beta}\big(\overline{\mathcal{D}(L^{\sigma'})}\big) \subset l^p_{\alpha^{\sigma'}\beta}(X)$ and as a consequence

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) \subset l^q_{\alpha^\sigma\beta}(X). \tag{2.14}$$

Thus, for $s \in \mathcal{D}(L^{\sigma'})$ and $s \neq 0$, we can see that $\{\gamma^n s\}_{n\geq 0} \in l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big) \setminus l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ by (2.13) and (2.14). Thus (2.11) follows.

Case 3: $\alpha^\sigma\beta < \gamma$. For this case, we choose $-\frac{\log \beta\gamma^{-1}}{\log \alpha} < \sigma_1 < \sigma < \sigma_2$, we notice that by Proposition 2.9,

$$l^p_{\alpha^{\sigma_i},\beta}\big(\overline{\mathcal{D}(L^{\sigma_i})},\gamma\big) = l^p_{\alpha^{\sigma_i},\beta}\big(\overline{\mathcal{D}(L^{\sigma_i})}\big) \oplus \Lambda(\gamma) S^{L,\varphi}_\alpha(X_{\sigma_i+\frac{\log\beta\gamma^{-1}}{\log\alpha},p}), i = 1,2.$$

Then by the reiteration theorem of real and complex interpolations, and the fact that $(A_1 \oplus B_1, A_2 \oplus B_2)_{\theta,q} = (A_1,A_2)_{\theta,q} \oplus (B_1,B_2)_{\theta,q}$ if $(A_1 + A_2) \cap (B_1 + B_1) = \{0\}$, we conclude that

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big) = l^{p,q}_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) \oplus \Lambda(\gamma) S^{L,\varphi}_\alpha(X_{\sigma+\frac{\log\beta\gamma^{-1}}{\log\alpha},q}).$$

So (2.10) and (2.11) follows immediately. □

3. A decomposition of Sobolev spaces

From now on, we return to the study of function spaces on fractals. We will always use K to denote a connected p.c.f. self-similar set equipped with a regular harmonic structure $(H,\boldsymbol{r})$, and a self-similar measure μ with weight $\mu_i = r_i^{d_H}$ as introduced in Section 1.

In this section, we will establish some useful characterizations of Sobolev spaces on products of fractals. We split this section into three parts. In the first part, we will give a brief discussion on the Laplacians on product spaces, and will provide a useful characterization of the Sobolev spaces. In the second part, we will study the relation between Sobolev spaces and the sequence spaces we described in Section 2. In particular, we will decompose a Sobolev space into the union of a kernel part and a sequence part. In the third part, we will fulfill the unprovided proofs in the last two parts.

3.1. The Laplacian.
First, we introduce some notations.

Notation. *Let $S_1, S_2, \cdots, S_d$ be some metric measure spaces, and let $\Omega = S_1 \times S_2 \times \cdots \times S_d$ be the product space with product topology and measure.*

(a). Let f_i be a measurable function on S_i for $i = 1,2,\cdots,d$. We define the tensor product $f_1 \otimes f_2 \otimes \cdots \otimes f_d : \Omega \to \mathbb{C}$ by

$$f_1 \otimes f_2 \otimes \cdots \otimes f_d(x) = f_1(x_1) f_2(x_2) \cdots f_d(x_d), \quad \forall x = (x_1,x_2,\cdots,x_d) \in \Omega.$$

(b). For each $x = (x_1,x_2,\cdots,x_d) \in \Omega$ and $1 \leq i \leq d$, write $x^{\wedge i} = (x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_d)$. Denote $\Omega^{\wedge i} = \{x^{\wedge i} : x \in \Omega\}$.

(c). For $1 \leq i \leq d$, $x = (x_1,x_2\cdots,x_d) \in \Omega$ and $f : \Omega \to \mathbb{C}$, write

$$f^{(i)}_{x^{\wedge i}}(x_i) = f(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_d).$$

In this way, for each $y \in \Omega^{\wedge i}$, we view $f^{(i)}_y(\cdot)$ as a function $S_i \to \mathbb{C}$, and write $f^{(i)}_\bullet$ for the corresponding map from $\Omega^{\wedge i}$ to functions on S_i.

On the other hand, for each $z \in S_i$, we view $f^{(i)}_\bullet(z)$ as a function $\Omega^{\wedge i} \to \mathbb{C}$, and write $f^{(i)}(\cdot)$ for the corresponding map from S_i to functions on $\Omega^{\wedge i}$.

(d). Following the notations of (c), for $p \in (1,\infty)$, for a function space $D(S_i)$ on S_i, we define

$$L^p\big(\Omega^{\wedge i}, D(S_i)\big) := \big\{f^{(i)}_\bullet \text{ is strongly measurable from } \Omega^{\wedge i} \text{ to } D(S_i), \text{ and } \|f^{(i)}_\bullet\|_{D(S_i)} \in L^p(\Omega^{\wedge i})\big\};$$

for a function space $D(\Omega^{\wedge i})$ on $\Omega^{\wedge i}$, we define

$$L^p\big(S_i, D(\Omega^{\wedge i})\big) := \big\{f^{(i)}(\cdot) \text{ is strongly measurable from } S_i \text{ to } D(\Omega^{\wedge i}), \text{ and } \|f^{(i)}(\cdot)\|_{D(\Omega^{\wedge i})} \in L^p(S_i)\big\}.$$

(e). Let $U : L^p(S_i) \to L^p(S_i)$ be a closed operator. Define $U^{(i)} : L^p(\Omega) \to L^p(\Omega)$ as the closed operator with $\mathcal{D}(U^{(i)}) = L^p\big(\Omega^{\wedge i}, \mathcal{D}(U)\big)$, and $U^{(i)}f(x) = Uf_{x^{\wedge i}}(x_i)$ (in almost everywhere sense).

Remark 1. If $\rho(U) \neq \emptyset$, we can show $\big(c - U^{(i)}\big)^{-1} = \big((c-U)^{-1}\big)^{(i)}$ for $c \in \rho(U)$, and thus

$$\mathcal{D}(U^{(i)}) = \big\{f \in L^p(\Omega) : f^{(i)}_y \in \mathcal{D}(U) \text{ for almost every } y \in \Omega^{\wedge i}, \text{ and } Uf_{\bullet^{\wedge i}}(\bullet_i) \in L^p\big(\Omega\big)\big)\big\}.$$

Remark 2. In the case that $S_1 = \cdots = S_d = S$, we will usually omit the index of S_i, but still keep the superscript in $\Omega^{\wedge i}$ to highlight i. See Proposition 3.3 for example.

Let's consider the particular case $\Omega = K^l \times \tilde{K}^{d-l}$. With the above notations, the $\Delta^{(i)}$ with $1 \leq i \leq d$ are the Laplacians acting on certain "directions". For short, we write $\boldsymbol{i} = i_1 i_2 \cdots i_m$ with each $1 \leq i_j \leq d$, and $\Delta^{(\boldsymbol{i})} = \Delta^{(i_m)}\Delta^{(i_{m-1})}\cdots\Delta^{(i_1)}$. Write $C^\infty(\Omega)$ for the space of "smooth" functions f such that $\Delta^{(\boldsymbol{i})}f \in C(\Omega)$ for any $\boldsymbol{i}$.

Remark 3. On $\Omega = K^l \times \tilde{K}^{d-l}$, for $l < i \leq d$, by Remark 1, we can equivalently define $-\Delta^{(i)}$ as the generator of the semigroup $\{P^{(i)}_t\}_{t\geq 0}$ on $L^p(\Omega)$, where $\{P_t\}_{t\geq 0}$ is the heat semigroup of $-\Delta$ on $L^p(\tilde{K})$.

Remark 4. On $\Omega = K^l \times \tilde{K}^{d-l}$, for $1 \leq i \leq l$, we have $f = -\Delta^{(i)}G^{(i)}f$, where G is the Green's operator on K. Clearly, $G^{(i)}f(x) = \int_K G(x_i,y)f^{(i)}_{x^{\wedge i}}(y)d\mu(y)$, where $G \in C(K\times K)$ is the Green's function. See [26] or [36] for detailed constructions of the Green's function G.

The definition of Δ on $\tilde{K}^d$ is a little more complicated. The heat operator on $\tilde{K}^d$ is naturally defined as the product $U_t = P^{(1)}_t \cdots P^{(d)}_t$. The $-\Delta$ on $\tilde{K}^d$, viewed as the generator of $\{U_t\}_{t\geq 0}$, is determined by the corresponding Bessel potential

$$(1-\Delta)^{-1} = \int_0^\infty e^{-t}U_t dt.$$

It is well-known that Hambly and Kumagai in [20], Kumagai and Sturm in [27], showed that the p.c.f. fractals under consideration satisfy the sub-Gaussian heat kernel estimates. As an application, Ionescu, Rogers and Strichartz [23] studied Calderón-Zygmund operators on product of p.c.f. fractals. Below is an immediate consequence of Corollary 5.5 in [23].

Proposition 3.1 ([23])**.** *The operators $\Delta^{(i)}(1-\Delta)^{-1}$ is bounded from $L^p(\tilde{K}^d)$ to $L^p(\tilde{K}^d)$.*

Lemma 3.2. *For $1 < p < \infty$ and $\Omega = \tilde{K}^d$, we have $\Delta = \sum_{i=1}^d \Delta^{(i)}$.*

Proof. First, we show $\sum_{i=1}^d \Delta^{(i)} \subset \Delta$. Let $f \in \bigcap_{i=1}^d \mathcal{D}(\Delta^{(i)})$. By Remark 3, we have $(P^{(i)}_t - 1)f = \int_0^t P^{(i)}_{t'}\Delta^{(i)}f dt'$ using the fundamental identity for semigroups (Proposition

A.8.2 in the book []). As a consequence, we have

$$U_tf-f=\sum_{i=1}^d\big(\prod_{j=i+1}^d P_t^{(j)}\big)(P^{(i)}f-f)=\sum_{i=1}^d\big(\int_0^t\big(\prod_{j=i+1}^d P_t^{(j)}\big)P_{t'}^{(i)}dt'\big)\Delta^{(i)}f.$$

Furthermore, using the fact that $\{P_t^{(i)}\}_{t\geq 0}$ is a bounded strongly continuous semigroup for any $1\leq i\leq d$, we can conclude that $\lim\limits_{t\to 0}\frac{1}{t}(U_tf-f)=\sum_{i=1}^d\Delta^{(i)}f$. This shows $f\in\mathcal{D}(\Delta)$ and $\Delta f=\sum_{i=1}^d\Delta^{(i)}f$.

Next, $\Delta\subset\sum_{i=1}^d\Delta^{(i)}$ follows from Proposition 3.1. □

Before ending this subsection, we return to Sobolev spaces. As an immediate consequence of Lemma 3.2 we have the following characterization of $H^p_{2k}(\tilde{K}^d)$ (Definition 1.1).

Proposition 3.3. *For $p\in(1,\infty)$, $k\in\mathbb{Z}_+$ and fixed $1\leq i\leq d$,*

$$\begin{aligned}H^p_{2k}(\tilde{K}^d)&=\big\{f\in L^p(\tilde{K}^d):f\in\mathcal{D}(\Delta^{(\boldsymbol{i})}),\forall\boldsymbol{i}\text{ with }|\boldsymbol{i}|\leq k\big\}\\&=\big\{f\in L^p(\tilde{K}^d):(\Delta^{(i)})^jf\in L^p\big(\tilde{K},H^p_{2k-2j}(\tilde{K}^{d\wedge i})\big),j=0,1,\cdots,k\big\}.\end{aligned}$$

Proof. The first identify is an immediate consequence of Lemma 3.2. In addition, using this, for any $k'\leq k$, we can see

$$\begin{aligned}&\big\{f\in L^p(\tilde{K}^d):f\in L^p\big(\tilde{K},H^p_{2k'}(\tilde{K}^{d\wedge i})\big)\big\}\\=&\big\{f\in L^p(\tilde{K}^d):f\in\mathcal{D}(\Delta^{(\boldsymbol{i})}),\forall\boldsymbol{i}=i_1i_2\cdots i_m,\text{ with }m\leq k'\text{ and }i_j\neq i,1\leq j\leq m\big\}.\end{aligned}$$

The second identity follows. □

Proposition 3.3 is referred as the definition of Sobolev spaces in many contexts. Thus, it is reasonable to define Sobolev spaces on $\Omega=K^l\times\tilde{K}^{d-l}$ as follows.

Definition 3.4. *Let $\Omega=K^l\times\tilde{K}^{d-l}$ for some $1\leq l\leq d$, and $1<p<\infty$.*

(a). For $k\in\mathbb{Z}_+$, we define

$$H^p_{2k}(\Omega)=\big\{f\in L^p(\Omega):f\in\mathcal{D}(\Delta^{(\boldsymbol{i})}),\forall\boldsymbol{i}\text{ with }|\boldsymbol{i}|\leq k\big\}$$

with norm $\|f\|_{H^p_{2k}(\Omega)}=\sum_{|\boldsymbol{i}|\leq k}\|\Delta^{(\boldsymbol{i})}f\|_{L^p(\Omega)}$.

(b). For $k\in\mathbb{Z}_+$, $0<\theta<1$, define $H^p_{2k+2\theta}(\Omega)=\big[H^p_{2k}(\Omega),H^p_{2k+2}(\Omega)\big]_\theta$.

Clearly, we still have the characterization

$$H^p_{2k}(\Omega)=\big\{f\in L^p(\Omega):f\in\mathcal{D}\big((\Delta^{(1)})^j\big),\ (\Delta^{(1)})^jf\in L^p\big(K,H^p_{2k-2j}(\Omega^{\wedge 1})\big),\ j=0,1,\cdots,k\big\},$$

We will revisit Proposition 3.3 and Definition 3.4 in Appendix B, where we enlarge the domain of Laplacians to distributions, and see what happens.

One of our goals in this section is to prove the following theorem, which is also referred as an equivalent definition of the Sobolev spaces. See the book [].

Theorem 3.5. *Let $\Omega=K^l\times\tilde{K}^{d-l}$ for some $1\leq l\leq d$. Then, we have*

(a). $H^p_\sigma(\Omega)=H^p_\sigma(\tilde{K}^d)|_\Omega$, for $1<p<\infty$ and $\sigma\geq 0$.

(b). $\big[H^{p_1}_{\sigma_1}(\Omega),H^{p_2}_{\sigma_2}(\Omega)\big]_\theta=H^{p_\theta}_{\sigma_\theta}(\Omega)$, where $\frac{1}{p_\theta}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$, $\sigma_\theta=(1-\theta)\sigma_1+\theta\sigma_2$, for $1<p_1,p_2<\infty$, $\sigma_1,\sigma_2\geq 0$ and $0<\theta<1$.

We will show that $H^p_\sigma(\Omega)$ is a retract of $H^p_\sigma(\tilde{K}^d)$, with the same extension map E and the natural restriction map R for $0\leq\sigma\leq 2k$ with $k\in\mathbb{Z}_+$. See Definition C.7 for the concept of retract.

It is easy to observe that $H^p_\sigma(\tilde{K}^d)|_\Omega\subset H^p_\sigma(\Omega)$ by complex interpolation. The other half can be directly proved in the case when K is the unit interval, where various cut-off and reflection techniques are available. However, for the fractal case, multiplication of functions do not preserve smoothness (see [6]), things will be difficult. We will come back to the proof in the last part of this section, as a first application of the following decomposition theorem.

3.2. **A decomposition theorem.** As the main part of this section, for $\Omega=K^l\times\tilde{K}^{d-l}$ with $1\leq l\leq d$, we will provide a decomposition theorem of $H^p_\sigma(\Omega)$ with $\sigma\geq 0$. First, we introduce some notations.

Definition 3.6. *(a). For $k\in\mathbb{Z}_+$ and on K, let $\mathcal{H}_{k-1}=\{f\in\mathcal{D}(\Delta^k):\Delta^k f=0\}$ be the space of k-multiharmonic functions on K.*

(b). Let $\Omega=K^l\times\tilde{K}^{d-l}$ with $1\leq l\leq d$. For $k\in\mathbb{Z}_+$, denote $\mathcal{H}_{k-1}\big(K,D(\Omega^{\wedge 1})\big):=\mathcal{H}_{k-1}\otimes D(\Omega^{\wedge 1})$ for some function space (Banach space) $D(\Omega^{\wedge 1})$ on $\Omega^{\wedge 1}$.

Note that the dimension of $\mathcal{H}_{k-1}$ is $k\#V_0$, $\mathcal{H}_{-1}=\{0\}$ and $\mathcal{H}_0$ is the space of *harmonic functions* on K. In addition, $\mathcal{H}_{k-1}\big(K,D(\Omega^{\wedge 1})\big)$ can be understood as "the space of multiharmonic functions on K taking values in $D(\Omega^{\wedge 1})$".

The following notations concern the contraction maps of K.

Definition 3.7. *Fix $w\in W_*$.*

(a). Define A_w by $A_wf(x)=f(F_wx)$ for any function f on K.

(b). Let $\Omega=K^l\times\tilde{K}^{d-l}$ for some $1\leq l\leq d$, and let $1\leq i\leq l$. We define $F^{(i)}_w$ as the contraction map

$$F^{(i)}_w(x_1,x_2,\cdots,x_d)=(x_1,x_2,\cdots,x_{i-1},F_wx_i,x_{i+1},\cdots,x_d),$$

and define $A^{(i)}_w$ by $A^{(i)}_wf(x)=f(F^{(i)}_wx)$ for any function f on Ω.

It is also helpful to extend some notations in Section 2 in this section, by replacing the number γ with the operator $A^{(1)}_w$.

Definition 3.8. *Let $1<p<\infty$, $\alpha,\beta>0$, $\Omega=K^l\times\tilde{K}^{d-l}$ for some $1\leq l\leq d$ and $D(\Omega),D_1(\Omega),D_2(\Omega)$ be some function spaces (Banach spaces) on Ω.*

(a). Define

$$l^p_\alpha\big(D(\Omega),A^{(1)}_w\big)=\big\{\boldsymbol{s}=\{s_n\}_{n\geq 0}:\{s_{n+1}-A^{(1)}_ws_n\}_{n\geq 0}\in l^p_\alpha\big(D(\Omega)\big)\big\},$$

with norm $\|\boldsymbol{s}\|_{l^p_\alpha(D(\Omega),A^{(1)}_w)}=\big\|\{s_{n+1}-A^{(1)}_ws_n\}_{n\geq 0}\big\|_{l^p_\alpha(D(\Omega))}+\|s_0\|_{D(\Omega)}$.

(b). In addition, define

$$l^p_{\alpha,\beta}\big(D_1(\Omega),D_2(\Omega),A^{(1)}_w\big)=l^p_{\alpha\beta}\big(D_1(\Omega),A^{(1)}_w\big)\cap l^p_\beta\big(D_2(\Omega),A^{(1)}_w\big),$$

with norm $\|\boldsymbol{s}\|_{l^p_{\alpha,\beta}(D_1(\Omega),D_2(\Omega),A^{(1)}_w)}=\|\boldsymbol{s}\|_{l^p_{\alpha\beta}(D_1(\Omega),A^{(1)}_w)}+\|\boldsymbol{s}\|_{l^p_\beta(D_2(\Omega),A^{(1)}_w)}$.

See Appendix A for a further discussion on these spaces under certain setting.

From now on, we aim to a decomposition of the Sobolev spaces $H^p_\sigma(\Omega)$. We will first deal with the half space $\tilde{K}^d_+:=K\times\tilde{K}^{d-1}$ and the full space $\tilde{K}^d$, and construct a **restriction**

map and an **extension map** for $H^p_\sigma(\tilde{K}^d_+)$ and $H^p_\sigma(\tilde{K}^d)$ respectively in the following. For convenience, throughout the rest of this paper, we always assume

(**C1**). *For any* $x\in V_0$*, we have* $\#\pi^{-1}(x)=1$.

For each $x\in V_0$, we fix to denote its address by $\tau_x\dot{w}_x$ with $\tau_x,w_x\in W_*$, i.e. $x=\pi(\tau_x\dot{w}_x)$, and require that

$$F_{\tau_x}K\cap F_{\tau_y}K=\emptyset,\quad \forall x\neq y\in V_0. \tag{3.1}$$

Remark. The condition (**C1**) is not necessary, but will bring simplification for the proof. See [11](Section 8) for a brief discussion on this condition and an example that (**C1**) fails.

1. The restriction map.

Since $\mathcal{H}_{k-1}$ with $k\in\mathbb{Z}_+$ is a finite dimensional subspace in $L^\infty(K)$, for $1<p<\infty$, the orthogonal projection $P_{\mathcal{H}_{k-1}}:L^2(K)\to\mathcal{H}_{k-1}$ extends to be a bounded map $P_{\mathcal{H}_{k-1}}:L^p(K)\to\mathcal{H}_{k-1}$. As usual, we denote $P^{(1)}_{\mathcal{H}_{k-1}}$ the operator on $L^p(\tilde{K}^d_+)$ as before.

Definition 3.9. *(a). Let* $w\in W_*\setminus\{\emptyset\}$, $k\in\mathbb{Z}_+$*. Define* $R_{w,k}f=\big\{P^{(1)}_{\mathcal{H}_{k-1}}(A^{(1)}_w)^nf\big\}_{n\geq 0}$ *for a function* f *in* $L^p(\tilde{K}^d_+)$.

(b). For $x\in V_0$ *with address* $\tau_x\dot{w}_x$*, define* $R_{x,k}f=R_{w_x,k}A^{(1)}_{\tau_x}f$ *for a function* f *in* $L^p(\tilde{K}^d_+)$.

(c). For $x\in V_0$ *with address* $\tau_x\dot{w}_x$, $1<p<\infty$*, denote* $\alpha_x=r_{w_x}^{(1+d_H)/2},\beta_x=\beta_x(p)=r_{w_x}^{-d_H/p}$.

Noticing that $H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}\subset H^p_\sigma(\tilde{K}^d_+)$, for a function $f\in H^p_\sigma(\tilde{K}^d)$, we simplify $R_{x,k}f|_{\tilde{K}^d_+}$ to $R_{x,k}f$ without causing confusion.

We will show the following proposition.

Proposition 3.10. *Let* $k\in\mathbb{Z}_+$, $0\leq\sigma\leq 2k$, $1<p<\infty$, $x\in V_0$ *and* $\Omega=\tilde{K}^d_+$*. Then* $R_{x,k}$ *is bounded from* $H^p_\sigma(\Omega)$ *(also* $H^p_\sigma(\tilde{K}^d)$*) to* $l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge 1})),A^{(1)}_{w_x}\big)$.

The proof relies on the following two lemmas.

Lemma 3.11. *Let* $k\geq j\geq 0$, $1<p<\infty$, $w\in W_*\setminus\{\emptyset\}$ *and* $\Omega=\tilde{K}^d_+$*. There exists a function* $g_{w,k,j}\in L^\infty(K\times K)$ *such that for any* $f\in\mathcal{D}((\Delta^{(1)})^j)$*, we have*

$$P^{(1)}_{\mathcal{H}_{k-1}}A^{(1)}_wf(\xi)-A^{(1)}_wP^{(1)}_{\mathcal{H}_{k-1}}f(\xi)=\int_K g_{w,k,j}(\xi_1,\eta)\big((\Delta^{(1)})^jf\big)^{(1)}_{\xi^{\wedge 1}}(\eta)d\mu(\eta).$$

Proof. For $j=0$, the lemma is obvious since $P_{\mathcal{H}_{k-1}}$ is realized with an integration kernel.

Now, assume $j\geq 1$. Let G be the Green's operator on K, and $G^{(1)}$ its associated operator on $\tilde{K}^d_+$ as before. See Remark 4 in the last subsection. Let $f\in\mathcal{D}((\Delta^{(1)})^j)$, then we have $g=f-(G^{(1)})^j(-\Delta^{(1)})^jf\in\mathcal{D}((\Delta^{(1)})^j)$ with $(\Delta^{(1)})^jg=0$. As a consequence, $g\in\mathcal{H}_{j-1}\big(K,L^p(\Omega^{\wedge 1})\big)$. Thus,

$$\mathcal{D}((\Delta^{(1)})^j)=\mathcal{H}_{j-1}\big(K,L^p(\Omega^{\wedge 1})\big)\oplus(G^{(1)})^j\big(L^p(\Omega)\big). \tag{3.2}$$

It is easy to see that $(P^{(1)}_{\mathcal{H}_{k-1}}A^{(1)}_w-A^{(1)}_wP^{(1)}_{\mathcal{H}_{k-1}})|_{\mathcal{H}_{j-1}(K,L^p(\Omega^{\wedge 1}))}=0$. So by (3.2), it suffices to prove the lemma for the function $(G^{(1)})^j(-\Delta^{(1)})^jf$. We only need to take

$$g_{w,k,j}(\xi,\eta)=(-1)^j\int_{K^{j-1}}(P_{\mathcal{H}_{j-1}}A_w-A_wP_{\mathcal{H}_{k-1}})G_{\eta_1}(\xi)G(\eta_1,\eta_2)\cdots G(\eta_{j-1},\eta_j)d\mu(\eta_1)\cdots d\mu(\eta_{j-1}),$$

where $G_\eta(\xi)=G(\xi,\eta)$ is the Green's function. □

Lemma 3.12. *Let $1<p<\infty$, $f\in L^p(K)$ and $g\in L^\infty(K)$. Define*

$$\boldsymbol{s}=\{\mu_w^{n/p}\int_K A_w^n f(\xi)g(\xi)d\mu(\xi)\}_{n=0}^\infty,$$

then we have

$$\|\boldsymbol{s}\|_{l^p}\lesssim\|f\|_{L^p(K)}\|g\|_{L^\infty(K)}.$$

Proof. Let $Z=K\setminus F_wK$. Then $\|f\|_{L^p(K)}=\big\|\|\mu_w^{n/p}A_w^nf\|_{L^p(Z)}\big\|_{l^p}$ by scaling, and

$$\begin{aligned}\big|\int_K f(\xi)g(\xi)d\mu(\xi)\big|&=\big|\sum_{m=0}^\infty\mu_w^m\int_Z A_w^mf(\xi)A_w^mg(\xi)d\mu(\xi)\big|\\&\leq\sum_{m=0}^\infty\mu_w^m\|A_w^mf\|_{L^p(Z)}\|g\|_{L^\infty(K)},\end{aligned}$$

where we use the fact $\mu(Z)\leq 1$. So using Minkowski inequality, we get

$$\begin{aligned}\|\boldsymbol{s}\|_{l^p}&\leq\|g\|_{L^\infty(K)}\big\|\sum_{m=0}^\infty\mu_w^{n/p}\mu_w^m\|A_w^{n+m}f\|_{L^p(Z)}\big\|_{l^p}\\&\leq\|g\|_{L^\infty(K)}\sum_{m=0}^\infty\mu_w^{m-\frac{m}{p}}\|f\|_{L^p(K)}.\end{aligned}$$

Since $\mu_w<1$, we get the lemma. □

Now we return to the proof of Proposition 3.10.

Proof of Proposition 3.10. It suffices to consider $R_{w,k}$ for any $w\in W_*\setminus\{\emptyset\}$. We write $\alpha=r_w^{(1+d_H)/2}$ and $\beta=\mu_w^{-1/p}=r_w^{-d_H/p}$ for short.

First, we show that $R_{w,k}$ is a bounded map from $H^p_\sigma(\Omega)$ to $l^p_\beta\big(\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge1})),A_w^{(1)}\big)$. First, we consider the case $\sigma=2j$ with $j\in\mathbb{Z}_+$ and $0\leq j\leq k$. Let $f\in H^p_{2j}(\Omega)\subset L^p\big(K,H^p_{2j}(\Omega^{\wedge1})\big)$. By applying Lemma 3.11, we see that

$$\begin{aligned}\big((R_{w,k}f)_{n+1}-A_w^{(1)}(R_{w,k}f)_n\big)(\xi)&=\Big(\big(P^{(1)}_{\mathcal{H}_{k-1}}A_w^{(1)}-A_w^{(1)}P^{(1)}_{\mathcal{H}_{k-1}}\big)(A_w^{(1)})^nf\Big)(\xi)\\&=\int_K g_{w,k,0}(\xi_1,\eta)\big((A_w^{(1)})^nf\big)^{(1)}_{\xi^{\wedge1}}(\eta)d\mu(\eta).\end{aligned}$$

The claim then follows from Lemma 3.12 since $g_{w,k,0}\in L^\infty(K\times K)$. For general $0\leq\sigma\leq 2k$, the claim follows from the fact that $H^p_\sigma(\Omega^{\wedge1})=H^p_\sigma(\tilde{K}^{d-1})$ is stable under complex interpolation, so is $\mathcal{H}_{k-1}\big(K,H^p_\sigma(\Omega^{\wedge1})\big)$.

Next, we need to show that $R_{w,k}$ is a bounded map from $H^p_\sigma(\Omega)$ to $l^p_{\alpha^\sigma\beta}\big(\mathcal{H}_{k-1}(L^p(\Omega^{\wedge1})),A_w^{(1)}\big)$. As the last paragraph, we only need to consider the case $\sigma=2j$ with $j\in\mathbb{Z}_+$ and $0\leq j\leq k$.

We apply Lemma 3.11 again, and see that

$$\begin{aligned}\big((R_{w,k}f)_{n+1}-A_w^{(1)}(R_{w,k}f)_n\big)(\xi)&=\Big(\big(P^{(1)}_{\mathcal{H}_{k-1}}A_w^{(1)}-A_w^{(1)}P^{(1)}_{\mathcal{H}_{k-1}}\big)(A_w^{(1)})^nf\Big)(\xi)\\&=\int_K g_{w,k,j}(\xi_1,\eta)\big((\Delta^{(1)})^j(A_w^{(1)})^nf\big)^{(1)}_{\xi^{\wedge1}}(\eta)d\mu(\eta)\\&=(r_w\mu_w)^{jn}\int_K g_{w,k,j}(\xi_1,\eta)\big((A_w^{(1)})^n(\Delta^{(1)})^jf\big)^{(1)}_{\xi^{\wedge1}}(\eta)d\mu(\eta)\end{aligned}$$

The claim then follows from Lemma 3.12 since $(\Delta^{(1)})^jf\in L^p(\Omega)=L^p(K,L^p(\Omega^{\wedge1}))$ and $g_{w,k,j}\in L^\infty(K\times K)$. □

2. The extension map.

Now for $\Omega=\tilde K^d_+$, $x\in V_0$, $1<p<\infty$, $k\in\mathbb{Z}_+$ and $0\le\sigma\le 2k$, we will construct a bounded map $E_{x,k}$ (or $\tilde E_{x,k}$) from $l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge1})),A^{(1)}_{w_x}\big)$ to $H^p_\sigma(\tilde K^d_+)$ $\big($or $H^p_\sigma(\tilde K^d)\big)$.

Let $w\in W_*\setminus\{\emptyset\}$, without loss of generality, we assume that F_wK is bounded away from $V_0\setminus\{\pi(\dot w)\}$. We introduce some maps that will be used.

1). For each $h\in\mathcal{H}_{k-1}$, clearly there is a smooth function $\breve h\in C^\infty(K)$ such that

$$A_w\breve h=h,\quad P_{\mathcal{H}_{k-1}}\breve h=0$$

and $\breve h$ vanishes in a neighbourhood of $V_0\setminus\{\pi(\dot w)\}$. By choosing $\breve h$ properly, $h\to\breve h$ becomes a linear map from $\mathcal{H}_{k-1}$ to $C^\infty(K)$.

In addition, for any $f=\sum_{i=1}^m h_i\otimes f_i$ with $m\in\mathbb{N}$, $h_i\in\mathcal{H}_{k-1}$ and f_i defined on $\Omega^{\wedge1}$, we write

$$\breve f=\sum_{i=1}^m\breve h_i\otimes f_i.$$

2). For each $h\in\mathcal{H}_{k-1}$, still clearly there is a smooth function $\bar h\in C^\infty(K)$ such that

$$A_w\bar h=A_wh,\quad P_{\mathcal{H}_{k-1}}\bar h=h$$

and $\bar h$ vanishes in a neighbourhood of $V_0\setminus\{\pi(\dot w)\}$. By choosing $\bar h$ properly, $h\to\bar h$ becomes a linear map from $\mathcal{H}_{k-1}$ to $C^\infty(K)$.

In addition, for any $f=\sum_{i=1}^m h_i\otimes f_i$ with $m\in\mathbb{N}$, $h_i\in\mathcal{H}_{k-1}$, and f_i defined on $\Omega^{\wedge1}$, we write

$$\bar f=\sum_{i=1}^m\bar h_i\otimes f_i.$$

Now, we define the extension map $E_{x,k}$ for $H^p_\sigma(\tilde K^d_+)$.

Definition 3.13. *(a). Let $w\in W_*\setminus\{\emptyset\}$ and assume that $F_wK\cap V_0=\{\pi(\dot w)\}\cap V_0$. For each sequence of $\boldsymbol{s}=\{s_n\}_{n\ge0}\in\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge1}))\big)^{\mathbb{Z}_+}$, we define formally*

$$E_{w,k}\boldsymbol{s}=\bar s_0+\sum_{n=1}^\infty\breve s'_n\circ(F_w^{(1)})^{-n+1},$$

where $s'_n=s_n-A_w^{(1)}s_{n-1}$.

*(b). Let $x\in V_0$ and $\tau_x\dot{w}_x$ be the address of x as we discussed after (**C1**). Define*

$$E_{x,k}\boldsymbol{s}=(E_{w_x,k}\boldsymbol{s})\circ F_{\tau_x}^{-1}.$$

Proposition 3.14. *Let $k\in\mathbb{Z}$, $0\leq\sigma\leq 2k$, $1<p<\infty$, $x\in V_0$ and $\Omega=\tilde{K}_+^d$. Then $E_{x,k}$ is bounded from $l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge 1})),A^{(1)}_{w_x}\big)$ to $H^p_\sigma(\Omega)$, with $\alpha_x=r_{w_x}^{(1+d_H)/2}$, $\beta_x=r_{w_x}^{-d_H/p}$.*

In addition, we have $E_{x,k}\boldsymbol{s}$ supported in $F_{\tau_x}K\times\tilde{K}^{d-1}$, and $R_{x,k}E_{x,k}=Id$.

Proof. It suffices to consider $E_{w,k}$ by assuming $F_wK\cap V_0=\{\pi(\dot{w})\}\cap V_0$. We write $\alpha=r_w^{(1+d_H)/2}$ and $\beta=\mu_w^{-1/p}=r_w^{-d_H/p}$ for short.

First, for any $j\in\mathbb{Z}_+$ with $0\leq j\leq k$ and $\theta\geq 0$, we show that $E_{w,k}$ is bounded from $l^p_{\alpha^{2j}\beta}\big(\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1})),A^{(1)}_w\big)$ to the function space

$$\big\{f\in\mathcal{D}\big((\Delta^{(1)})^j\big):(\Delta^{(1)})^jf\in L^p\big(K,H^p_\theta(\Omega^{\wedge 1})\big)\big\}.$$

Let $\boldsymbol{s}\in l^p_{\alpha^{2j}\beta}\big(\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1})),A^{(1)}_w\big)$. Write $s'_n=s_n-A^{(1)}_ws_{n-1}$ for $n\geq 1$, and for convenience, write $f_0=\bar{s}_0$ and $f_n=\breve{s}'_n\circ(F^{(1)}_w)^{-n+1}$ for short. Then we see that

$$\begin{aligned}&\big\|\{(\Delta^{(1)})^jf_n\}_{n\geq 0}\big\|_{l^p_\beta(L^\infty(K,H^p_\theta(\Omega^{\wedge 1})))}\\ \lesssim&\|s_0\|_{\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1}))}+\big\|\{s_{n+1}-A^{(1)}_ws_n\}_{n\geq 0}\big\|_{l^p_{\alpha^{2j}\beta}(\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1})))}\\ \asymp&\|\boldsymbol{s}\|_{l^p_{\alpha^{2j}\beta}(\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1})),A^{(1)}_w)}.\end{aligned}$$

Write $Z=K\setminus F_wK$. Then we have

$$\begin{aligned}&\big\|\sum_{m=0}^{\infty}|(\Delta^{(1)})^jf_m|\big\|_{L^p(K,H^p_\theta(\Omega^{\wedge 1}))}\\ =&\big\|\mu_w^{n/p}\|(A^{(1)}_w)^n\sum_{m=0}^{n+1}|(\Delta^{(1)})^jf_m|\|_{L^p(Z,H^p_\theta(\Omega^{\wedge 1}))}\big\|_{l^p}\lesssim\big\|\mu_w^{n/p}\sum_{m=0}^{n+1}\|(\Delta^{(1)})^jf_m\|_{L^\infty(Z,H^p_\theta(\Omega^{\wedge 1}))}\big\|_{l^p}\\ =&\big\|\sum_{m=-1}^{\infty}1_{n\geq m}\mu_w^{m/p}\mu_w^{(n-m)/p}\|(\Delta^{(1)})^jf_{n-m}\|_{L^\infty(Z,H^p_\theta(\Omega^{\wedge 1}))}\big\|_{l^p}\lesssim\big\|\{(\Delta^{(1)})^jf_n\}_{n\geq 0}\big\|_{l^p_\beta(L^\infty(K,H^p_\theta(\Omega^{\wedge 1})))}.\end{aligned}$$

Now, noticing that

$$l^p_{\alpha^{2j}\beta}\big(\mathcal{H}_{k-1}(K,H^p_\theta(\Omega^{\wedge 1})),A^{(1)}_w\big)\subset l^p_\beta\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),A^{(1)}_w\big),$$

we clearly have $E_{w,k}\boldsymbol{s}\in L^p(\Omega)$ by applying the above two estimates (take $j=0,\theta=0$). In addition, the claim also follows from above.

Next, for $j\in\mathbb{Z}_+$ with $0\leq j\leq k$, we observe that

$$l^p_{\alpha^{2j},\beta}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_{2j}(\Omega^{\wedge 1})),A^{(1)}_w\big)=\bigcap_{j'=0}^{j}l^p_{\alpha^{2j'}\beta}\big(\mathcal{H}_{k-1}(K,H^p_{2(j-j')}(\Omega^{\wedge 1})),A^{(1)}_w\big),$$

using the fact that the Sobolev space $H^p_\sigma(\tilde{K}^{d-1})$ is stable under complex interpolation.

Then by using the fact that

$$H^p_{2j}(\Omega)=\big\{f\in L^p(\Omega):f\in\mathcal{D}\big((\Delta^{(1)})^{j'}\big),\ (\Delta^{(1)})^{j'}f\in L^p\big(K,H^p_{2(j-j')}(\Omega^{\wedge 1})\big),\ j'=0,1,\cdots,j\big\},$$

we find that $E_{w,k}$ is a bounded map from $l^p_{\alpha^{2j},\beta}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_{2j}(\Omega^{\wedge 1})),A^{(1)}_w\big)$ to $H^p_{2j}(\Omega)$, by combining the above two parts.

On the other hand, it is easy to verify that $R_{w,k}E_{w,k}=Id$ on $l^p_\beta\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),A^{(1)}_w\big)$, and thus it holds on subspaces.

Till now, we have proved the proposition for $\sigma=2j$ with $0\leq j\leq k$. Since $\{(1-\Delta)^{it}\}_{t\in\mathbb{R}}$ is a C_0-group as a consequence of Proposition?? in [], where Δ is the Laplacian on $\tilde{K}^{d-1}$, by applying Lemma 2.11 to $-\Delta$, noticing that $H^p_\sigma(\tilde{K}^{d-1})=\mathcal{D}(\Delta^{\sigma/2})$ for $\sigma\geq 0$, the result for general $0\leq\sigma\leq 2k$ then follows by using complex interpolation. □

Next, we construct the extension map $\tilde{E}_{x,k}$ for $H^p_\sigma(\tilde{K}^d)$. The only thing we need to do is to extend each function $\bar{s}_0$ and $\breve{s}'_n$ to $\tilde{K}^d$. To make things clear, we introduce the following notations.

3). For each $h\in\mathcal{H}_{k-1}$ and $x=\pi(\tau_x\dot{w}_x)\in V_0$, there is clearly a function $M_x(h)\in\mathcal{H}_{k-1}$ such that

$$\Delta^jM_x(h)(x)=\Delta^jh(x),\quad \partial_n\Delta^jM_x(h)(x)=-\partial_n\Delta^jh(x),\quad \forall 0\leq j\leq k-1.$$

By choosing $M_x(h)$ properly, $h\to M_x(h)$ becomes a linear map from $\mathcal{H}_{k-1}$ to $\mathcal{H}_{k-1}$.

For convenience, we write $M_x(h\otimes f)=M_x(h)\otimes f$ and thus M_x extends to be a linear map $M_x:\mathcal{H}_{k-1}\big(K,L^p(\Omega^{\wedge 1})\big)\to\mathcal{H}_{k-1}\big(K,L^p(\Omega^{\wedge 1})\big)$.

With this map, we define $\tilde{E}_{x,k}$ as follows.

Definition 3.15. *Let $x=\pi(\tau_x\dot{w}_x)\in V_0$ and $\boldsymbol{s}=\{s_n\}_{n\geq 0}\in\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1}))\big)^{\mathbb{Z}_+}$. Define $\boldsymbol{s}_-=\{M_x(s_n)\}_{n\geq 0}$, and we formally define $\tilde{E}_{x,k}\boldsymbol{s}$ on $\tilde{K}^d$ as*

$$\tilde{E}_{x,k}\boldsymbol{s}|_{\tilde{K}^d_+}=E_{x,k}\boldsymbol{s},\qquad \tilde{E}_{x,k}\boldsymbol{s}|_{\tilde{K}^d_-}=E_{x,k}\boldsymbol{s}_-.$$

Using a same argument as Proposition 3.14, we have

Proposition 3.16. *Let $k\in\mathbb{Z}_+$, $0\leq\sigma\leq 2k$, $1<p<\infty$, $x\in V_0$. Then $\tilde{E}_{x,k}$ is bounded from $l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge 1})),A^{(1)}_w\big)$ to $H^p_\sigma(\tilde{K}^d)$, with $\alpha_x=r_{w_x}^{(1+d_H)/2},\beta_x=r_{w_x}^{-d_H/p}$.*

In addition, we have $\tilde{E}_{x,k}\boldsymbol{s}$ supported in $F^{(1)}_{\tau_x}\tilde{K}^d_+\cup F^{(1)}_{\tau_x}\tilde{K}^d_-$, and $R_{x,k}\tilde{E}_{x,k}=Id$.

3. The decomposition.

Still assume $1<p<\infty$, $k\in\mathbb{Z}_+$ and $0\leq\sigma\leq 2k$. Notice that the maps $R_{x,k}$, $E_{x,k}$ and $\tilde{E}_{x,k}$ for $x\in V_0$ can be defined naturally on $K\times\Omega'$ for $\Omega'=K^l\times\tilde{K}^{s-l-1}$ (also on $\tilde{K}\times\Omega'$). For simplicity, we introduce some notations.

Notations.

(a). Let $\Omega=K^l\times\tilde{K}^{d-l}$ with $0\leq l\leq d$. We define $\mathcal{K}^p_{\sigma,k}(\Omega)=\big\{f\in H^p_\sigma(\Omega):R_{x,k}f=0,\forall x\in V_0\big\}$.

(b). Let $\Omega=K^l\times\tilde{K}^{d-l}$ with $1\leq l\leq d$. We define

$$\mathcal{T}^p_{\sigma,k}(\Omega)=\big\{f\in H^p_\sigma(\Omega):f=\sum_{x\in V_0}E_{x,k}\boldsymbol{s}_x,\ \textit{with}\quad \boldsymbol{s}_x\in l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge 1})),A^{(1)}_{w_x}\big),\forall x\in V_0\big\}.$$

Also define

$$\mathcal{T}^p_{\sigma,k}(\tilde{K}^d)=\big\{f\in H^p_\sigma(\tilde{K}^d):f=\sum_{x\in V_0}\tilde{E}_{x,k}\boldsymbol{s}_x,\ \textit{with}\quad \boldsymbol{s}_x\in l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d-1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d-1})),A^{(1)}_{w_x}\big),\forall x\in V_0\big\}.$$

As an easy consequence of the Proposition 3.10, 3.14 and 3.16. We conclude this subsection with the following theorem.

Theorem 3.17. *Let $1<p<\infty$, $k\in\mathbb{Z}_+$ and $0\leq\sigma\leq 2k$. We have*
(a). $H^p_\sigma(\tilde{K}^d)=\mathcal{K}^p_{\sigma,k}(\tilde{K}^d)\oplus\mathcal{T}^p_{\sigma,k}(\tilde{K}^d)$.
(b). $H^p_\sigma(\tilde{K}^d_+)=\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)\oplus\mathcal{T}^p_{\sigma,k}(\tilde{K}^d_+)$.
(c). for $\Omega=K^l\times\tilde{K}^{d-l}$ with $2\leq l\leq d$, $H^p_\sigma(\Omega)=\mathcal{K}^p_{\sigma,k}(\Omega)\oplus\mathcal{T}^p_{\sigma,k}(\Omega)$.

Proof. Noticing the requirement (3.1), part (a) is a consequence of Proposition 3.10 and 3.16, part (b) is a consequence of Proposition 3.10 and 3.14. We will prove (c) in the next subsection. □

3.3. **Proof of Theorem 3.5 and 3.17 (c).** In the last part of this section, we will prove Theorem 3.5, and also fullfill the proof of the decomposition theorem. Let's start from the simple case where $\Omega=\tilde{K}^d_+$.

Lemma 3.18. *For each f on $\tilde{K}^d_+$, we define $\Theta(f)$ to be the function on $\tilde{K}^d$ such that*

$$\Theta(f)(x)=\begin{cases}f(x), & \text{if } x\in\tilde{K}^d_+,\\ 0, & \text{if } x\in\tilde{K}^d_-.\end{cases}$$

Then Θ is bounded from $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$ to $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d)$. As a consequence, $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d)|_{\tilde{K}^d_+}=\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$.

Proof. First, for $j=0,1,2,\cdots,k$, we can easily see that Θ is bounded from $\mathcal{K}^p_{2j,k}(\tilde{K}^d_+)$ to $\mathcal{K}^p_{2j,k}(\tilde{K}^d)$. Next, by Theorem 3.17 (a) and (b), Lemma 2.11, and the fact that

$$\big[H^p_{2j}(\tilde{K}^d),H^p_{2j+2}(\tilde{K}^d)\big]_\theta=H^p_{2j+2\theta}(\tilde{K}^d),\quad \big[H^p_{2j}(\tilde{K}^d_+),H^p_{2j+2}(\tilde{K}^d_+)\big]_\theta=H^p_{2j+2\theta}(\tilde{K}^d_+),$$

we conclude

$$\big[\mathcal{K}^p_{2j,k}(\tilde{K}^d),\mathcal{K}^p_{2j+2,k}(\tilde{K}^d)\big]_\theta=\mathcal{K}^p_{2j+2\theta,k}(\tilde{K}^d),\quad \big[\mathcal{K}^p_{2j,k}(\tilde{K}^d_+),\mathcal{K}^p_{2j+2,k}(\tilde{K}^d_+)\big]_\theta=\mathcal{K}^p_{2j+2\theta,k}(\tilde{K}^d_+),$$

for $j=0,1,\cdots,k-1$ and $\theta\in(0,1)$. Thus, Θ is bounded from $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$ to $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d)$ for all $0\leq\sigma\leq 2k$ by using complex interpolation. □

Lemma 3.19. *For $1<p<\infty$ and $\sigma\geq 0$, we have $H^p_\sigma(\tilde{K}^d_+)=H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$. In fact, $H^p_\sigma(\tilde{K}^d_+)$ is a retract of $H^p_\sigma(\tilde{K}^d)$.*

Proof. Obviously, we have $H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}\subset H^p_\sigma(\tilde{K}^d_+)$. On the other hand, define the extension map Λ_k from $L^p(\tilde{K}^d_+)$ to $L^p(\tilde{K}^d)$ as

$$\Lambda_k=\Theta(1-\sum_{x\in V_0}E_{x,k}R_{x,k})+\sum_{x\in V_0}\tilde{E}_{x,k}R_{x,k}.$$

Using Proposition 3.10 and 3.16 and Lemma 3.18, we can see that Λ_k is bounded from $H^p_\sigma(\tilde{K}^d_+)$ to $H^p_\sigma(\tilde{K}^d)$. The lemma follows. □

For the special case $\Omega=\tilde{K}^d_+$, Theorem 3.5 follows immediately from Lemma 3.19. Now we prove Theorem 3.5 for the general case.

Proof of Theorem 3.5 and Theorem 3.17 (c). For $d=1$, Theorem 3.5 is true by Lemma 3.19, and there is nothing to prove for Theorem 3.17 (c).

For $d\geq 2$, we prove by induction. We assume that both results are true for $d-1$, and thus could be applied to $\Omega'=K^l\times\tilde{K}^{d-l-1}\subset\tilde{K}^{d-1}$. As a consequence, for $\sigma_1,\sigma_2\geq 0$, we have

$$\big[H^p_{\sigma_1}(\Omega'),H^p_{\sigma_2}(\Omega')\big]_\theta=H^p_{\sigma_\theta}(\Omega'),\text{ with }\sigma_\theta=(1-\theta)\sigma_1+\theta\sigma_2\text{ and }\theta\in(0,1).$$

Let $\Omega=K\times\Omega'\subset\tilde{K}^d$. We can use a same proof as Proposition 3.10 to see that $R_{x,k}$ is bounded from $H^p_\sigma(\Omega)$ (also $H^p_\sigma(\tilde{K}\times\Omega')$) to $l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega')),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega')),A^{(1)}_{w_x}\big)$.

Moreover, we have $l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega')),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega')),A^{(1)}_{w_x}\big)$ is a retract of

$$l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d-1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d-1})),A^{(1)}_{w_x}\big).$$

In fact, there is an extension map E (independent of σ) from $H^p_\sigma(\Omega')$ to $H^p_\sigma(\tilde{K}^{d-1})$ and let R be the restriction map from $\tilde{K}^{d-1}$ to Ω', such that $RE=Id$. The extension map E naturally extends to the sequence spaces.

As a consequence $l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega')),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega')),A^{(1)}_{w_x}\big)$ is stable under complex interpolation. Thus, we can use a same proof as Proposition 3.14 (or 3.16) to see that $E_{x,k}$ (or $\tilde{E}_{x,k}$) is bounded from $l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega')),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega')),A^{(1)}_{w_x}\big)$ to $H^p_\sigma(\Omega)$ (or $H^p_\sigma(\tilde{K}\times\Omega')$). As an immediate consequence, we have Theorem 3.17 (c) proved for Ω.

In addition, following the same proof of Lemma 3.18 and Lemma 3.19, we have $H^\sigma_p(K\times\Omega')$ is a retract of $H^p_\sigma(\tilde{K}\times\Omega')$. Now, we apply the claim to different Ω' and different directions, we have

$$H^p_\sigma(\tilde{K}^d)\curvearrowright H^p_\sigma(K\times\tilde{K}^{d-1})\curvearrowright\cdots\curvearrowright H^p_\sigma(K^{d-1}\times\tilde{K})\curvearrowright H^p_\sigma(K^d),$$

which are realized by the extension map Λ_k and the restricition map in different directions. As a consequence, we have $H^p_\sigma(\tilde{K}^d)\curvearrowright H^p_\sigma(\Omega)$. Theorem 3.5 is proved for d. The proof is completed by induction. □

4. Embedding Theorems and boundary behavior

In this section, we study the embedding theorems of function spaces on product of fractals. Recall that in Theorem 3.5 we have shown that the Sobolev spaces $H^p_\sigma(\Omega)$ on $\Omega=K^l\otimes\tilde{K}^{d-l}$ with $\sigma\geq 0$ are stable under complex interpolation. We can then define the Besov spaces $B^{p,q}_\sigma(\Omega)$ by real interpolation as follows.

Definition 4.1. *Let $\Omega = K^l \times \tilde{K}^{d-l}$, $p,q\in(1,\infty)$ and $\sigma>0$, we define*
$$B^{p,q}_\sigma(\Omega) = \big(L^p(\Omega), H^p_{\sigma'}(\Omega)\big)_{\frac{\sigma}{\sigma'},q}, \text{ with } \sigma'>\sigma.$$

Note that the definition is independent of the choice of σ'. In addition, $B^{p,q}_\sigma(\Omega)$ is a retract of $B^{p,q}_\sigma(\tilde{K}^d)$, noticing that the couple $\big(L^p(\Omega), H^p_{\sigma'}(\Omega)\big)$ is a retract of $\big(L^p(\tilde{K}^d), H^p_{\sigma'}(\tilde{K}^d)\big)$. One can equivalently define $B^{p,q}_\sigma(\Omega)$ on Ω as the restriction of $B^{p,q}_\sigma(\tilde{K}^d)$ on $\tilde{K}^d$.

Proposition 4.2. *Let $\Omega = K^l\times\tilde{K}^{d-l}$, $p,q\in(1,\infty)$ and $\sigma>0$. We have $B^{p,q}_\sigma(\Omega) = B^{p,q}_\sigma(\tilde{K}^d)|_\Omega$.*

In this section, we will introduce some related spaces, $\mathring{H}^p_\sigma(\Omega)$, $\mathring{B}^{p,q}_\sigma(\Omega)$, $\tilde{H}^p_\sigma(\Omega)$, $\tilde{B}^{p,q}_\sigma(\Omega)$ and some others, which will play important roles in the next section. At the end of this section, we will extend the embedding theorems, Theorem 3.5 and Proposition 4.2, to *real orders*. For convenience, we will mostly focus on $\Omega=\tilde{K}^d_+$ in this section and the next section, though quite a large portion of the theorems can be extended to the general case.

4.1. **A trace theorem.** We begin this section with a trace theorem of $H^p_\sigma(\Omega)$ and $B^{p,q}_\sigma(\Omega)$ on the boundary $\partial\Omega$ of $\Omega = K^l\times\tilde{K}^{d-l}$. In this part, for simplicity, we only take care of a face of $\partial\Omega$, which is identified with $K^{l-1}\times\tilde{K}^{d-l}$.

It is well-known that on K, we have $H^p_\sigma(K)\subset C(K)$ if and only if $\sigma>\frac{2d_H}{p(1+d_H)}=\frac{d_S}{p}$, where $d_S := \frac{2d_H}{1+d_H}$ is the *spectral dimension* on K, see [] for a proof. Similarly, we also have that $\partial_n f(x), \forall x\in V_0$ is well-defined for any $f\in H^p_\sigma(K)$ if and only if $\sigma>2-\frac{d_S}{p'}$. Readers can compare the following theorem with the classical trace theorem in monographs [] and [].

Theorem 4.3. *Let $\Omega=K^l\times\tilde{K}^{d-l}$ with $1\leq l\leq d$, $1<p<\infty$, $p'=\frac{p}{p-1}$ and $x\in V_0$.*

(a). For $2k-\frac{d_S}{p'}<\sigma\leq 2k+\frac{d_S}{p}$ with $k\in\mathbb{Z}_+$, the trace map
$$T^{(x,1)}_\sigma f = \big(f|_{\{x\}\times\Omega^{\wedge 1}},\cdots,(\Delta^{(1)})^{k-1}f|_{\{x\}\times\Omega^{\wedge 1}}, \partial^{(1)}_n f|_{\{x\}\times\Omega^{\wedge 1}},\cdots,\partial^{(1)}_n(\Delta^{(1)})^{k-1}f|_{\{x\}\times\Omega^{\wedge 1}}\big)$$
is bounded and surjective from $H^p_\sigma(\Omega)$ to $\prod_{i=0}^{k-1}B^{p,p}_{\sigma-2i-d_S/p}(\Omega^{\wedge 1})\times\prod_{i=0}^{k-1}B^{p,p}_{\sigma-2i-2+d_S/p'}(\Omega^{\wedge 1})$, and is bounded and surjective from $B^{p,q}_\sigma(\Omega)$ to $\prod_{i=0}^{k-1}B^{p,q}_{\sigma-2i-d_S/p}(\Omega^{\wedge 1})\times\prod_{i=0}^{k-1}B^{p,q}_{\sigma-2i-2+d_S/p'}(\Omega^{\wedge 1})$.

(b). For $2k+\frac{d_S}{p}<\sigma\leq 2k+2-\frac{d_S}{p'}$ with $k\in\mathbb{Z}_+$, the trace map
$$T^{(x,1)}_\sigma f = \big(f|_{\{x\}\times\Omega^{\wedge 1}},\cdots,(\Delta^{(1)})^{k}f|_{\{x\}\times\Omega^{\wedge 1}}, \partial^{(1)}_n f|_{\{x\}\times\Omega^{\wedge 1}},\cdots,\partial^{(1)}_n(\Delta^{(1)})^{k-1}f|_{\{x\}\times\Omega^{\wedge 1}}\big)$$
is bounded and surjective from $H^p_\sigma(\Omega)$ to $\prod_{i=0}^{k}B^{p,p}_{\sigma-2i-d_S/p}(\Omega^{\wedge 1})\times\prod_{i=0}^{k-1}B^{p,p}_{\sigma-2i-2+d_S/p'}(\Omega^{\wedge 1})$, and is bounded and surjective from $B^{p,q}_\sigma(\Omega)$ to $\prod_{i=0}^{k}B^{p,q}_{\sigma-2i-d_S/p}(\Omega^{\wedge 1})\times\prod_{i=0}^{k-1}B^{p,q}_{\sigma-2i-2+d_S/p'}(\Omega^{\wedge 1})$.

Remark. It is known that $B^{2,2}_\sigma(\tilde{K}^d)=H^2_\sigma(\tilde{K}^d)$ by the standard interpolation theory, so that $B^{2,2}_\sigma(\Omega)=H^2_\sigma(\Omega)$ for any $\Omega=K^l\times\tilde{K}^{d-l}$.

To prove Theorem 4.3, it suffices to show that for $i<k, 2i+\frac{d_S}{p}<\sigma\leq 2k$ and $2i+2-\frac{d_S}{p'}<\sigma'\leq 2k$, and for any $f\in H^p_\sigma(\Omega)$, $f'\in H^p_{\sigma'}(\Omega)$, $x=\pi(\tau\dot{w})\in V_0$ and a.e. $\xi\in\Omega^{\wedge 1}$, we always have
$$\begin{aligned}(\Delta^{(1)})^i f(x,\xi) &= \lim_{n\to\infty}(r_\tau\mu_\tau)^{-i}(r_w\mu_w)^{-in}(\Delta^{(1)})^i(R_{x,k}f)_n(F_\tau^{-1}x,\xi),\\ \partial^{(1)}_n(\Delta^{(1)})^i f'(x,\xi) &= \lim_{n\to\infty}r_\tau^{-1}r_w^{-n}(r_\tau\mu_\tau)^{-i}(r_w\mu_w)^{-in}\partial^{(1)}_n(\Delta^{(1)})^i(R_{x,k}f')_n(F_\tau^{-1}x,\xi),\end{aligned}\tag{4.1}$$

where $R_{x,k}$ is defined in Definition 3.9. Then Theorem 4.3 follows immediately from Theorem 3.17 and Proposition 2.9 for $H^p_\sigma(\Omega)$ case, and from Proposition 2.13 for $B^{p,q}_\sigma(\Omega)$ case by using real interpolation. Indeed, the identities in (4.1) follow from the following lemma, noticing that $H^p_\sigma(\Omega)\subset(1-\Delta^{(1)})^{-\sigma/2}\big(L^p(\Omega)\big)$ and $(1-\Delta^{(1)})^{-\sigma/2}\big(L^p(\Omega)\big)\subset L^\infty\big(K,L^p(\Omega^{\wedge 1})\big)$ for $\sigma>\frac{d_S}{p}$.

Lemma 4.4. *Let $\Omega=K^l\times\tilde{K}^{d-l}$, $k\in\mathbb{Z}_+$, $0<\sigma\leq 2k$ and $x=\pi(\tau\dot{w})\in V_0$. For each $f\in H^p_\sigma(\Omega)$, we have*

$$\lim_{n\to\infty}(r_w\mu_w)^{-\sigma n/2}\mu_w^{n/p}\big\|(A^{(1)}_w)^nA^{(1)}_\tau f-(R_{x,k}f)_n\big\|_{\mathcal{D}((\Delta^{(1)})^{\sigma/2})}=0;$$

For each $f\in B^{p,q}_\sigma(\Omega)$, we have

$$\lim_{n\to\infty}(r_w\mu_w)^{-\sigma n/2}\mu_w^{n/p}\big\|(A^{(1)}_w)^nA^{(1)}_\tau f-(R_{x,k}f)_n\big\|_{\big(\mathcal{D}((\Delta^{(1)})^0),\mathcal{D}((\Delta^{(1)})^k)\big)_{\sigma/(2k),q}}=0.$$

Proof. Write $\mathring{l}(\sigma)=\big\{\boldsymbol{s}=\{s_n\}_{n\geq0}\in l^\infty(\mathcal{D}((\Delta^{(1)})^{\sigma/2})):\lim_{n\to\infty}s_n=0\big\}$ for short, and denote $\mathring{l}_\alpha(\sigma)=\big\{\boldsymbol{s}=\{s_n\}_{n\geq0}:\{\alpha^{-n}s_n\}_{n\geq0}\in\mathring{l}(\sigma)\big\}$ with $\alpha>0$. One can easily check that the map $f\to\big\{(A^{(1)}_w)^nA^{(1)}_\tau f-(R_{x,k}f)_n\big\}_{n\geq0}$ is bounded from $L^p(\Omega)$ to $\mathring{l}_\beta(0)$, and is bounded from $H^p_{2k}(\Omega)$ to $\mathring{l}_{\alpha^{2k}\beta}(2k)$, where $\alpha=(r_w\mu_w)^{1/2}$ and $\beta=\mu_w^{-1/p}$ as in the last section.

In addition, using Theorem 3.5 (b), one can easily check that

$$\big[\mathring{l}_\beta(0),\mathring{l}_{\alpha^{2k}\beta}(2k)\big]_{\sigma/(2k)}=\mathring{l}_{\alpha^\sigma\beta}(\sigma),\quad\forall\sigma\in(0,2k).$$

The identity for $H^p_\sigma(\Omega)$ case then follows by using complex interpolation.

The identity for $B^{p,q}_\sigma(\Omega)$ follows by the real interpolation of the couple $\big(\mathring{l}_\beta(0),\mathring{l}_{\alpha^{2k}\beta}(2k)\big)$. □

Remark. We can get better estimates if we use the norm $L^\infty\big(K,L^p(\Omega^{\wedge1})\big)$ for the remainder term in Lemma 4.4 for σ large enough. See the authors' previous work [] for a discussion on $\Omega=K$ in the L^2 setting.

4.2. **The spaces $\tilde{H}^p_\sigma(\Omega)$ and $\tilde{B}^{p,q}_\sigma(\Omega)$.** In this part, we focus on $\tilde{H}^p_\sigma(\Omega)$ and $\tilde{B}^{p,q}_\sigma(\Omega)$ for $\Omega=K^l\times\tilde{K}^{d-l}$ with $1\leq l\leq d$, which are viewed as functions in $H^p_\sigma(\tilde{K}^d)$ and $B^{p,q}_\sigma(\tilde{K}^d)$ with support in Ω. The notations $\tilde{H}^p_\sigma(\Omega)$ and $\tilde{B}^{p,q}_\sigma(\Omega)$ follow from Tribel [].

For convenience, we write $\Theta:L^p(\Omega)\to L^p(\tilde{K}^d)$ the extension map by zero.

Definition 4.5. *(a). For $1<p<\infty$ and $\sigma\geq0$. Define*

$$\tilde{H}^p_\sigma(\Omega)=\big\{f\in H^p_\sigma(\Omega):\Theta f\in H^p_\sigma(\tilde{K}^d)\big\}.$$

(b). For $1<p,q<\infty$ and $\sigma>0$. Define

$$\tilde{B}^{p,q}_\sigma(\Omega)=\big\{f\in B^{p,q}_\sigma(\Omega):\Theta f\in B^{p,q}_\sigma(\tilde{K}^d)\big\}.$$

In fact, by introducing the Dirichlet Laplacian Δ_D and the Neumann Laplacian Δ_N on Ω, using the same idea in Section 3.1, we have

$$\tilde{H}^p_\sigma(\Omega)=H^p_{\sigma,D}(\Omega)\cap H^p_{\sigma,N}(\Omega):=(1-\Delta_D)^{-\sigma/2}\big(L^p(\Omega)\big)\cap(1-\Delta_N)^{-\sigma/2}\big(L^p(\Omega)\big).$$

This can be shown easily using symmetric extension. Moreover, we have the following characterizations.

Theorem 4.6. *Let $1<p,q<\infty$, $p'=\frac{p}{p-1}$ and $\sigma\notin\{\frac{d_S}{p},2-\frac{d_S}{p'}\}+2\mathbb{Z}_+$. Then*

(a). $\tilde{H}^p_\sigma(\tilde{K}^d_+)=\big\{f\in H^p_\sigma(\tilde{K}^d_+):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\}$.

(b). $\tilde{B}^{p,q}_\sigma(\tilde{K}^d_+)=\big\{f\in B^{p,q}_\sigma(\tilde{K}^d_+):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\}$.

Proof. Write $\Omega=\tilde{K}^d_+$ for simplicity.

(a). The statement is clearly true when $\sigma\in 2\mathbb{Z}_+$, so we only need to extend the result to general σ using interpolation.

Let's fix $k\in\mathbb{N}$. We show the following two claims.

Claim 1: The interpolation couple $\big(\tilde{H}^p_0(\Omega),\tilde{H}^p_{2k}(\Omega)\big)$ is a retract of $\big(H^p_0(\tilde{K}^d),H^p_{2k}(\tilde{K}^d)\big)$.

Proof of Claim 1. We have the extension map Θ already. We define the restriction map R using the mappings Λ_k defined in Lemma 3.19. Let $f\in L^p(\tilde{K}^d)$, we define $f'=\Lambda_k(f|\tilde{K}^d_-)$. Then, we define $Rf=(f-f')|_\Omega$. Clearly, we have $R:\big(H^p_0(\tilde{K}^d),H^p_{2k}(\tilde{K}^d)\big)\to\big(\tilde{H}^p_0(\Omega),\tilde{H}^p_{2k}(\Omega)\big)$, and $R\Theta$ is the identity on $\tilde{H}^p_0(\Omega)$.

Claim 2: We have $\big[\tilde{H}^p_0(\Omega),\tilde{H}^p_{2k}(\Omega)\big]_{\sigma/2k}=\big\{f\in H^p_\sigma(\Omega):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\}$.

Proof of Claim 2. Recall that in Theorem 3.17, we have developed the decomposition $H^p_\sigma(\Omega)=\mathcal{K}^p_{\sigma,k}(\Omega)\oplus\mathcal{T}^p_{\sigma,k}(\Omega)$ for $0\le\sigma\le 2k$. Note that

$$\begin{aligned}\mathcal{T}^p_{\sigma,k}(\Omega)=\big\{f\in H^p_\sigma(\Omega):f=\sum_{x\in V_0}E_{x,k}\boldsymbol{s}_x,\ \text{with}\\ \boldsymbol{s}_x\in l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge1})),A^{(1)}_{w_x}\big),\forall x\in V_0\big\},\end{aligned}$$

we then define

$$\begin{aligned}\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)=\big\{f\in H^p_\sigma(\Omega):f=\sum_{x\in V_0}E_{x,k}\boldsymbol{s}_x,\ \text{with}\\ \boldsymbol{s}_x\in \tilde{l}^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge1})),A^{(1)}_{w_x}\big),\forall x\in V_0\big\},\end{aligned}\tag{4.2}$$

with

$$\begin{aligned}\tilde{l}^p_{\cdots}(\cdots)=\big\{\boldsymbol{s}\in l^p_{\cdots}(\cdots):\{(\Delta^{(1)})^i s_n(F_\tau^{-1}x,\bullet)\}_{n\ge0}\in l^p_{\alpha^\sigma_x,\beta_x}(L^p(\Omega^{\wedge1}),H^p_\sigma(\Omega^{\wedge1})),\\ \{\partial^{(1)}_n(\Delta^{(1)})^i s_n(F_\tau^{-1}x,\bullet)\}_{n\ge0}\in l^p_{\alpha^\sigma_x,\beta_x}(L^p(\Omega^{\wedge1}),H^p_\sigma(\Omega^{\wedge1})),\forall 0\le i<k,x\in V_0\big\}.\end{aligned}\tag{4.3}$$

We can check that $\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)$ is stable under complex interpolation, i.e. $\big[\tilde{\mathcal{T}}^p_{0,k}(\Omega),\tilde{\mathcal{T}}^p_{2k,k}(\Omega)\big]_{\sigma/2k}=\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)$, using Lemma C.9 (b) and Lemma 2.11. Furthermore, by applying Proposition 2.9, we see that

$$\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)=\big\{f\in\mathcal{T}^p_{\sigma,k}(\Omega):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\},\ \text{if }\sigma\notin\{\frac{d_S}{p},2-\frac{d_S}{p'}\}+2\mathbb{Z}_+.$$

and $\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)=\mathcal{T}^p_{\sigma,k}(\Omega)$ if $\sigma<\frac{d_S}{p}$. Clearly, we have $\tilde{H}^\sigma_0(\Omega)=\mathcal{K}^p_{0,k}(\Omega)\oplus\tilde{\mathcal{T}}^p_{0,k}(\Omega)$ and $\tilde{H}^\sigma_{2k}(\Omega)=\mathcal{K}^p_{2k,k}(\Omega)\oplus\tilde{\mathcal{T}}^p_{2k,k}(\Omega)$. Thus

$$\big[\tilde{H}^p_0(\Omega),\tilde{H}^p_{2k}(\Omega)\big]_{\sigma/2k}=\mathcal{K}^p_{\sigma,k}(\Omega)\oplus\tilde{\mathcal{T}}^p_{\sigma,k}(\Omega)=\big\{f\in H^p_\sigma(\Omega):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\}.$$

Now, using Claim 1, we conclude that $\tilde{H}^p_\sigma(\Omega)$ is stable under complex interpolation. Then (a) follows from Claim 2.

(b). It suffices to show that $\big(\tilde{H}^p_0(\Omega),\tilde{H}^p_{2k}(\Omega)\big)_{\sigma/2k,q}=\big\{f\in B^{p,q}_\sigma(\Omega):T^{(x,1)}_\sigma f=0,\forall x\in V_0\big\}$ by using Proposition 2.13. This follows from a same proof of Claim 2. □

Remark 1. We can use the above argument iteratively to show a same result for general $\Omega=K^l\times\tilde{K}^{d-l}$.

4.3. **The spaces $\mathring{H}^p_\sigma(\Omega)$ and $\mathring{B}^{p,q}_\sigma(\Omega)$.** In this section, we introduce the function spaces $\mathring{H}^p_\sigma(\Omega)$ and $\mathring{B}^{p,q}_\sigma(\Omega)$. We focus on $\Omega=\tilde{K}^d_+$ for simplicity.

Definition 4.7. *(a). For $1<p<\infty$ and $\sigma\geq 0$. Define $\mathring{H}^p_\sigma(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $H^p_\sigma(\Omega)$.*

(b). For $1<p,q<\infty$ and $\sigma>0$. Define $\mathring{B}^{p,q}_\sigma(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $B^{p,q}_\sigma(\Omega)$.

In classical analysis, on a smooth bounded domain $\Omega\subset\mathbb{R}^d_+$, we always have $\tilde{H}^p_\sigma(\Omega)=\mathring{H}^p_\sigma(\Omega)$ for all orders $\sigma\geq 0$ except for the critical orders $\{\frac{d_S}{p},2-\frac{d_S}{p'}\}+2\mathbb{Z}_+$. However, in fractal setting, things will be different since the boundary behavior of functions will be more complicated. In fact, other than the normal derivative, some other 'higher order derivatives' emerge, which do not play a role in the matching of pieces of functions at junction points, but really reflect the boundary behavior of functions. See [11, 33] for example.

To be more precise, we need to introduce the so-called tangents of functions. Recall the definition of multiharmonic functions in Definition 3.6 (a) and the map A_w in Definition 3.7 (a). We introduce the following notations.

Notations. *1). Denote $\mathcal{H}_\#=\bigcup_{k=1}^\infty\mathcal{H}_{k-1}$.*

2). Let λ be a generalized eigenvalue of A_w on $\mathcal{H}_\#$, and write $U_{\lambda,w}$ for the generalized eigenspace, i.e. $U_{\lambda,w}=\bigcup_{m=0}^\infty\ker(A_w-\lambda)^m$.

*3). Let $1=\gamma_{0,w}>\gamma_{1,w}>\cdots>\gamma_{l,w}>\cdots$ be the absolute values of **nonzero** eigenvalues of $A_w:\mathcal{H}_\#\to\mathcal{H}_\#$, which is ordered in decreasing order.*

4). Write $\bar{U}_{i,w}=\bigcup_{|\lambda|=\gamma_{i,w}}U_{\lambda,w}$ for each $i\geq 0$.

For convenience, for $x=\pi(\tau\dot{w})\in V_0$, we write $\gamma_{i,x}=\gamma_{i,w}$, $U_{\lambda,x}=U_{\lambda,w}$ and $\bar{U}_{i,x}=\bar{U}_{i,w}$.

These notations are analogous to those in the Appendix A, and we will use the results there. Nevertheless, we recommend readers to consider a simple case that A_w is diagonalizable at this stage, so that all the sequence spaces that we will consider later are essentially direct sums of sequence spaces illustrated in Section 2, and only the results in Section 2 are needed.

For $x\in V_0$, $\sigma\geq 0$, in contrast with the trace map $T^{(x,1)}_\sigma$ in Theorem 4.3, we introduce the following *tangent map* $Tan^{(x,1)}_\sigma$ on $L^p(\Omega)$ with $\Omega=\tilde{K}^d_+$. Readers may also read [12, 32, 33, 37] for some disucssions on the definition of tangents.

Definition 4.8. *Let $x=\pi(\tau\dot{w})\in V_0$, $1<p<\infty$, $\sigma\geq 0$, $\alpha_x=(r_w\mu_w)^{1/2}$ and $\beta_x=\mu_w^{-1/p}$. Let $i\geq 0$ be such that $\gamma_{i+1,x}\leq\alpha_x^\sigma\beta_x<\gamma_{i,x}$. For $f\in L^p(\Omega)$, we write $Tan^{(x,1)}_\sigma f=s$ with $s\in\bigoplus_{j=0}^i\bar{U}_{j,x}\otimes L^p(\Omega^{\wedge 1})$ if*

$$\lim_{n\to\infty}\gamma_{i,x}^{-n}\big\|\big(A^{(1)}_w\big)^n\big(A^{(1)}_\tau f-s\big)\big\|_{L^\infty(K,L^p(\Omega^{\wedge 1}))}=0.$$

For convenience, we set $Tan^{(x,1)}_\sigma f=0$ for $0\leq\sigma\leq d_S/p$ (i.e. $\alpha_x^\sigma\beta_x\geq 1$).

In fact, by applying Theorem 3.17, Lemma 4.4 and using Lemma A.1 (or Lemma 2.3 if A_w is diagonalizable), we can easily see the existence of the tangents at x for functions in $H^p_\sigma(\Omega)$.

Lemma 4.9. *The tangent map* $Tan_\sigma^{(x,1)}$ *is bounded from* $H^p_\sigma(\Omega)\to\bigoplus_{\gamma_{j,x}>\alpha_x^\sigma\beta_x}\bar U_{j,x}\otimes L^p(\Omega^{\wedge 1})$.

Remark. If A_w is diagonalizable, we can easily see a trace theorem for the tangent as Theorem 4.3. However, when A_w is not diagonalizable, the trace space is not so clear at present. It is of interest to see whether it is related with some interpolation functor.

In this subsection, we will derive the following characterization of Sobolev spaces $\mathring{H}^p_\sigma(\tilde K^d_+)$ and Besov spaces $\mathring{B}^{p,q}_\sigma(\tilde K^d_+)$.

Theorem 4.10. *Let* $1<p,q<\infty$.
(a). For $\sigma\geq 0$*, we have* $\mathring{H}^p_\sigma(\tilde K^d_+)=\big\{f\in H^p_\sigma(\tilde K^d_+): Tan_\sigma^{(x,1)}f=0,\forall x\in V_0\big\}$.
(b). For $\sigma>0$*, we have* $\mathring{B}^{p,q}_\sigma(\tilde K^d_+)=\big\{f\in B^{p,q}_\sigma(\tilde K^d_+): Tan_\sigma^{(x,1)}f=0,\forall x\in V_0\big\}$.

We will prove this theorem in the remaining part of this subsection. First, let's look at some easy lemmas.

Lemma 4.11. *Let* $1<p<\infty$ *and* $0\leq\sigma<\sigma'\leq 2k$. *The space* $\mathcal{K}^p_{\sigma',k}(\tilde K^d_+)$ *is dense in* $\mathcal{K}^p_{\sigma,k}(\tilde K^d_+)$.

Proof. Since $H^p_{\sigma'}(\tilde K^d)$ is dense in $H^p_\sigma(\tilde K^d)$, we have $H^p_{\sigma'}(\tilde K^d_+)$ is dense in $H^p_\sigma(\tilde K^d_+)$ by Theorem 3.5. The lemma then follows by using Theorem 3.17 (b). □

Lemma 4.12. *Let* $x=\pi(\tau\dot w)\in V_0$, $1<p<\infty$ *and* $k\in\mathbb{Z}_+$. *Then there exists a constant* $C>0$ *only depending on* k *and* p*, such that for any* $f\in C^\infty(\tilde K^d_+)$*, we can find a function* $g\in C^\infty(\tilde K^d_+)$ *supported in* $F^{(1)}_\tau\tilde K^d_+$ *satisfying*
$$\|g\|_{H^p_{2k}(\tilde K^d_+)}\leq C\|f\|_{H^p_{4k}(\tilde K^d_+)},$$
and
$$T^{(x,1)}_{2k'}g=T^{(x,1)}_{2k'}f,\quad\forall k'\in\mathbb{Z}_+.$$

Proof. For convenience, we write $T^{(x,1)}_{2k}f=\big((T^{(x,1)}_{2k}f)_0,\cdots,(T^{(x,1)}_{2k}f)_{2k-1}\big)$, i.e. $(T^{(x,1)}_{2k}f)_i=(\Delta^{(1)})^if|_{\{x\}\times\tilde K^{d-1}}$ for $0\leq i<k$, and $(T^{(x,1)}_{2k}f)_i=(\partial^{(1)}\Delta^{(1)})^{i-k}f|_{\{x\}\times\tilde K^{d-1}}$ for $k\leq i<2k$. Clearly, for $0\leq i<2k$, we have
$$\big\|(T^{(x,1)}_{2k}f)_i\big\|_{H^p_{2k}(\tilde K^{d-1})}\lesssim\|f\|_{H^p_{4k}(\tilde K^d_+)}\tag{4.4}$$
by using Theorem 4.3, noticing that $B^{p,p}_\sigma(\tilde K^{d-1})\subset H^p_{\sigma-\varepsilon}(\tilde K^{d-1})$.

First, we look for a smooth function g_1 supported in $F^{(1)}_\tau\tilde K^d_+$ satisfying $\|g_1\|_{H^p_{2k}(\tilde K^d_+)}\leq C_1\|f\|_{H^p_{4k}(\tilde K^d_+)}$ for some constant $C_1>0$, and $T^{(x,1)}_{2k}g_1=T^{(x,1)}_{2k}f$. For this, by applying Proposition B.2 (due to L. Rogers, R.S. Strichartz and A. Teplyaev []), we choose $h_{j,1},h_{j,2}\in C^\infty(K), j=0,1,\cdots,k-1$ supported in $F_\tau K$ and
$$\begin{cases}\Delta^ih_{j,1}(x)=\delta_{i,j},\\ \partial_n\Delta^ih_{j,1}(x)=0,\end{cases}\qquad\begin{cases}\Delta^ih_{j,2}(x)=0,\\ \partial_n\Delta^ih_{j,2}(x)=\delta_{i,j},\end{cases}\qquad\forall i\geq 0\text{ and }0\leq j<k.$$

Then we take $g_1=\sum_{i=0}^{k-1}h_{i,1}\otimes(T^{(x,1)}_{2k}f)_i+\sum_{i=0}^{k-1}h_{i,2}\otimes(T^{(x,1)}_{2k}f)_{i+k}$, which is obviously supported in $F^{(1)}_\tau\tilde K^d_+$ and $T^{(x,1)}_{2k}g_1=T^{(x,1)}_{2k}f$. Using (4.4), one can easily see that
$$\|g_1\|_{H^p_{2k}(\tilde K^d_+)}\leq C_1\|f\|_{H^p_{4k}(\tilde K^d_+)},$$

where C_1 only depends on k and p.

So it remains to consider higher order Laplacians and normal derivatives of f at x. Choose $h_{j,1},h_{j,2}\in C^\infty(K)$ for $j\geq k$, supported in $F_\tau K$, such that

$$\begin{cases}\Delta^i h_{j,1}(x)=\delta_{i,j},\\ \partial_n\Delta^i h_{j,1}(x)=0,\end{cases}\qquad \begin{cases}\Delta^i h_{j,2}(x)=0,\\ \partial_n\Delta^i h_{j,2}(x)=\delta_{i,j},\end{cases}\qquad \forall i\geq 0 \text{ and } j\geq k.$$

We define

$$\begin{aligned}g_2=&\sum_{j\geq k}(r_w\mu_w)^{jn_j}(A_\tau h_{j,1}\circ F_w^{-n_j}\circ F_\tau^{-1})\otimes(\Delta^{(1)})^j f(x,\bullet)\\ &+\sum_{j\geq k}(r_w\mu_w)^{jn_j}(A_\tau h_{j,2}\circ F_w^{-n_j}\circ F_\tau^{-1})\otimes\partial_n^{(1)}(\Delta^{(1)})^j f(x,\bullet),\end{aligned}$$

with each $n_j\in\mathbb{N}$ to be determined. Obviously, for each $j\geq k$, by choosing n_j sufficiently large, we can make $\|g_2\|_{H^p_{2k}(\tilde K^d)}$ as small as possible. In addition, $g_2\in C^\infty(\tilde K^d_+)$ and is supported in $F_\tau^{(1)}\tilde K^d_+$. See [] for a discussion for the one dimensional case. The lemma is proven with $g=g_1+g_2$. □

The following scaling property is obvious.

Lemma 4.13. *For $f\in H^p_{2k}(F_w^{(1)}\tilde K^d_+)$, we have*

$$(r_w\mu_w)^k\mu_w^{-1/p}\|f\|_{H^p_{2k}(F_w^{(1)}\tilde K^d_+)}\lesssim\|A_w^{(1)}f\|_{H^p_{2k}(\tilde K^d_+)}\lesssim\mu_w^{-1/p}\|f\|_{H^p_{2k}(F_w^{(1)}\tilde K^d_+)}.$$

Lemma 4.14. *For $x=\pi(\tau\dot w)\in V_0$ and $f\in H^p_{8k}(\tilde K^d_+)$, we have*

$$\lim_{n\to\infty}(r_w\mu_w)^{-2kn}\mu_w^{n/p}\big\|(A_w^{(1)})^nA_\tau^{(1)}f-(R_{x,4k}f)_n\big\|_{H^p_{4k}(\tilde K^d_+)}=0.$$

Proof. First, we can easily see that

$$\lim_{n\to\infty}(r_w\mu_w)^{-4kn}\mu_w^{n/p}\big\|(A_w^{(1)})^nA_\tau^{(1)}f-(R_{x,4k}f)_n\big\|_{L^p(\tilde K^d_+)}=0.$$

In fact, it is helpful to consider the case $\tilde K^d_+=K$. Next, we can directly check that

$$\lim_{n\to\infty}\mu_w^{n/p}\big\|(A_w^{(1)})^nA_\tau^{(1)}f-(R_{x,4k}f)_n\big\|_{H^p_{8k}(\tilde K^d_+)}=0,$$

since by Lemma 4.13,

$$\begin{aligned}\big\|(A_w^{(1)})^nA_\tau^{(1)}f-(R_{x,4k}f)_n\big\|_{H^p_{8k}(\tilde K^d_+)}&\lesssim\mu_w^{-n/p}\big\|A_\tau^{(1)}f-\big(R_{w,4k}(A_\tau^{(1)}f\cdot 1|_{(F_w^{(1)})^n\tilde K^d_+})\big)_0\big\|_{H^p_{8k}((F_w^{(1)})^n\tilde K^d_+)}\\ &\lesssim\mu_w^{-n/p}\|A_\tau^{(1)}f\|_{H^p_{8k}((F_w^{(1)})^n\tilde K^d_+)}=o(\mu_w^{-n/p}).\end{aligned}$$

The lemma then follows by complex interpolation. □

Proof of Theorem 4.10. (a). For convenience, write $\Omega=\tilde K^d_+$. Fix $k\in\mathbb{N}$ and let $0\leq\sigma\leq 8k$, we let

$$\dot H^p_\sigma(\Omega)=\mathcal{K}^p_{\sigma,4k}(\Omega)\oplus\dot{\mathcal{T}}^p_{\sigma,4k}(\Omega),$$

where

$$\dot T^p_{\sigma,4k}(\Omega)=\sum_{x\in V_0}E_{x,4k}l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\Omega^{\wedge 1})),\mathcal{H}_{k-1}(K,H^p_\sigma(\Omega^{\wedge 1}))\big).\tag{4.5}$$

See Definition 3.13 for the meaning of the symbols.

Claim 1: $\dot{H}^p_{8k}(\Omega) \subset \mathring{H}^p_{2k}(\Omega)$.

Proof of Claim 1. Let $f \in \dot{H}^p_{8k}(\Omega)$ and fix $x = \pi(\tau\dot{w}) \in V_0$. By using Lemma 4.14, we can see that

$$\lim_{n\to\infty} (r_w\mu_w)^{-2kn}\mu_w^{n/p}\big\|(A_w^{(1)})^n A_\tau^{(1)} f\big\|_{H^p_{4k}(\Omega)} = 0.$$

For fixed $\varepsilon > 0$, by choosing n_x large enough, we have

$$\big\|(A_w^{(1)})^{n_x} A_\tau^{(1)} f\big\|_{H^p_{4k}(\Omega)} \leq \varepsilon(r_w\mu_w)^{2kn_x}\mu_w^{-n_x/p}.$$

By applying the Dirichlet heat kernel P_t with t sufficiently small, we have $P_tf \in C^\infty(\Omega)$ with

$$\big\|(A_w^{(1)})^{n_x} A_\tau^{(1)} P_t f\big\|_{H^p_{4k}(\Omega)} \leq 2\varepsilon(r_w\mu_w)^{2kn_x}\mu_w^{-n_x/p}.$$

By applying Lemma 4.12, noticing the requirement (3.1), we can find a function $g_x \in C^\infty(\Omega)$ supported outside a neighbourhood of the point $F_\tau^{-1}(x)$, satisfying the same boundary conditions (values and normal derivatives of $(\Delta^{(1)})^i$'s) as $(A_w^{(1)})^{n_x} A_\tau^{(1)} P_t f$ at $(V_0 \setminus \{F_\tau^{-1}x\}) \times \Omega^{\wedge 1}$, and

$$\|g_x\|_{H^p_{2k}(\Omega)} \lesssim \big\|(A_w^{(1)})^{n_x} A_\tau^{(1)} P_t f\big\|_{H^p_{4k}(\Omega)} \lesssim \varepsilon(r_w\mu_w)^{2kn_x}\mu_w^{-n_x/p}.$$

Apply Lemma 4.13, we have

$$\begin{cases} \big\|g_x \circ (F_w^{(1)})^{-n_x} \circ (F_\tau^{(1)})^{-1}\big\|_{H^p_{2k}(F_\tau^{(1)}\circ(F_w^{(1)})^{n_x}\Omega)} \lesssim \varepsilon, \\ \|P_tf\|_{H^p_{2k}(F_\tau^{(1)}\circ(F_w^{(1)})^{n_x}\Omega)} \lesssim \varepsilon. \end{cases}$$

We replace P_tf with $g_x \circ (F_w^{(1)})^{-n_x} \circ (F_\tau^{(1)})^{-1}$ on the cell $F_\tau^{(1)}(F_w^{(1)})^{n_x}\Omega$ for each x by using the matching condition at $(F_\tau F_w^{n_x} V_0 \setminus \{x\}) \times \Omega^{\wedge 1}$, and denote the result function as f_ε. Obviously, $f_\varepsilon \in C_c^\infty(\Omega)$ and $\|f - f_\varepsilon\|_{H^p_{2k}(\Omega)} \lesssim \varepsilon$. The claim then follows since 2ε can be chosen arbitrarily small. □

The following claim is an easy consequence of Claim 1 and Lemma 4.11.

Claim 2: Define

$$\mathring{\mathcal{T}}^p_{\sigma,4k}(\Omega) = \sum_{x\in V_0} E_{x,4k}\overline{l^p_{\alpha_x^\sigma,\beta_x}\big(\mathcal{H}_{k-1}(K, L^p(\Omega^{\wedge 1})), \mathcal{H}_{k-1}(K, H^p_\sigma(\Omega^{\wedge 1}))\big)}^{A_w^{(1)}}. \tag{4.6}$$

(Here $\overline{l^p_{\dots}(\cdots)}^{A_w^{(1)}}$ *is the closure of* $l^p_{\dots}(\cdots)$ *in* $l^p_{\dots}(\cdots, A_w^{(1)})$*, see Appendix A for details.) Then for* $0 \leq \sigma \leq 2k$*, we have* $\mathcal{K}^p_{\sigma,4k}(\Omega) \oplus \mathring{\mathcal{T}}^p_{\sigma,4k}(\Omega) \subset \mathring{H}^p_\sigma(\Omega)$.

The following claim is a consequence of Claim 2 and Proposition A.5 in Appendix A (or Proposition 2.9 if A_w is diagonalizable).

Claim 3: $\big\{f \in H^p_\sigma(\Omega) : Tan_\sigma^{(x,1)} f = 0, \forall x \in V_0\big\} \subset \mathring{H}^p_\sigma(\Omega)$.

Claim 3 implies half of part (a). The other direction of the containment is obvious by Lemma 4.9 and the fact that $C_c^\infty(\Omega) \subset \big\{f \in H^p_\sigma(\Omega) : Tan_\sigma^{(x,1)} f = 0, \forall x \in V_0\big\}$.

(b). This follows by a similar proof as part (a), using real interpolation and Proposition A.6 instead. □

4.4. **An embedding theorem with $\sigma \in \mathbb{R}$.** In this last subsection, we will extend the Sobolev spaces and Besov spaces to $\sigma \in \mathbb{R}$. Let's first look at the full space $\tilde{K}^d$. Formally, we can extend Definition 1.1 and Definition 1.2 directly. To make things meaningful, we need to use the distributions on $\tilde{K}^d$ (or equivalently the uniform extrapolation space of Δ, see [] for details). See Definition B.1 in Appendix B for the definition and [] for a detailed discussion.

Definition 4.15. *Let $p, q \in (1,\infty)$, $\sigma \in \mathbb{R}$, and consider the $\Delta : \mathfrak{D}'(\tilde{K}^d) \to \mathfrak{D}'(\tilde{K}^d)$.*

(a). Define the Sobolev space

$$H^p_\sigma(\tilde{K}^d) = (1-\Delta)^{-\sigma/2} L^p(\tilde{K}^d),$$

with norm $\|f\|_{H^p_\sigma(\tilde{K}^d)} = \big\|(1-\Delta)^{\sigma/2} f\big\|_{L^p(\tilde{K}^d)}$.

(b). Define the heat Besov space

$$B^{p,q}_\sigma(\tilde{K}^d) = \Big\{ f \in L^p(\tilde{K}^d) : \Big(\int_0^\infty \big(t^{-\sigma/2} \big\|(t\Delta)^k e^{t\Delta} f\big\|_{L^p(\tilde{K}^d)}\big)^q dt/t \Big)^{1/q} < \infty \Big\},$$

with $k \in \mathbb{N} \cap (\sigma/2, \infty)$, and norm $\|f\|_{B^{p,q}_\sigma(\tilde{K}^d)} = \|f\|_{L^p(\tilde{K}^d)} + \big(\int_0^\infty (t^{-\sigma/2} \|(t\Delta)^k e^{t\Delta}(f)\|_{L^p(\tilde{K}^d)})^q dt/t \big)^{1/q}$.

Since $(1-\Delta) : L^p(\tilde{K}^d) \to L^p(\tilde{K}^d)$ is invertible, we can apply Theorem **??** in book [].

Proposition 4.16. *Let $p, q \in (1,\infty)$ and $\sigma \in \mathbb{R}$. Then $(1-\Delta)$ is an isomorphism from $H^p_\sigma(\tilde{K}^d)$ to $H^p_{\sigma-2}(\tilde{K}^d)$ and from $B^{p,q}_\sigma(\tilde{K}^d)$ to $B^{p,q}_{\sigma-2}(\tilde{K}^d)$.*

Thus, for a lot of properties, we only need to study the $\sigma \geq 0$ case. In particular, the interpolation property, Lemma 1.3, holds for $\sigma \in \mathbb{R}$. Also, since $L^p(\tilde{K}^d)$ is the dual space of $L^{p'}(\tilde{K}^d)$ for $\frac{1}{p} + \frac{1}{p'} = 1$, it is not hard to see the following dual property for $H^p_\sigma(\tilde{K}^d)$ and $B^{p,q}_\sigma(\tilde{K}^d)$, where for $B^{p,q}_\sigma(\tilde{K}^d)$ we apply the property of real interpolation.

Proposition 4.17. *Let $p, q \in (1,\infty)$ and $\sigma \in \mathbb{R}$. Also, let $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$. Then*

(a). $H^p_\sigma(\tilde{K}^d)$ is the dual of $H^{p'}_{-\sigma}(\tilde{K}^d)$.

(b). $B^{p,q}_\sigma(\tilde{K}^d)$ is the dual of $B^{p',q'}_{-\sigma}(\tilde{K}^d)$.

However, to define Sobolev spaces and Besov spaces with $\sigma \in \mathbb{R}$ on a domain **with boundary** is a more delicate question. In history, basically there are two methods for the Sobolev spaces, by trace or by dual. See the monographs [30] and [38] for example. The two are almost the same, except for delicate difference at some critical orders. In this paper, we will admit the definition by dual due to J.L. Lions and E. Magenes [30].

For $D(\tilde{K}^d_+) \subset L^p(\tilde{K}^d_+)$ a function space on $\tilde{K}^d_+$, we denote $\big(D(\tilde{K}^d_+)\big)^*$ as the *dual* of $D(\tilde{K}^d_+)$, embedded in $\mathfrak{D}'(\tilde{K}^d_+)$ with a natural meaning.

Definition 4.18. *Let $p, q \in (1,\infty)$, $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$ and $\sigma \in \mathbb{R}$.*

(a). If $\sigma \geq 0$, we define $H^p_\sigma(\tilde{K}^d_+) = H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$ and $B^{p,q}_\sigma(\tilde{K}^d_+) = B^{p,q}_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$ as before.

(b). If $\sigma < 0$, we define $H^p_\sigma(\tilde{K}^d_+) = \big(\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)\big)^*$ and $B^{p,q}_\sigma(\tilde{K}^d_+) = \big(\mathring{B}^{p',q'}_{-\sigma}(\tilde{K}^d_+)\big)^*$.

We end this section with an embedding theorem concerning the function spaces on $\tilde{K}^d_+$ and $\tilde{K}^d$.

Theorem 4.19. *Let $p, q \in (1,\infty)$, $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$ and $\sigma \in \mathbb{R}$. We have*

$$H^p_\sigma(\tilde{K}^d_+) = H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+} \quad \text{and} \quad B^{p,q}_\sigma(\tilde{K}^d_+) = B^{p,q}_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$$

if and only if $\sigma\notin\{\frac{d_S}{p'},2-\frac{d_S}{p}\}-2\mathbb{N}$*, where the restriction is in the sense of distribution.*

Remark. For each $f\in\mathfrak{D}'(\tilde{K}^d)$, we *restrict* it to $\mathfrak{D}'(\tilde{K}^d_+)$ with the equation

$$<f|_{\tilde{K}^d_+},\varphi>=<f,\Theta\varphi>,\forall\varphi\in\mathfrak{D}(\tilde{K}^d_+),$$

where Θ is the extension by zero map introduced in Lemma 3.18.

Proof. It suffices to consider the $\sigma<0$ case, and we focus on Sobolev spaces. Since $\mathfrak{D}(\tilde{K}^d_+)$ is dense in $\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$, each $f\in H^p_\sigma(\tilde{K}^d_+)$ is uniquely determined by the distribution $f|_{\mathfrak{D}(\tilde{K}^d_+)}$, which we still denote by f for convenience. Thus, the Sobolev spaces $H^p_\sigma(\tilde{K}^d_+)$ is well defined as a space of distributions, and the theorem makes sense. We consider two cases in the following.

First, we consider $\sigma\notin\{\frac{d_S}{p'},2-\frac{d_S}{p}\}-2\mathbb{N}$. In this case, combining Theorem 4.6 and 4.10, we can see that

$$\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)\subset\tilde{H}^{p'}_{-\sigma}(\tilde{K}^d_+).$$

As a consequence, we can identify $\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$ with the closed subspace $\Theta\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$ of $H^{p'}_{-\sigma}(\tilde{K}^d)$. Thus, each $f\in H^p_\sigma(\tilde{K}^d)=\big(H^{p'}_{-\sigma}(\tilde{K}^d)\big)^*$ naturally restricts to be in $H^p_\sigma(\tilde{K}^d_+)=\big(\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)\big)^*$. Also, each $f\in H^p_\sigma(\tilde{K}^d_+)=\big(\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)\big)^*$ can be extended to $H^p_\sigma(\tilde{K}^d)=\big(H^{p'}_{-\sigma}(\tilde{K}^d)\big)^*$ by the Hahn-Banach Theorem. This proves the non-critical order case.

Next, we consider the critical order case, i.e. $\sigma\in\{\frac{d_S}{p'},2-\frac{d_S}{p}\}-2\mathbb{N}$. For this case, we notice that

$$\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)\nsubseteq\tilde{H}^{p'}_{-\sigma}(\tilde{K}^d_+),$$

since $\mathring{\tilde{\mathcal{T}}}^{p'}_{-\sigma,k}(\tilde{K}^d_+)\nsubseteq\tilde{\mathcal{T}}^{p'}_{-\sigma,k}(\tilde{K}^d_+)$ for $k\geq-\sigma$ in $\mathbb{N}$. As a consequence, let $\mathring{\tilde{H}}^{p'}_{-\sigma}(\tilde{K}^d_+)$ be the closure of $\mathfrak{D}(\tilde{K}^d_+)$ in $\tilde{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$, one see that $\mathring{\tilde{H}}^{p'}_{-\sigma}(\tilde{K}^d_+)$ is a proper dense subspace of $\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$ and $\big(\mathring{\tilde{H}}^{p'}_{-\sigma}(\tilde{K}^d_+)\big)^*=H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}$ by a same argument as the first case. Thus, we conclude that

$$H^p_\sigma(\tilde{K}^d_+)\subsetneq H^p_\sigma(\tilde{K}^d)|_{\tilde{K}^d_+}.$$

This finishes the proof. The proof for Besov spaces is essentially the same. $\square$

Remark. There is a natural analog of Theorem 4.19 in [38] Section 2.10. We remark here that our method is essentially different from the classical case in the following senses.

1). Our proof is self-contained. In particular, our proof derives the observation that functions in $H^p_\sigma(\tilde{K}^d_+)$ can extend by zero to $H^p_\sigma(\tilde{K}^d)$ (i.e. $H^p_\sigma(\tilde{K}^d_+)=\tilde{H}^p_\sigma(\tilde{K}^d_+)$) for $0\leq\sigma<\frac{d_S}{p}$ without further characterization of the spaces. (See Section 2.10 for the role of this observation in the proof, and see papers [] and [] for a proof of this obvervation on $\mathbb{R}^d_+$).

2). The normal derivatives and the tangents are defined in a pointwise way on fractal domains, compared with the classical domains. In addition, much information in the tangent does not take part in the matching conditions. So there is not an easy restriction mapping from the whole space to the half space, analogous to the reflection method in the classical domains.

5. Interpolation theorems

The interpolation theorems for Sobolev spaces and Besov spaces on domains in $\mathbb{R}^d$ have been extensively studied in a sequence of monographs [30],[7] and [38].

It is of interest to derive the corresponding results in the fractal setting. In this section, we will provide a full list of interpolation results of Sobolev spaces and Besov spaces on $\tilde{K}^d$ and $\tilde{K}^d_+$. We are particularly concern those involving critical orders of σ, see Figure **??**.

5.1. **The full space $\tilde{K}^d$ case.** We will first consider the interpolation property of $H^p_\sigma(\tilde{K}^d)$ and $B^{p,q}_\sigma(\tilde{K}^d)$ on $\tilde{K}^d$ with $p,q\in(1,\infty)$ and $\sigma\in\mathbb{R}$. For p,q fixed, of cause, the spaces are both complex and real interpolation stable, by Lemma 1.3, Proposition 4.16, with a standard reiteration argument of interpolations. So we are particular interested in the case that p,q are not fixed. Readers please refer to the book [] for a classical theorem on $\mathbb{R}^d$.

Theorem 5.1. *Let $0<\theta<1$, $\sigma,\sigma_0,\sigma_1\in\mathbb{R}$ and $1<p_0,p_1,q_0,q_1<\infty$. Put $\sigma_\theta\in\mathbb{R}$, $1<p_\theta,q_\theta<\infty$ sasifying*

$$\sigma_\theta=(1-\theta)\sigma_0+\theta\sigma_1,\quad \frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Then we have

$$\big[H^{p_0}_{\sigma_0}(\tilde{K}^d),H^{p_1}_{\sigma_1}(\tilde{K}^d)\big]_\theta = H^{p_\theta}_{\sigma_\theta}(\tilde{K}^d), \tag{5.1}$$

$$\big(H^{p_0}_{\sigma}(\tilde{K}^d),H^{p_1}_{\sigma}(\tilde{K}^d)\big)_{\theta,p_\theta} = H^{p_\theta}_{\sigma}(\tilde{K}^d), \tag{5.2}$$

$$\big[B^{p_0,q_0}_{\sigma_0}(\tilde{K}^d),B^{p_1,q_1}_{\sigma_1}(\tilde{K}^d)\big]_\theta = B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d), \tag{5.3}$$

$$\big(B^{p_0,q_0}_{\sigma_0}(\tilde{K}^d),B^{p_1,q_1}_{\sigma_1}(\tilde{K}^d)\big)_{\theta,q_\theta} = B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d). \qquad (\text{if } p_\theta=q_\theta) \tag{5.4}$$

Proof. The first two identities can be proven with a same argument as that for the $\mathbb{R}^d$ case, using the fact that $\big\{(1-\Delta)^{it}\big\}_{t\in\mathbb{R}}$ is a C_0-group, see [] for example.

We now prove (5.3) and (5.4). The key idea follows from Peetre [], by introducing a retraction from the spaces of the form $l^q_{\alpha^\sigma}(L^p)$ to Besov spaces $B^{p,q}_\sigma(\tilde{K}^d)$. We will realize a similar retraction using heat kernel instead of Fourier transform here.

Let's first recall Lemma 2.6 and 2.8, which together imply the following claim. For convenience, we write $L=1-\Delta:L^p(\tilde{K}^d)\to L^p(\tilde{K}^d)$ in the proof.

Claim 1: Fix $k\in\mathbb{N}$ and $0<\alpha<1$, and let $0<\sigma<2k$, $p,q\in(1,\infty)$. We have $B^{p,q}_\sigma(\tilde{K}^d)$ is a retract of $l^q_{\alpha^k,\alpha^{\sigma/2-k}}\big(\overline{\mathcal{D}(L^k)},1\big)$, with the restriction map being $R=\Gamma_1$, and extension map being $E=S^{L,\varphi}_\alpha$ with suitable φ. (See Lemma 2.6 and 2.8 for the notations.)

To proceed, we need to use some properties of the Laplacian Δ.

Claim 2: For $k\in\mathbb{N}$ and $L:L^p(\tilde{K}^d)\to L^p(\tilde{K}^d)$, we have

$$\sup_{t>0}\big\|t\big(t+L^k\big)^{-1}\big\|<\infty,\quad \sup_{t>0}\big\|L^k\big(t+L^k\big)^{-1}\big\|<\infty.$$

By Lemma C.5 in Appendix C, $L=1-\Delta:L^p(\tilde{K}^d)\to L^p(\tilde{K}^d)$ is sectorial of angle 0. So we can apply Lemma C.2 to conclude that L^k is also sectorial of angle 0. The claim follows from the definition of sectorial operators and the equality $L^k\big(t+L^k\big)^{-1}=1-t\big(t+L^k\big)^{-1}$.

As a consequence of Claim 2, we have

Claim 3: The map $I : l^q_{\alpha^{\sigma/2-k}}\big(L^p(\tilde{K}^d)\big) \to l^q_{\alpha^k,\alpha^{\sigma/2-k}}\big(\overline{\mathcal{D}(L^k)}\big)$ *defined by*

$$I\big(\{s_n\}_{n\geq 0}\big) = \big\{(\alpha^{-kn} + L^k)^{-1}(s_n)\big\}_{n\geq 0}$$

is an isomorphism for $0 < \sigma < 2k$.

By using Claim 1 and Claim 3, noticing that $l^q_{\alpha^k,\alpha^{\sigma/2-k}}\big(\overline{\mathcal{D}(L^k)},1\big)$ is in fact isomorphic to $l^q_{\alpha^k,\alpha^{\sigma/2-k}}\big(\overline{\mathcal{D}(L^k)}\big)$, we conclude that $B^{p,q}_\sigma(\tilde{K}^d)$ is a retract of $l^q_{\alpha^{\sigma/2-k}}\big(L^p(\tilde{K}^d)\big)$. Hence (5.3) and (5.4) follows using Lemma C.9. □

Remark. It is possible to establish a Littlewood-Paley type decomposition of the Besov spaces, basing on Theorem in [], which will also provide a suitable retract.

We end this part with some corollaries for spaces on $\tilde{K}^d_+$.

Definition 5.2. *For* $k \in \mathbb{N}$, $0 < \sigma < 2k$, $1 < p < \infty$, *recall the definition of* $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$ *and* $\mathcal{T}^p_{\sigma,k}(\tilde{K}^d_+)$ *defined before Theorem 3.17, the space* $\dot{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ *defined in equation (4.5), the space* $\mathring{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ *defined in equation (4.6), and the space* $\tilde{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ *defined in equations (4.2) and (4.3). For* $1 < q < \infty$, *we define*

$$\mathcal{K}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\mathcal{K}^p_{0,k}(\tilde{K}^d_+),\mathcal{K}^p_{2k,k}(\tilde{K}^d_+)\big)_{\sigma/(2k),q}, \qquad \mathcal{T}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\mathcal{T}^p_{0,k}(\tilde{K}^d_+),\mathcal{T}^p_{2k,k}(\tilde{K}^d_+)\big)_{\sigma/(2k),q},$$
$$\dot{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\dot{\mathcal{T}}^p_{0,k}(\tilde{K}^d_+),\dot{\mathcal{T}}^p_{2k,k}(\tilde{K}^d_+)\big)_{\sigma/(2k),q}, \qquad \mathring{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\mathring{\mathcal{T}}^p_{0,k}(\tilde{K}^d_+),\mathring{\mathcal{T}}^p_{2k,k}(\tilde{K}^d_+)\big)_{\sigma/(2k),q},$$
$$\text{and } \tilde{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\tilde{\mathcal{T}}^p_{0,k}(\tilde{K}^d_+),\tilde{\mathcal{T}}^p_{2k,k}(\tilde{K}^d_+)\big)_{\sigma/(2k),q}.$$

Corollary 5.3. *(a). For* $0 \leq \sigma \leq 2k$, *equations (5.1) and (5.2) hold if we replace* $H^p_\sigma(\tilde{K}^d)$ *with the space* $\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$ *(or* $\mathcal{T}^p_{\sigma,k}(\tilde{K}^d_+)$ *or* $\dot{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ *or* $\tilde{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$*).*

(b). For $0 < \sigma < 2k$, *equations (5.3) and (5.4) hold if we replace* $B^{p,q}_\sigma(\tilde{K}^d)$ *with the space* $\mathcal{K}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ *(or* $\mathcal{T}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ *or* $\dot{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ *or* $\tilde{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$*).*

Proof. (a). One can see that $H^p_\sigma(\tilde{K}^d) \curvearrowright H^p_\sigma(\tilde{K}^d_+) \curvearrowright \mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)$ and $H^p_\sigma(\tilde{K}^d) \curvearrowright H^p_\sigma(\tilde{K}^d_+) \curvearrowright \mathcal{T}^p_{\sigma,k}(\tilde{K}^d_+)$ by Theorem 3.5 and 3.17. In addition, $\mathcal{T}^p_{\sigma,k}(\tilde{K}^d_+)$ is isomorphic to $\dot{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ (by the isomorphism of sequence spaces). Lastly, the result for $\tilde{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ holds by using the retraction Lemma C.9.

Part (b) follows from a same argument as (a), noticing that $B^{p,q}_\sigma(\tilde{K}^d) \curvearrowright B^{p,q}_\sigma(\tilde{K}^d_+) \curvearrowright \mathcal{K}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d) \curvearrowright B^{p,q}_\sigma(\tilde{K}^d_+) \curvearrowright \mathcal{T}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ by real interpolation. □

Definition 5.4. *(a) For* $k \in \mathbb{N}$, $0 \leq \sigma \leq 2k$ *and* $1 < p < \infty$, *define* $\dot{H}^p_\sigma(\tilde{K}^d_+) = \mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+) \oplus \dot{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$.

(b) For $k \in \mathbb{N}$, $0 < \sigma < 2k$ *and* $1 < p, q < \infty$, *define* $\dot{B}^p_\sigma(\tilde{K}^d_+) = \mathcal{K}^{p,q}_{\sigma,k}(\tilde{K}^d_+) \oplus \dot{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$.

The following corollary follows immediately.

Corollary 5.5. *For* $\sigma \geq 0$, *equations (5.1) and (5.2) hold if we replace* $H^p_\sigma(\tilde{K}^d)$ *with* $\tilde{H}^p_\sigma(\tilde{K}^d_+)$ *or* $\dot{H}^p_\sigma(\tilde{K}^d_+)$*; for* $\sigma > 0$, *equations (5.3) and (5.4) hold if we replace* $B^{p,q}_\sigma(\tilde{K}^d)$ *with* $\tilde{B}^{p,q}_\sigma(\tilde{K}^d_+)$ *or* $\dot{B}^{p,q}_\sigma(\tilde{K}^d_+)$.

5.2. **The spaces $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and $\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$.** The interpolation theorems for the spaces $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and $\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$ are more delicate. We will rely on Proposition A.5 and Proposition A.6 to fullfill the story (or Proposition 2.9 and Proposition 2.13 if the involved operator A_{w_x} is diagonalizable).

For $\sigma \geq 0$, $1 < p < \infty$, we need to consider **the critical set**

$$\begin{aligned}\mathcal{C}_{O_+} :&= \big\{(\sigma, \frac{1}{p}) : \exists x = \pi(\tau_x \dot{w}_x) \in V_0, i \geq 0, \text{ such that } (\mu_{w_x} r_{w_x})^{\sigma/2} \mu_{w_x}^{-1/p} = \gamma_{i,x}\big\} \\ &= \big\{(\sigma, \frac{1}{p}) : \sigma = \frac{2 \log \gamma_{i,x}}{\log r_{w_x}\mu_{w_x}} + \frac{d_S}{p} \text{ for some } x = \pi(\tau_x \dot{w}_x) \in V_0 \text{ and } i \geq 0\big\}\end{aligned} \tag{5.5}$$

contained in the $(\sigma, 1/p)$-plane, which clearly consists of a sequence of lines, called *critical lines*. See Figure 2. Moreover, we can see that

$$\big\{(\sigma, \frac{1}{p}) : \sigma = \frac{d_S}{p} + 2\mathbb{Z}_+\big\} \cup \big\{(\sigma, \frac{1}{p}) : \sigma = 2 - \frac{d_S}{p'} + 2\mathbb{Z}_+, \text{ with } p' = \frac{p}{p-1}\big\},$$

which we call *Dirichlet critical lines* and *Neumann critical lines* that appear in Theorem 4.6, are contained in $\mathcal{C}_{O_+}$.

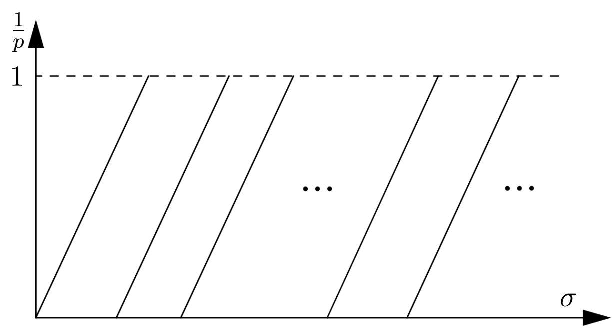

FIGURE 2. An illustration for $\mathcal{C}_{O_+}$.

For interpolations on $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and $\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$, we need to consider three different cases.

Definition 5.6. *Let $\sigma_0, \sigma_1 \geq 0$ and $1 < p_0, p_1 < \infty$, and put*

$$\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

for $0 < \theta < 1$. We need to consider the following three cases:

(O_+1). *$(\sigma_\theta, \frac{1}{p_\theta}) \notin \mathcal{C}_{O_+}$;*

(O_+2). *$(\sigma_0, \frac{1}{p_0})$, $(\sigma_1, \frac{1}{p_1})$ and $(\sigma_\theta, \frac{1}{p_\theta})$ lie on a same critical line in $\mathcal{C}_{O_+}$;*

(O_+3). *otherwise.*

Remark. In the third case, it should be $(\sigma_\theta, \frac{1}{p_\theta}) \in \mathcal{C}_{O_+}$, but $(\sigma_0, \frac{1}{p_0})$ and $(\sigma_1, \frac{1}{p_1})$ could not lie on the same critical line as $(\sigma_\theta, \frac{1}{p_\theta})$, although they may belong to $\mathcal{C}_{O_+}$ separately or simultaneously.

We will show that (**O_+1**) is a safe case of the interpolations, where $\dot{H} = \mathring{H}$ and $\dot{B} = \mathring{B}$ for $(\frac{1}{p_\theta}, \sigma_\theta)$, (**$O_+$2**) is 'locally stable' for $\mathring{H}$ and $\mathring{B}$, while (**O_+3**) is the unstable case where the interpolation spaces of $\mathring{H}$ or $\mathring{B}$ becomes $\dot{H}$ or $\dot{B}$. See Figure 3 for an illustration of the three cases, and Theorem 5.7 for the detailed interpolation theorem.

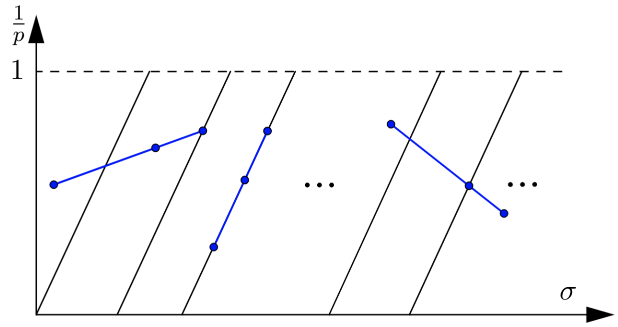

FIGURE 3. An illustration for the three cases of interpolations.

Theorem 5.7. *Let the coefficients $\theta, \sigma_0, \sigma_1, \sigma_\theta, p_0, p_1, p_\theta$ be chosen as in Definition 5.6, and let $q_0, q_1, q_\theta, q \in (1,\infty)$ with $\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$. The interpolation results for $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and $\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$ are given by the following table:*

	(O_+1)	*(O_+2)*	*(O_+3)*
$\big[\mathring{H}^{p_0}_{\sigma_0}(\tilde{K}^d_+), \mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta$	$\mathring{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\mathring{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$
$\big(\mathring{H}^{p_0}_{\sigma}(\tilde{K}^d_+), \mathring{H}^{p_1}_{\sigma}(\tilde{K}^d_+)\big)_{\theta,p_\theta}$	$\mathring{H}^{p_\theta}_{\sigma}(\tilde{K}^d_+) = \dot{H}^{p_\theta}_{\sigma}(\tilde{K}^d_+)$	/	$\dot{H}^{p_\theta}_{\sigma}(\tilde{K}^d_+)$
$\big(\mathring{H}^{p}_{\sigma_0}(\tilde{K}^d_+), \mathring{H}^{p}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q}$	$\mathring{B}^{p,q}_{\sigma_\theta}(\tilde{K}^d_+) = \dot{B}^{p,q}_{\sigma_\theta}(\tilde{K}^d_+)$	/	$\dot{B}^{p,q}_{\sigma_\theta}(\tilde{K}^d_+)$,
$\big[\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+), \mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta$	$\mathring{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\mathring{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$
$\big(\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+), \mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q_\theta}$ *(if $p_\theta = q_\theta$)*	$\mathring{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+) = \dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\mathring{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$	$\dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$
$\big(\mathring{B}^{p,q_0}_{\sigma}(\tilde{K}^d_+), \mathring{B}^{p,q_1}_{\sigma}(\tilde{K}^d_+)\big)_{\theta,q_\theta}$	$\mathring{B}^{p,q_\theta}_{\sigma}(\tilde{K}^d_+) = \dot{B}^{p,q_\theta}_{\sigma}(\tilde{K}^d_+)$	$\mathring{B}^{p,q_\theta}_{\sigma}(\tilde{K}^d_+)$	/

where we have the restriction $\sigma = \sigma_0 = \sigma_1$ for the second equation, and the restriction $p = p_0 = p_1$ for the third equation. Also '/' in the above table means there exists no case or only the trivial case.

Remark. One can apply reiteration theorems of real interpolation to get more interpolation formulas.

Proof. We will use '$\cdots$' to stand for some unimportant information for simplification.

First, for each $x = \pi(\tau_x \dot{w}_x) \in V_0$ with $\alpha_x = (r_{w_x}\mu_{w_x})^{1/2}$ and $\beta_x(p) = \mu_{w_x}^{-1/p}$, by using Proposition A.5 and A.6, we conclude that

$$\overline{l^p_{\alpha_x^\sigma,\beta_x(p)}(\cdots)}^{A^{(1)}_{w_x}} = l^p_{\alpha_x^\sigma,\beta_x(p)}(\cdots), \qquad \overline{l^{p,q}_{\alpha_x^\sigma,\beta_x(p)}(\cdots)}^{A^{(1)}_{w_x}} = l^{p,q}_{\alpha_x^\sigma,\beta_x(p)}(\cdots),$$

if and only if $\sigma \notin \frac{d_S}{p} + \{\frac{2\log\gamma_{i,x}}{\log r_{w_x}\mu_{w_x}}\}_{i\geq 0}$. As a consequence, one can easily see that

$$\mathring{H}^p_\sigma(\tilde{K}^d_+) = \dot{H}^p_\sigma(\tilde{K}^d_+) \quad \text{and} \quad \mathring{B}^{p,q}_\sigma(\tilde{K}^d_+) = \dot{B}^{p,q}_\sigma(\tilde{K}^d_+), \quad \text{if and only if } (\sigma, \frac{1}{p}) \notin \mathcal{C}_{O_+}. \tag{5.6}$$

Now we proceed to prove the interpolation results. For convenience, we number the lines in the critical set $\mathcal{C}_{O_+}$ in increasing order, $\ell_1,\ell_2,\ell_3,\cdots$, from left to right, and call the region

$$\mathcal{B}_0 := \cup_{0\leq t\leq 1}\big((1-t)+t\ell_1\big),\quad \mathcal{B}_i := \cup_{0<t\leq 1}\big((1-t)\ell_i+t\ell_{i+1}\big),\quad i\geq 1$$

the i-th *stable bands*.

Fix $k\in\mathbb{N}$, we will then prove the interpolation formulas with $0\leq\sigma\leq 2k$ for Sobolev spaces and $0<\sigma<2k$ for Besov spaces. We have $\mathring{H}^p_\sigma(\tilde{K}^d_+)=\mathcal{K}^p_{\sigma,k}(\tilde{K}^d_+)\oplus\mathring{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ with

$$\mathring{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)=\sum_{x\in V_0}E_{x,k}\overline{l^p_{\alpha^\sigma_x,\beta_x(p)}(\cdots)}^{A^{(1)}_{w_x}}.$$

Applying Proposition A.5, we see that

$$l^p_{\alpha^\sigma_x,\beta_x(p)}(\cdots,A^{(1)}_{w_x})=\overline{l^p_{\alpha^\sigma_x,\beta_x(p)}(\cdots)}^{A^{(1)}_{w_x}}\oplus\big(\oplus_{i'=0}^i\mathcal{S}^{-\Delta,\varphi}_{\alpha,A^{(1)}_{w_x}}(\bar{U}_{i',x}\otimes\cdots)\big),$$

if $\frac{d_S}{p}+\frac{2\log\gamma_{i,x}}{\log r_{w_x}\mu_{w_x}}<\sigma\leq\frac{d_S}{p}+\frac{2\log\gamma_{i+1,x}}{\log r_{w_x}\mu_{w_x}}$. As a consequence, and for short, we can write

$$H^p_\sigma(\tilde{K}^d_+)=\mathring{H}^p_\sigma(\tilde{K}^d_+)\oplus\big(\oplus_{i'=0}^i X_{i',k}(\sigma,p,p)\big)\text{ for }(\sigma,1/p)\in\mathcal{B}_i\cap\big\{(\sigma,\frac{1}{p}):0\leq\sigma\leq 2k\big\},\quad(5.7)$$

and similarly by applying Proposition A.6 instead,

$$B^{p,q}_\sigma(\tilde{K}^d_+)=\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)\oplus\big(\oplus_{i'=0}^i X_{i',k}(\sigma,p,q)\big)\text{ for }(\sigma,1/p)\in\mathcal{B}_i\cap\big\{(\sigma,\frac{1}{p}):0<\sigma<2k\big\}.\quad(5.8)$$

We consider two possible cases.

Case 1: $(\sigma_0,\frac{1}{p_0})$ and $(\sigma_1,\frac{1}{p_1})$ are in a same stable band. (This includes the ($\boldsymbol{O}_+2$) case).

In fact, for those $(\sigma,\frac{1}{p})$ in a same stable band $\mathcal{B}_i$, the spaces have the same remainder terms on the right hand side of (5.7) and (5.8). This means

$$\big\{H^p_\sigma(\tilde{K}^d_+)\big\}_{(\sigma,\frac{1}{p})\in\mathcal{B}_i}\curvearrowright\big\{\mathring{H}^p_\sigma(\tilde{K}^d_+)\big\}_{(\sigma,\frac{1}{p})\in\mathcal{B}_i},\quad\big\{B^p_\sigma(\tilde{K}^d_+)\big\}_{(\sigma,\frac{1}{p})\in\mathcal{B}_i}\curvearrowright\big\{\mathring{B}^p_\sigma(\tilde{K}^d_+)\big\}_{(\sigma,\frac{1}{p})\in\mathcal{B}_i}.$$

Noticing that we have all the interpolation results for $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ by Corollary 5.5 and Definition 4.1, the interpolation results concerning $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ and $\mathring{B}^{p,q}_\sigma(\tilde{K}^d_+)$ follows.

Case 2: $(\sigma_0,\frac{1}{p_0})$ and $(\sigma_1,\frac{1}{p_1})$ are not in a same stable band.

We assume $(\sigma_0,\frac{1}{p_0})\in\mathcal{B}_{i_0}$ and $(\sigma_1,\frac{1}{p_1})\in\mathcal{B}_{i_1}$ with $i_1>i_0$. The hard situation is when $(\sigma_0,\frac{1}{p_0})$ or $(\sigma_1,\frac{1}{p_1})$ is in $\mathcal{C}_{O_+}$. We will apply the reiteration argument to overcome this difficulty.

Claim 1: If $(\sigma_\theta,\frac{1}{p_\theta})\in\mathcal{B}_i\setminus\mathcal{C}_{O_+}$ for some i, then we have

$$\begin{cases}\big[\mathring{H}^{p_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta\subset\dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^i X_{i',k}(\sigma_\theta,p_\theta,p_\theta),\\ \big[\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta\subset\dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^i X_{i',k}(\sigma_\theta,p_\theta,q_\theta).\end{cases}$$

Proof of Claim 1. The claim is an easy consequence of (5.6), (5.7) and (5.8). In fact, we have

$$\big[\mathring{H}^{p_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta\subset\big[H^{p_0}_{\sigma_0}(\tilde{K}^d_+),H^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta=H^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)=\mathring{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^i X_{i',k}(\sigma,p_\theta,p_\theta),$$

and the identify for Besov spaces follows similarly. □

Claim 2: Assume $(\sigma_\theta,\frac{1}{p_\theta})\in\mathcal{B}_{i_1}\setminus\mathcal{C}_{O_+}$*, then we have* $\big[\mathring{H}^{p_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta=\dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$ *and* $\big[\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta=\dot{B}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)$.

Proof of Claim 2. We choose $\theta'<\theta$ such that $(\sigma_{\theta'},\frac{1}{p_{\theta'}})\in\mathcal{B}_{i_1}$. Then using the reiteration theorem, Corollary 5.5 for $\dot{H}$, Claim 1 and Case 1, we get

$$\begin{aligned}\dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)\subset\big[\mathring{H}^{p_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\theta&\subset\big[\mathring{H}^{p_{\theta'}}_{\sigma_{\theta'}}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^{i_1}X_{i',k}(\sigma,p_{\theta'},p_{\theta'}),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\eta\\&=\big[\mathring{H}^{p_{\theta'}}_{\sigma_{\theta'}}(\tilde{K}^d_+),\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)\big]_\eta=\dot{H}^{p_\theta}_{\sigma_\theta}(\tilde{K}^d_+)\end{aligned}\tag{5.9}$$

with $\eta=\frac{\theta-\theta'}{1-\theta'}$, where in the second inclusion we use the fact that $\oplus_{i'=0}^{i_1}X_{i',k}(\sigma,p_{\theta'},p_{\theta'})\cap\mathring{H}^{p_1}_{\sigma_1}(\tilde{K}^d_+)=\{0\}$. Thus we get the desired identity for Sobolev spaces. The Besov space case follows from a same idea. □

Let's return to the general situation of Case 2. We choose to prove the 5-th interpolation identity for example, while the proofs of the others are essentially the same. We choose θ_0,θ_1 such that

$$0<\theta_0<\theta<\theta_1<1,\quad(\sigma_{\theta_0},\frac{1}{p_{\theta_0}})\notin\mathcal{C}_{O_+}\text{ and }(\sigma_{\theta_1},\frac{1}{p_{\theta_1}})\in\mathcal{B}_{i_1}\setminus\mathcal{C}_{O_+}.$$

Choose $i\geq 0$ such that $(\sigma_{\theta_0},\frac{1}{p_{\theta_0}})\in\mathcal{B}_i$. By Claim 1 and Claim 2, we already have

$$\begin{cases}\big[\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_{\theta_0}\subset\dot{B}^{p_{\theta_0},q_{\theta_0}}_{\sigma_{\theta_0}}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^{i}X_{i',k}(\sigma_{\theta_0},p_{\theta_0},q_{\theta_0}),\\\big[\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big]_{\theta_1}=\dot{B}^{p_{\theta_1},q_{\theta_1}}_{\sigma_{\theta_1}}(\tilde{K}^d_+).\end{cases}$$

As a consequence, with $\eta=(\theta-\theta_0)/(\theta_1-\theta_0)$, we have

$$\begin{aligned}\dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+)&=\big(\dot{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\dot{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q_\theta}\subset\big(\mathring{B}^{p_0,q_0}_{\sigma_0}(\tilde{K}^d_+),\mathring{B}^{p_1,q_1}_{\sigma_1}(\tilde{K}^d_+)\big)_{\theta,q_\theta}\\&\subset\big(\dot{B}^{p_{\theta_0},q_{\theta_0}}_{\sigma_{\theta_0}}(\tilde{K}^d_+)\oplus\oplus_{i'=0}^{i}X_{i',k}(\sigma_{\theta_0},p_{\theta_0},q_{\theta_0}),\dot{B}^{p_{\theta_1},q_{\theta_1}}_{\sigma_{\theta_1}}(\tilde{K}^d_+)\big)_{\eta,q_\theta}\\&=\big(\dot{B}^{p_{\theta_0},q_{\theta_0}}_{\sigma_{\theta_0}}(\tilde{K}^d_+),\dot{B}^{p_{\theta_1},q_{\theta_1}}_{\sigma_{\theta_1}}(\tilde{K}^d_+)\big)_{\eta,q_\theta}=\dot{B}^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde{K}^d_+),\end{aligned}$$

where we use reiteration theorem in the second line, the fact that $\oplus_{i'=0}^{i}X_{i',k}(\sigma_{\theta_0},p_{\theta_0},q_{\theta_0})\cap\dot{B}^{p_{\theta_1},q_{\theta_1}}_{\sigma_{\theta_1}}(\tilde{K}^d_+)=\{0\}$ in the first equality of the third line, and the interpolation result for $\dot{H}$ in the first and last equalities, noticing that $\frac{\eta}{p_{\theta_0}}+\frac{1-\eta}{p_{\theta_1}}=\frac{1}{p_\theta}=\frac{1}{q_\theta}$. Then the desired result follows. □

5.3. Extend to real orders on $\tilde{K}^d_+$.

We will extend the interpolation theorems to real order σ's for function spaces on $\tilde{K}^d_+$. We need to appropriately combine the results in previous two subsections.

The difficulty comes from the fact that most information of tangents of functions at boundary takes part in when we clarify $\mathring{H}^p_\sigma(\tilde{K}^d_+)$ in $H^p_\sigma(\tilde{K}^d_+)$. In $\mathbb{R}^d_+$ case, it can be shown that $H^p_\sigma(\mathbb{R}^d_+)$ is a retract of $H^p_\sigma(\mathbb{R}^d)$ for $-2k\leq\sigma\leq 2k$ with any fixed $k\in\mathbb{N}$(except the critical order cases using a reflection technique). See [38] for a discussion on $\mathbb{R}^d_+$, and a proof in the L^2 setting for spaces on K in the authors' previous work [11].

However, the idea of using retract is not convenient for the general L^p setting on p.c.f. fractals. We will instead to apply the idea of decomposing function spaces again.

Throughout this section, we will use $< f, g >$ to abbreviate the integral $\int_K f \cdot \bar{g} d\mu$, and write $f \perp g$ if $< f, g >= 0$.

1. A decomposition by projection. To deal with negative orders, it is convenient to introduce a suitable decomposition of spaces that is well behaved in dual spaces.

Lemma 5.8. *For $k \in \mathbb{N}$ and $\dot{w} \in \mathcal{P}$, assuming $F_w K \cap V_0 = \{\pi(\dot{w})\}$ without loss of generality, there is a linear map $h \to \hat{h}_w : \mathcal{H}_{k-1} \to C^\infty(K)$ such that $A_w \hat{h}_w = A_w h$, $P_{\hat{\mathcal{H}}_{k-1,w}} \hat{h}_w = \hat{h}_w$ and $\hat{h}_w$ is supported away from $V_0 \setminus \{\pi(\dot{w})\}$, where we denote $\hat{\mathcal{H}}_{k-1,w}$ for the range of this map.*

Proof. To achieve this, we choose a basis $\{h_1, h_2, \cdots, h_m\}$ of $\mathcal{H}_{k-1}$, and denote

$$a_{ij} = \int_{F_w K} h_i \cdot \bar{h}_j d\mu, \quad 1 \le i, j \le m.$$

It is clear that we can find $g_i \in C^\infty(K), 1 \le i \le m$, such that $A_w g_i = A_w h_i$, the support of g_i is a small neighbourhood of $F_w K$, and $< g_i, h_j >_{L^2(K)} = a_{ij}$. In addition, we can assume that

$$< g_i, g_j >= \varepsilon_{ij} + a_{ij}, \quad 1 \le i, j \le m$$

with $\varepsilon = \max_{i,j}\{|\varepsilon_{ij}|\}$ small enough so that we can find $f_i \in C^\infty(K)$ supported in some compact subsets of $K \setminus F_w K$ away from the boundary, satisfying

$$\begin{cases} < f_i, g_j >= 0, & \forall 1 \le i, j \le m, \\ < f_i, f_j >= \delta_{ij}\varepsilon, & \forall 1 \le i, j \le m, \\ < f_i, h_j >= \varepsilon_{ij} + \delta_{ij}\varepsilon, & \forall 1 \le i, j \le m. \end{cases}$$

Set $\hat{h}_{i,w} = g_i + f_i$ for each $1 \le i \le m$, and extend this to be a linear map $\mathcal{H}_{k-1} \to C^\infty(K)$. One can then check that

$$< h_i, \hat{h}_{j,w} >=< \hat{h}_{i,w}, \hat{h}_{j,w} >= a_{ij} + \varepsilon_{ij} + \delta_{ij}\varepsilon,$$

and thus $P_{\hat{\mathcal{H}}_{k-1,w}}(h_i) = \hat{h}_{i,w}$ for any $1 \le i \le m$. The lemma follows immediately. □

Remark. We omit a subscript k for $\hat{h}_w$ since obviously we can make the choice of $\hat{h}_w$ consistent for different k's. Later, we sometimes write $\hat{h_w}$ instead of $\hat{h}_w$.

Definition 5.9. *Let $x = \pi(\tau\dot{w}) \in V_0$ and $k \in \mathbb{N}$.*

(a). Let $S^n_{w,k}, n \ge 0$ be the subspace of $L^2(K)$ spanned by functions $\{\hat{h}_w \circ F_w^{-n} : h \in \mathcal{H}_{k-1}\}$, and $P_{S^n_{k,w}}$ be the orthogonal projection from $L^2(K)$ to $S^n_{w,k}$. Clearly, $P_{S^n_{w,k}}$ extends to be from $L^p(K)$ to $S^n_{w,k}$.

(b). Let $S_{w,k}$ be the subspace of $L^2(K)$ spanned by $\bigcup_{n\ge 0} S^n_{w,k}$, and $P_{S_{w,k}}$ be the orthogonal projection from $L^2(K)$ to $S_{w,k}$.

(c). Let $S^n_{x,k} = S^n_{w,k} \circ F_\tau^{-1}$ and $S_{x,k} = S_{w,k} \circ F_\tau^{-1}$. Denote $P_{S^n_{x,k}}, P_{S_{x,k}}$ the map defined by $P_{S^n_{x,k}} f = (P_{S^n_{w,k}} A_\tau f) \circ F_\tau^{-1}$ respectively.

We have the following proposition.

Proposition 5.10. *Let $x=\pi(\tau\dot{w})\in V_0$, $k\in\mathbb{N}, 0\leq\sigma\leq 2k$ and $1<p<\infty$.*

(a). Let $f\in H^p_\sigma(K)$ and if $A_\tau f\perp S_{x,k}$, we have $Tan^{(x)}_\sigma f=0$.

(b). $P_{S_{x,k}}$ extends to be a bounded map: $L^p(K)\to L^p(K)$.

(c). $P_{S_{x,k}}$ is bounded from $H^p_\sigma(K)$ to $H^p_\sigma(K)$, and $1-\sum_{x\in V_0}P_{S_{x,k}}$ is bounded from $H^p_\sigma(K)$ to $\mathring{H}^p_\sigma(K)$.

Proof. Without loss of generality, we assume $\tau=\emptyset$ and consider $P_{S_{w,k}}$ instead.

(a). Let $h=Tan^{(x)}_\sigma f$ and $f'=f-h$. Then we have

$$\begin{aligned}< (A^n_w h)\breve{}_w\circ F^{-n}_w, f> &=\mu^n_w < (A^n_w h)\breve{}_w, A^n_w f>\\ &=\mu^n_w\big(< (A^n_w h)\breve{}_w, A^n_w h>+< (A^n_w h)\breve{}_w, A^n_w f'>\big)\\ &=\mu^n_w\big(\|(A^n_w h)\breve{}_w\|^2_{L^2(K)}+o(\gamma^n_{j,x})\|(A^n_w h)\breve{}_w\|_{L^2(K)}\big),\end{aligned}$$

where $\gamma_{j,x}=\min\{\gamma_{i,x}:\gamma_{i,x}>(r_w\mu_w)^{\sigma/2}\mu_w^{-1/p}\}$. Thus the left side equals 0 for any $n\geq 0$ only if $Tan^{(x)}_\sigma f=h=0$.

(b). For each $f\in L^p(K)$, we will construct a series $\sum_{n=0}^\infty f_n$ converging in $L^p(K)$, with $f_n\in S^n_{w,k}$ for each $n\geq 0$, so that $P_{S_{w,k}}f=\sum_{n=0}^\infty f_n$.

Let's look at the L^2 setting first. We start with a special situation. Let $f\in L^2(K)$ such that $A^l_w f\in\mathcal{H}_{k-1}$ for some $l\geq 0$. Denote $S^{[0,l]}_{w,k}=\oplus_{n=0}^l S^n_{w,k}$, and write $P_{S^{[0,l]}_{w,k}}f=\sum_{n=0}^l f_n$, with $f_n\in S^n_{w,k}$. Clearly, $g=f-\sum_{n=0}^{l-1}f_n$ is k-multiharmonic in F^l_wK, and so $f_l=(A^l_w g)\breve{}_w\circ F^{-l}_w$ by Lemma 5.8. As a consequence, we have $f-P_{S^{([0,l])}_{k,w}}f=0$ on F^{l+1}_wK, which shows that $f-P_{S^{[0,l]}_{w,k}}f\perp S_{w,k}$. By this observation, we make the following construction.

Step 1: For any $f\in L^2(K)$ such that $A^l_w f\in\mathcal{H}_{k-1}$ for some $l\geq 0$, we can write $P_{S_{w,k}}f=\sum_{n=0}^l f_n$ with $f_n\in S^n_{w,k}$ for $0\leq n\leq l$.

Step 2: For any $f\in L^2(K)$ such that $A^l_w f\in\mathcal{H}_{k-1}$ for some $l\geq 0$, we have by induction

$$\|f_n\|_{L^\infty(K)}\lesssim\Big\|\sum_{m=0}^n f_m\Big\|_{L^\infty(F^n_wK\setminus F^{n+1}_wK)}+\sum_{m=0}^{n-1}\|f_m\|_{L^\infty(K)}\leq C_n\|f\|_{L^2(K)},\ \textit{for } n\geq 0.$$

So we can continuously extend the definition of $f_n, n\geq 0$ to general functions f in $L^2(K)$.

We have the following observations on the sequence $\{f_n\}_{n\geq 0}$.

Observation 1: For any $f\in L^2(K)$ and $n\geq 1$, $A^{n-1}_w f_n=(A^{n-1}_w f)_1$.

Proof of Observation 1. Only need to consider the case that $f\in L^2(K)$ with $A^l_w f\in\mathcal{H}_{k-1}$ for some $l\geq 0$. Let $g=f-\sum_{m=0}^{n-2}f_m$. Then we have

$$A^{n-1}_w P_{S_{w,k}}g=A^{n-1}_w P_{S^{[n-1,\infty)}_{w,k}}g=P_{S_{w,k}}\big(A^{n-1}_w g\big),$$

where $S^{[n-1,\infty)}_{w,k}$ is the space spanned by $\bigcup_{m\geq n-1}S^m_{w,k}$. So we have $A^{n-1}_w f_n=A^{n-1}_w g_n=(A^{n-1}_w g)_1$. On the other hand, we have $(A^{n-1}_w f)_1=(A^{n-1}_w g)_1$ as $A^{n-1}_w(f-g)\in\mathcal{H}_{k-1}$. ☐

Observation 2: There are kernels $\psi_0, \psi_1 \in L^\infty(K\times K)$ such that for any $f \in L^2(K)$, we have

$$\begin{cases} f_0(\xi) = \int_K \psi_0(\xi,\eta) f(\eta) d\mu(\eta), \\ A_w^{n-1} f_n(\xi) = \int_K \psi_1(\xi,\eta)\big(A_w^{n-1} f(\eta)\big) d\mu(\eta), \forall n \geq 1. \end{cases} \tag{5.10}$$

Proof of Observation 2. We choose a basis $\{u_1, u_2, \cdots, u_l\}$ of $S^0_{w,k}$, and let $u'_i = u_i - P_{S^{[1,\infty)}_{w,k}} u_i$. Since u_i is multiharmonic in $F_w K$, we have $u'_i = u_i - P_{S^1_{w,k}} u_i \in L^\infty(K)$.

Now, let $f \in L^2(K)$ and write $P_{S_{w,k}} f = \sum_{n=0}^\infty f_n$ as before. We can see that $< f, u'_i > = < f_0, u'_i >$. If we write $f_0 = \sum_{i=1}^l c_i u_i$, then we have the linear equations

$$< f, u'_j > = \sum_{i=1}^l c_i < u_i, u'_j >, \quad 1 \leq j \leq l.$$

One can see that $\sum_{i=0}^l < c_i u_i, u'_j > = 0, \forall 1 \leq j \leq l$ implies that $\sum_{i=0}^l c_i u_i = 0$, since $< \sum_{i=0}^l c_i u_i, \sum_{i=0}^l c_i u'_i > = 0$ if and only if $\sum_{i=0}^l c_i u_i = 0$. Thus f_0 is completely determined by $< f, u'_j >, 1 \leq j \leq l$, and thus we can write $f_0 = \sum_{i=1}^l c_i u_i$ with $c_i = \sum_{j=1}^l a_{ij} < f, u'_j >$ and coefficients a_{ij} independent of f. We then define $\psi_0(\xi,\eta) = \sum_{i=1}^l \sum_{j=1}^l a_{ij} u_i(\xi) \bar{u}'_j(\eta)$, which clearly satisfies the first formula of (5.10).

Similarly, one can find $\psi_{[0,1]} \in L^\infty(K\times K)$ such that $f_0(\xi) + f_1(\xi) = \int_K \psi_{[0,1]}(\xi,\eta) f(\eta) d\mu(\eta)$ for any $f \in L^2(K)$. It suffices to take $\psi_1 = \psi_{[0,1]} - \psi_0$, then the second formula of (5.10) holds for $n = 1$. For larger n, we apply Observation 1.

As a consequence of Observation 2, one can extend formulas (5.10) to any $f \in L^p(K), 1 < p < \infty$. By similar estimates as in Section 3.2, we can see that for $f \in L^p(K)$, $\sum_{n=0}^\infty f_n$ converges in $L^p(K)$, and additionally $P_{S_{w,k}} f = \sum_{n=0}^\infty f_n$ since the functions that are multiharmonic in some $F^l_w K$, $l \geq 0$ is dense in $L^p(K)$.

(c). This part follows from the following observation.

Observation 3: There is a kernel $\psi \in L^\infty(K \times K)$ such that

$$A_w^{n-1} f_n(\xi) = \int_K \psi(\xi,\eta) \Delta^k \big(A_w^{n-1} f(\eta)\big) d\mu(\eta), \tag{5.11}$$

for any $f \in H^p_{2k}(K)$ and $n \geq 1$.

Proof of Observation 3. We only need to take

$$\psi(\xi,\eta) = (-1)^k \int_{K^k} \psi_1(\xi,\eta_0) G(\eta_0,\eta_1) G(\eta_1,\eta_2) \cdots G(\eta_{k-1},\eta) d\mu(\eta_0) \cdots d\mu(\eta_{k-1}). \qquad \square$$

Following a similar argument as above, we can see that $P_{S_{w,k}}$ is bounded from $H^p_{2k}(K)$ to $H^p_{2k}(K)$. Then by Theorem 4.10 and (a), $1 - \sum_{x \in V_0} P_{x,k}$ is bounded from $H^p_{2k}(K)$ to $\mathring{H}^p_{2k}(K)$. For general $0 \leq \sigma \leq 2k$, the result follows from complex interpolation. $\square$

Proposition 5.10 can be extended to higher dimensional $\tilde{K}^d_+$ case without difficulty by using the same argument in Section 3.2. We omit the details.

In particular, for $x=\pi(\tau\dot{w})\in V_0$ and $f\in L^p(\tilde{K}^d_+)$, we have

$$P^{(1)}_{S_{x,k}}f=\sum_{n=0}^{\infty}f_n\circ(F^{(1)}_w)^{-n}\circ(F^{(1)}_\tau)^{-1},$$

with $f_n\in\hat{\mathcal{H}}_{k-1,w}\big(K,L^p(\tilde{K}^{d\wedge 1}_+)\big):=\hat{\mathcal{H}}_{k-1,w}\otimes L^p(\tilde{K}^{d\wedge 1}_+)$, and we can naturally relate each f_n with an $f'_n\in\mathcal{H}_{k-1,w}\big(K,L^p(\tilde{K}^{d\wedge 1}_+)\big)$ if we assume without loss of generality that the map $\hat{}_w$ is one to one from $\mathcal{H}_{k-1}$ to $\hat{\mathcal{H}}_{k-1,w}$.

Definition 5.11. *For $x=\pi(\tau\dot{w})\in V_0$, $k\in\mathbb{N}$ and $1<p<\infty$, we define $I_{x,k}$ from $P^{(1)}_{S_{x,k}}L^p(\tilde{K}^d_+)$ to $\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+))\big)^{\mathbb{Z}_+}$ by*

$$I_{x,k}P^{(1)}_{S_{x,k}}f=\big\{(A^{(1)}_w)^n\sum_{m=0}^{n}f'_m\circ(F^{(1)}_w)^{-m}\big\}_{n\geq 0}$$

for each element $P^{(1)}_{S_{x,k}}f=\sum_{n=0}^\infty f_n\circ(F^{(1)}_w)^{-n}\circ(F^{(1)}_\tau)^{-1}$ in $P_{S_{x,k}}L^p(\tilde{K}^d_+)$.

Lemma 5.12. *Let $x=\pi(\tau\dot{w})\in V_0$, $k\in\mathbb{N},0\leq\sigma\leq 2k$ and $1<p<\infty$.*
(a). $I_{x,k}$ is isomorphic from $P^{(1)}_{S_{x,k}}H^p_\sigma(\tilde{K}^d_+)$ to $l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+)),\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d\wedge 1}_+)),A^{(1)}_w\big)$.
(b). $I_{x,k}$ is isomorphic from $P^{(1)}_{S_{x,k}}\mathring{H}^p_\sigma(\tilde{K}^d_+)$ to $\overline{l^p_{\alpha^\sigma_x,\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+)),\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d\wedge 1}_+))\big)}^{A^{(1)}_w}$.

Proof. Part (a) follows from a similar argument in Section 3.2, by using (5.10), (5.11), and the fact that $\hat{h}_w\in C^\infty(K)$ with good support of $\hat{h}_w$ and $\Delta^k\hat{h}_w$.

For part (b), one has $Tan^{(x,1)}_\sigma f=Tan^{(x,1)}_\sigma\big(P^{(1)}_{S_{x,k}}f\big)$ by Proposition 5.10 (a) for functions $f\in H^p_\sigma(\tilde{K}^d_+)$. So $Tan^{(x,1)}_\sigma f=0$ if and only if $\|I_{x,k}P^{(1)}_{S_{x,k}}f\|_{L^\infty(K,L^p(\tilde{K}^{d\wedge 1}_+))}=o(\gamma^n_{j,x})$ with $\gamma_{j,x}=\min\{\gamma_{i,x}:\alpha^\sigma_x\beta_x<\gamma_{i,x}\}$. The claim then follows from Theorem 4.10 and Proposition A.5. □

Now we introduce a new decomposition of the space $H^p_\sigma(\tilde{K}^d_+)$ with $-2k\leq\sigma\leq 2k$ for $k\in\mathbb{N}$(extend to negative σ's). In the following, we view each $f\in H^p_\sigma(\tilde{K}^d_+)$ as a functional on $\mathring{H}^{p'}_{2k}(\tilde{K}^d_+)$ with $p'=\frac{p}{p-1}$. Noticing that for $f\in L^p(\tilde{K}^d_+)$ and $\varphi\in\mathring{H}^{p'}_{2k}(\tilde{K}^d_+)$, it holds that

$$<P^{(1)}_{S_{x,k}}f,\varphi>=<f,P^{(1)}_{S_{x,k}}\varphi>,$$

so we can naturally extend the map $P^{(1)}_{S_{x,k}}$ to $H^p_\sigma(\tilde{K}^d_+)$. In addition, since $P^{(1)}_{S_{x,k}}:\mathring{H}^{p'}_\sigma(\tilde{K}^d_+)\to\mathring{H}^{p'}_\sigma(\tilde{K}^d_+)$ for $0\leq\sigma\leq 2k$, we have $P^{(1)}_{S_{x,k}}:H^p_\sigma(\tilde{K}^d_+)\to H^p_\sigma(\tilde{K}^d_+)$ for $-2k\leq\sigma<0$ by dual. The same works for Besov spaces (one may be cautious about the order $\sigma=0$ for Besov spaces).

Definition 5.13. *(a). For $k\in\mathbb{N}$, $-2k<\sigma<2k$ and $1<p,q<\infty$, define $\hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)=\big(1-\sum_{x\in V_0}P^{(1)}_{S_{x,k}}\big)H^p_\sigma(\tilde{K}^d_+)$ and $\hat{\mathcal{K}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)=\big(1-\sum_{x\in V_0}P^{(1)}_{S_{x,k}}\big)B^{p,q}_\sigma(\tilde{K}^d_+)$.*

(b). For $k\in\mathbb{N}$, $-2k<\sigma<2k$ and $1<p,q<\infty$, define $\hat{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)=\big(\sum_{x\in V_0}P^{(1)}_{S_{x,k}}\big)H^p_\sigma(\tilde{K}^d_+)$, $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)=\big(\sum_{x\in V_0}P^{(1)}_{S_{x,k}}\big)B^{p,q}_\sigma(\tilde{K}^d_+)$.

Remark 1. One may worry about the case $\sigma = 0$ for Besov spaces since we did not deal with this previously. Luckily, we have $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ are retracts of $H^p_\sigma(\tilde{K}^d)$ and $B^{p,q}_\sigma(\tilde{K}^d)$, with extension maps being Θ (extension by 0), for $-\frac{d_S}{p'} < \sigma < \frac{d_S}{p}$. So the properties for $\sigma = 0$ naturally follows once we have the properties for $\sigma \in (-\frac{d_S}{p'}, \frac{d_S}{p}) \setminus \{0\}$.

Remark 2. Clearly, all the spaces $\hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$, $\hat{\mathcal{K}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$, $\hat{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ and $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$ are closed spaces of $H^p_\sigma(\tilde{K}^d_+)$ or $B^{p,q}_\sigma(\tilde{K}^d_+)$ since the operator $\sum_{x\in V_0} P^{(1)}_{S_{x,k}}$ is idempotent, which means that $\big(\sum_{x\in V_0} P^{(1)}_{S_{x,k}}\big)^2 = \sum_{x\in V_0} P^{(1)}_{S_{x,k}}$.

2. Interpolation properties of the decomposition.

Lemma 5.14. *Let $k \in \mathbb{N}$, $-2k < \sigma < 2k$ and $1 < p, q < \infty$. In addition, assume $(-\sigma, \frac{1}{p'}) \notin \mathcal{C}_{O_+}$, with $p' = \frac{p}{p-1}$. Then equations (5.1),(5.2) hold if we replace $H^p_\sigma(\tilde{K}^d)$ with the space $\hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$; equations (5.3),(5.4) hold if we replace $B^{p,q}_\sigma(\tilde{K}^d)$ with the space $\hat{\mathcal{K}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$. In addition, $\hat{\mathcal{K}}^{p,q}_{\sigma,k}(\tilde{K}^d_+) = \big(\hat{\mathcal{K}}^p_{-2k,k}(\tilde{K}^d_+), \hat{\mathcal{K}}^p_{2k,k}(\tilde{K}^d_+)\big)_{\theta,q}$ for $0 < \theta = \frac{\sigma+2k}{4k} < 1$.*

Proof. We can see that, for $0 \leq \sigma < 2k$, $\big(1 - \sum_{x\in V_0} P^{(1)}_{S_{x,k}}\big)(f|_{\tilde{K}^d_+}) \in \mathring{H}^{p'}_\sigma(\tilde{K}^d_+)$ for any $f \in H^{p'}_\sigma(\tilde{K}^d)$ by Theorem 3.5 and Proposition 5.10 (c). Thus $\hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$ is a retract of $H^p_\sigma(\tilde{K}^d)$ for any $(\sigma, \frac{1}{p})$ satisfying the assumption. In fact, we can take the extension map $E : \hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+) \to H^p_\sigma(\tilde{K}^d)$ by $Ef =< f, \big(1 - \sum_{x\in V_0} P^{(1)}_{S_{x,k}}\big)(\bullet|_{\tilde{K}^d_+}) >$ for $f \in \hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$, where $\bullet$ represents a function in $H^{p'}_{2k}(\tilde{K}^d)$; and the restriction map $R : H^p_\sigma(\tilde{K}^d) \to \hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$ is defined in the following way: for $f \in H^p_\sigma(\tilde{K}^d)$, we write $g =< f, \Theta\bullet >$ where $\bullet$ represents a function in $\mathring{H}^{p'}_{2k}(\tilde{K}^d_+)$; then noticing that $g \in H^p_\sigma(\tilde{K}^d_+)$, we define $Rf := (1 - \sum_{x\in V_0} P^{(1)}_{S_{x,k}})g$. One can check that R is bounded by Theorem 4.19, and it is direct to see that RE is the identity map on $\hat{\mathcal{K}}^p_{\sigma,k}(\tilde{K}^d_+)$.

The same extension and restriction maps work for Besov spaces, and the result follows. □

It remains to study the interpolation property of $\hat{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ and $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$. This time we will use duality instead of retract. The argument is based on the following discussion.

Lemma 5.15. *Fix $k \in \mathbb{N}$ and $x = \pi(\tau\dot{w}) \in V_0$. Let $f = \sum_{n=0}^\infty f_n \in P_{S_{x,k}} L^p(K)$, $g = \sum_{n=0}^\infty g_n \in P_{S_{x,k}} L^{p'}(K)$ with $1 < p < \infty, p' = \frac{p}{p-1}$, and $f_n, g_n \in S^n_{x,k}$ for $n \geq 0$. In addition, we write $\{h_n\}_{n\geq 0} = I_{x,k} f$, $\{e_n\}_{n\geq 0} = I_{x,k} g$ as defined in Definition 5.11. Then we have*

$$\begin{aligned} < f, g >=& \mu_\tau \sum_{n=0}^\infty \mu_w^n < \hat{h}_{n,w}, \hat{e}_{n,w} >_{K\setminus F_w K} \\ &+ \mu_\tau \sum_{n=0}^\infty \mu_w^{n+1} < A_w h_n - (A_w h_n)\check{}_w, A_w e_n - (A_w e_n)\check{}_w >_{K\setminus F_w K} . \end{aligned}$$

Proof. By a direct computation, we get

$$\begin{aligned}
< f,g > =& \sum_{n=0}^{\infty} < f,g >_{F_\tau F_w^n(K\setminus F_wK)}\\
=&\mu_\tau \sum_{n=0}^{\infty} \mu_w^n < A_w^n A_\tau \sum_{m=0}^{n} f_n, A_w^n A_\tau \sum_{m=0}^{n} g_n >_{K\setminus F_wK}\\
=&\mu_\tau \sum_{n=0}^{\infty} \mu_w^n < A_w h_{n-1} + A_w^n A_\tau f_n, A_w e_{n-1} + A_w^n A_\tau g_n >_{K\setminus F_wK}\\
=&\mu_\tau \sum_{n=0}^{\infty} \mu_w^n < A_w h_{n-1} - (A_w h_{n-1})\hat{}_w + \hat{h}_{n,w}, A_w e_{n-1} - (A_w e_{n-1})\hat{}_w + \hat{e}_{n,w} >_{K\setminus F_wK} .
\end{aligned}$$

Then the desired result follows. □

Lemma 5.15 provides a characterization of $< f,g >$ with a conjugate symmetric form on $\{h_n\},\{e_n\}$. Then comparing Lemma 5.12, we have the following lemma for negative σ's.

Lemma 5.16. *Let $k\in\mathbb{N}$, $-2k<\sigma<0$, $1<p<\infty$, and in addition assume that $(-\sigma,\frac{1}{p'})\notin\mathcal{C}_{O_+}$ with $p'=\frac{p}{p-1}$. Fix $x=\pi(\tau\dot{w})\in V_0$, then we can extend $I_{x,k}$ to be an isomorphism from $P^{(1)}_{S_{x,k}}H^p_\sigma(\tilde{K}^d_+)$ to $l^p_{\alpha_x^\sigma\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+))\big)+l^p_{\beta_x}\big(\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d\wedge 1}_+))\big)$, with $\alpha_x=(r_w\mu_w)^{1/2}$ and $\beta_w=\mu_w^{-1/p}$.*

Proof. For convenience, we focus on the case $d=1$. By using Lemma 5.12 (b) and Lemma 5.15, and using Proposition A.5, we can see that

$$\|f\|_{H^p_\sigma(K)}\asymp\|I_{x,k}f\|_{l^p_{\alpha_x^\sigma\beta_x}(\mathcal{H}_{k-1})},$$

for any $f\in P_{S_{x,k}}L^p(K)$. One can easily prove that $P_{S_{x,k}}L^p(K)$ is dense in $P_{S_{x,k}}H^p_\sigma(K)$, noticing that $L^p(K)$ is dense in $H^p_\sigma(K)$. So the claim follows for $d=1$ case.

For $d>1$ cases, a similar argument will work, applying the fact that $(A\cap B)^*=A^*+B^*$ if $A\cap B$ is dense in both A and B (See Theorem 2.7.1. in book [7]). We omit the details. □

Noticing that for $-2k<\sigma<0$, we have

$$\begin{aligned}
&l^p_{\alpha_x^\sigma\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+))\big)+l^p_{\beta_x}\big(\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d\wedge 1}_+))\big)\\
=&l^p_{\alpha_x^\sigma\beta_x}\big(\mathcal{H}_{k-1}(K,L^p(\tilde{K}^{d\wedge 1}_+)),A^{(1)}_w\big)+l^p_{\beta_x}\big(\mathcal{H}_{k-1}(K,H^p_\sigma(\tilde{K}^{d\wedge 1}_+)),A^{(1)}_w\big).
\end{aligned}$$

Now, using Lemma 5.12 (a), Lemma 5.16, combining with Lemma A.7, we have the following result.

Lemma 5.17. *Let $k\in\mathbb{N}$, $-k<\sigma<k$ and $1<p,q<\infty$, and in addition assume that $(-\sigma,\frac{1}{p'})\notin\mathcal{C}_{O_+}$ with $p'=\frac{p}{p-1}$. There exists a map Λ that is an isomorphism from $\hat{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)\to\hat{\mathcal{T}}^p_{\sigma+k,k}(\tilde{K}^d_+)$. In addition, Λ is also an isomorphism from $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)\to\hat{\mathcal{T}}^{p,q}_{\sigma+k,k}(\tilde{K}^d_+)$ for $-k<\sigma<0$ or $0<\sigma<k$.*

Proof. In fact, we only need to construct the isomorphism Λ that works for $\hat{\mathcal{T}}^p_{\sigma,k}(\tilde{K}^d_+)$ with $\sigma=-k,0,k$, and then use complex and real interpolations. The map can be easily constructed with Lemma A.7. We omit the details. □

As a consequence, we get the interpolation properties of $\hat{\mathcal{T}}^{p}_{\sigma,k}(\tilde{K}^d_+)$ and $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$.

Lemma 5.18. *Let $k\in\mathbb{N}$, $-k<\sigma<k$ and $1<p,q<\infty$. In addition, we assume that $(-\sigma,\frac{1}{p'})\notin\mathcal{C}_{O_+}$ with $p'=\frac{p}{p-1}$. Then equations (5.1),(5.2) hold if we replace $H^p_\sigma(\tilde{K}^d)$ with the space $\hat{\mathcal{T}}^{p}_{\sigma,k}(\tilde{K}^d_+)$; equations (5.3),(5.4) hold if we replace $B^{p,q}_\sigma(\tilde{K}^d)$ with the space $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)$. In addition, $\hat{\mathcal{T}}^{p,q}_{\sigma,k}(\tilde{K}^d_+)=\big(\hat{\mathcal{T}}^{p}_{-k,k}(\tilde{K}^d_+),\hat{\mathcal{T}}^{p}_{k,k}(\tilde{K}^d_+)\big)_{\theta,q}$ for $0<\theta=\frac{\sigma+k}{2k}<1$.*

3. The final interpolation theorem. Now we are ready to present the interpolation theorem for $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ with $\sigma\in\mathbb{R}$. As we have noticed, since for $\sigma<0$, $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ are duals of $\mathring{H}^{p'}_{-\sigma}(\tilde{K}^d_+)$ and $\mathring{B}^{p',q'}_{\sigma}(\tilde{K}^d_+)$, the **critical set** are now reflected, i.e.

$$\mathcal{C}_O=\big\{(\sigma,\frac{1}{p}):(-\sigma,1-\frac{1}{p})\in\mathcal{C}_{O_+}\big\}.$$

Also, we have three cases of interpolations.

Definition 5.19. *Let $\sigma_0,\sigma_1\in\mathbb{R}$ and $1<p_0,p_1<\infty$, and put*

$$\sigma_\theta=(1-\theta)\sigma_0+\theta\sigma_1,\quad \frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$$

for $0<\theta<1$. We need to consider the following three cases:
(O1). *$(\sigma_\theta,\frac{1}{p_\theta})\notin\mathcal{C}_O$;*
(O2). *$(\sigma_0,\frac{1}{p_0})$, $(\sigma_1,\frac{1}{p_1})$ and $(\sigma_\theta,\frac{1}{p_\theta})$ lie on a same critical line in $\mathcal{C}_O$;*
(O3). *otherwise.*

See Figure 4 for an illustration.

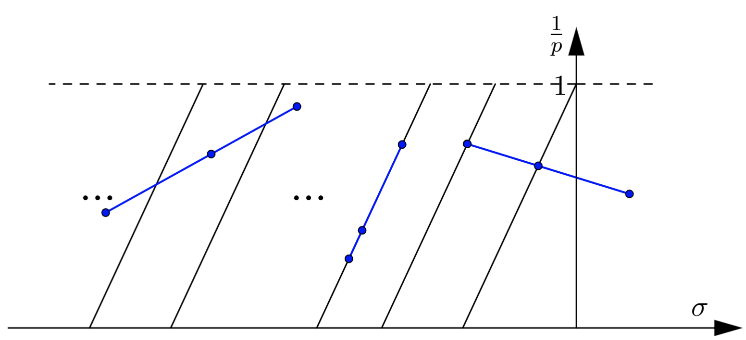

FIGURE 4. An illustration for $\mathcal{C}_O$ and the three cases of interpolations.

Theorem 5.20. *Let the coefficients $\theta,\sigma_0,\sigma_1,\sigma_\theta,p_0,p_1,p_\theta$ be chosen as in Definition 5.19, and let $q_0,q_1,q_\theta,q\in(1,\infty)$ with $\frac{1}{q_\theta}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}$. We also write $q'=\frac{q}{q-1}$, $p'_\theta=\frac{p_\theta}{p_\theta-1}$ and $q'_\theta=\frac{q_\theta}{q_\theta-1}$. The interpolation results for $H^p_\sigma(\tilde{K}^d_+)$ and $B^{p,q}_\sigma(\tilde{K}^d_+)$ are given by the following table:*

	(O1)	***(O2)***	***(O3)***
$\big[H^{p_0}_{\sigma_0}(\tilde K^d_+),H^{p_1}_{\sigma_1}(\tilde K^d_+)\big]_\theta$	$H^{p_\theta}_{\sigma_\theta}(\tilde K^d_+)=\big(\dot H^{p'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$	$H^{p_\theta}_{\sigma_\theta}(\tilde K^d_+)$	$\big(\dot H^{p'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$
$\big(H^{p_0}_{\sigma}(\tilde K^d_+),H^{p_1}_{\sigma}(\tilde K^d_+)\big)_{\theta,p_\theta}$	$H^{p_\theta}_{\sigma}(\tilde K^d_+)=\big(\dot H^{p'_\theta}_{-\sigma}(\tilde K^d_+)\big)^*$	/	$\big(\dot H^{p'_\theta}_{-\sigma}(\tilde K^d_+)\big)^*$
$\big(H^{p}_{\sigma_0}(\tilde K^d_+),H^{p}_{\sigma_1}(\tilde K^d_+)\big)_{\theta,q}$	$B^{p,q}_{\sigma_\theta}(\tilde K^d_+)=\big(\dot B^{p',q'}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$	/	$\big(\dot B^{p',q'}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$,
$\big[B^{p_0,q_0}_{\sigma_0}(\tilde K^d_+),B^{p_1,q_1}_{\sigma_1}(\tilde K^d_+)\big]_\theta$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde K^d_+)=\big(\dot B^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde K^d_+)$	$\big(\dot B^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$
$\big(B^{p_0,q_0}_{\sigma_0}(\tilde K^d_+),B^{p_1,q_1}_{\sigma_1}(\tilde K^d_+)\big)_{\theta,q_\theta}$ *(if $p_\theta=q_\theta$)*	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde K^d_+)=\big(\dot B^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$	$B^{p_\theta,q_\theta}_{\sigma_\theta}(\tilde K^d_+)$	$\big(\dot B^{p'_\theta,q'_\theta}_{-\sigma_\theta}(\tilde K^d_+)\big)^*$
$\big(B^{p,q_0}_{\sigma}(\tilde K^d_+),B^{p,q_1}_{\sigma}(\tilde K^d_+)\big)_{\theta,q_\theta}$	$B^{p,q_\theta}_{\sigma}(\tilde K^d_+)=\big(\dot B^{p',q'_\theta}_{-\sigma}(\tilde K^d_+)\big)^*$	$B^{p,q_\theta}_{\sigma}(\tilde K^d_+)$	/

where we have the restriction $\sigma=\sigma_0=\sigma_1$ for the second equation, and the restriction $p=p_0=p_1$ for the third equation. Also '/' in the above table means there exists no case or only the trivial case.

Proof. First, we assume $(\sigma_0,\frac{1}{p_0}),(\sigma_1,\frac{1}{p_1})$ and $(\sigma_\theta,\frac{1}{p_\theta})$ are not in $\mathcal{C}_O$. Then the interpolation results follow from Lemma 5.14 and Lemma 5.18. We can fill in the cases that involve $\mathcal{C}_O$ by applying Theorem 5.7 and reiteration. □

Appendix A. On sequence spaces

This appendix is a supplement to Section 2. We will take the same setting.

1). Let X be a Banach space, and L be a sectorial operator on X satisfying (**L1**) and (**L2**).

2). Let $\alpha\in(0,1)$, $\beta\in(1,\infty)$, $p\in(1,\infty)$, $\sigma\geq 0$.

In addition, we need more in this appendix.

3). Let $\mathcal{H}$ be a finitely dimensional space over $\mathbb{C}$, and A be a linear operator $\mathcal{H}\to\mathcal{H}$, with its largest eigenvalue (in absolute value) no larger than 1.

We define the tensor product

$$\mathcal{H}\otimes X=\big\{\sum_{i=1}^m h_i\otimes x_i : h_i\in\mathcal{H},x_i\in X,\forall 1\leq i\leq m \text{ and } m\in\mathbb{N}\big\},$$

with the cross project norm

$$\|s\|_{\mathcal{H}\otimes X}=\inf\big\{\sum_{i=1}^m\|h_i\|_{\mathcal{H}}\|x_i\|_X : s=\sum_{i=1}^m h_i\otimes x_i, m\in\mathbb{N}\big\}.$$

Since $\mathcal{H}$ is finitely dimensional, let $\{\phi_i\}_{i=1}^N$ be a basis of $\mathcal{H}$, one can easily see

$$\mathcal{H}\otimes X=\big\{\sum_{i=1}^N\phi_i\otimes x_i : x_i\in X,\forall 1\leq i\leq N\big\},$$

with $\big\|\sum_{i=1}^N\phi_i\otimes x_i\big\|_{\mathcal{H}\otimes X}\asymp\sum_{i=1}^N\|x_i\|_X$ by the open mapping theorem. In this sense, we can view $\mathcal{H}\otimes X$ merely as a N-dimensional vector space over X.

We then extend the operator A to $\mathcal{H}\times X$, by

$$A(s)=\sum_{i=1}^m A(h_i)\otimes x_i$$

for $s=\sum_{i=1}^m h_i\otimes x_i$. It is easy to check that A is well-defined on $\mathcal{H}\otimes X$, i.e. $A(s)$ is independent of the choice of the expansions of s.

In this appendix, we will consider the sequence spaces

$$l^p_\alpha(\mathcal{H}\otimes X,A):=\big\{\boldsymbol{s}=\{s_n\}_{n\geq 0}:\{s_{n+1}-As_n\}_{n\geq 0}\in l^p_\alpha(\mathcal{H}\otimes X)\big\},$$

with norm $\|\boldsymbol{s}\|_{l^p_\alpha(\mathcal{H}\otimes X,A)}=\|s_0\|_{\mathcal{H}\otimes X}+\big\|\{s_{n+1}-As_n\}_{n\geq 0}\big\|_{l^p_\alpha(\mathcal{H}\otimes X)}$, and the spaces

$$l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big):=l^p_{\alpha^\sigma\beta}(\mathcal{H}\otimes X)\cap l^p_\beta\big(\mathcal{H}\otimes\mathcal{D}(L^\sigma)\big),$$
$$l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)},A\big):=l^p_{\alpha^\sigma\beta}(\mathcal{H}\otimes X,A)\cap l^p_\beta\big(\mathcal{H}\otimes\mathcal{D}(L^\sigma),A\big),$$

where we use $\overline{\mathcal{D}(L^\sigma)}$ to denote the couple $\big(X,\mathcal{D}(L^\sigma)\big)$ for short as in Section 2.

In the case that A is **diagonalizable**, each of the above spaces is isomorphic to a direct sum of one of the three kinds of sequence spaces: $l^p_\alpha(X,\gamma)$, $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ and $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$. Thus, all the results in Section 2 apply with no difficulty. What we are of interest in this appendix is the case that A is **not diagonalizable**. Similar as before, we use $\overline{l^p_\alpha(\mathcal{H}\otimes X)}^A$ and $\overline{l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A$ to denote the closures of $l^p_\alpha(\mathcal{H}\otimes X)$ and $l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)$ in $l^p_\alpha\big(\mathcal{H}\otimes X,A\big)$ and $l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)},A\big)$ respectively.

Notations *(a). Let λ be a generalized eigenvalue of A on $\mathcal{H}$, and we write $U_\lambda:=U_{\lambda,A}$ for the generalized eigenspace, i.e. $U_\lambda=U_{\lambda,A}=\bigcup_{m=0}^\infty\ker(A-\lambda)^m$.*

*(b). Let $1\geq\gamma_0>\gamma_1>\cdots>\gamma_l>0$ be the absolute values of **nonzero** eigenvalues of $A:\mathcal{H}\to\mathcal{H}$, which is ordered in decreasing order.*

(c). Write $\bar{U}_i:=\bar{U}_{i,A}=\bigcup_{|\lambda|=\gamma_i}U_{\lambda,A}$ for $0\leq i\leq l$.

(d). Write $\vec{A}(s)=\{s,As,A^2s,\cdots\}$ for $s\in\mathcal{H}\otimes X$.

The following lemma is easy to derive, analogous to Lemma 2.3.

Lemma A.1. *For $1<p<\infty$, we have*

$$l^p_\alpha(\mathcal{H}\otimes X,A)=\begin{cases}\overline{l^p_\alpha(\mathcal{H}\otimes X)}^A, & \text{if } \alpha\geq\gamma_0,\\ \overline{l^p_\alpha(\mathcal{H}\otimes X)}^A\oplus\bigoplus_{i'=0}^i\vec{A}(\bar{U}_i\otimes X), & \text{if } \gamma_{i+1}\leq\alpha<\gamma_i,\\ \overline{l^p_\alpha(\mathcal{H}\otimes X)}^A\oplus\bigoplus_{i=0}^l\vec{A}(\bar{U}_i\otimes X), & \text{if } \alpha<\gamma_l.\end{cases}\tag{A.1}$$

In addition, $\overline{l^p_\alpha(\mathcal{H}\otimes X)}^A=l^p_\alpha(\mathcal{H}\otimes X)$ if and only if $\alpha\notin\{\gamma_0,\gamma_1,\cdots,\gamma_l\}$.

Proof. Since $\mathcal{H}$ can be decomposed into Jordan blocks of A, it suffices to consider a Jordan block only. Without loss of generality, we assume $\sigma_A=\{\lambda\}$ with $\gamma=|\lambda|$. Thus, we need to consider three cases $\alpha<\gamma$, $\alpha=\gamma$ and $\alpha>\gamma$ separately. The cases $\alpha<\gamma$ and $\alpha>\gamma$ are very similar to those of Lemma 2.3, but we need a little more effort for the case $\alpha=\gamma$.

Let $\boldsymbol{s}\in l^p_\alpha(\mathcal{H}\otimes X,A)$, and write $\boldsymbol{t}\in l^p_\alpha(\mathcal{H}\otimes X)$ with $t_0=s_0$ and $t_n=s_n-As_{n-1}$.

Case 1: $\alpha<\gamma$. Clearly $s_\infty:=\lim\limits_{n\to\infty}A^{-n}s_n=\sum_{m=0}^\infty A^{-m}t_m$ is well defined. In addition,

$$\big\|\boldsymbol{s}-\vec{A}(s_\infty)\big\|_{l^p_\alpha(\mathcal{H}\otimes X)}=\big\|\alpha^{-n}\|s_n-A^ns_\infty\|_{\mathcal{H}\otimes X}\big\|_{l^p}=\big\|\alpha^{-n}\|\sum_{m=n+1}^\infty A^{n-m}t_m\|_{\mathcal{H}\otimes X}\big\|_{l^p}$$
$$\leq\big\|\sum_{m=1}^\infty(\alpha^m\|A^{-m}\|)\alpha^{-n-m}\|t_{m+n}\|_{\mathcal{H}\otimes X}\big\|_{l^p}\leq\big(\sum_{m=1}^\infty\alpha^m\|A^{-m}\|\big)\|\boldsymbol{t}\|_{l^p_\alpha(\mathcal{H}\otimes X)}\lesssim\|\boldsymbol{s}\|_{l^p_\alpha(\mathcal{H}\otimes X,A)}.$$

Thus, $l^p_\alpha(\mathcal{H}\otimes X,A)\subset l^p_\alpha(\mathcal{H}\otimes X)\oplus\vec{A}(\mathcal{H}\otimes X)$. The other direction estimate is obvious.

Case 2: $\alpha=\gamma$. Comparing with Case 2 in the proof of Lemma 2.3, it suffices to show that $\vec{A}(s)\in\overline{l^p_\alpha(\mathcal{H}\otimes X)}^A$ for any $s=h\otimes x$. We assume that $(A-\lambda)^m h=0$ and $(A-\lambda)^{m-1}h\neq 0$, and let $r_1=s$, $r_2=(A-\lambda)r_1,\cdots,r_m=(A-\lambda)^{m-1}r_1$. For any $c_1,c_2,\cdots,c_m>0$, we define a sequence $\boldsymbol{s}=\{s_n\}_{n\geq 0}\in l^p_\alpha(\mathcal{H}\otimes X)$ according to the following rule:

1). let $s_0=r_1=s$;

2). if $As_n=\sum_{i=m'}^m d_ir_i$ with $d_{m'}\neq 0$, we define $s_{n+1}=\max\{0,1-\frac{c_{m'}}{|d_{m'}|}\}d_{m'}r_{m'}+\sum_{i=m'+1}^m d_ir_i$;

3). take $s_{n+1}=0$ if $s_n=0$.

It is easy to see that $\lim_{c_1\to 0}\lim_{c_2\to 0}\cdots\lim_{c_m\to 0}\|\boldsymbol{s}-\vec{A}(s)\|_{l^p_\alpha(\mathcal{H}\otimes X,A)}=0$, and thus the desired result holds.

Case 3: $\alpha>\gamma$ (including the case $\gamma=0$). In this case, we have the estimate that

$$\begin{aligned}\|\boldsymbol{s}\|_{l^p_\alpha(\mathcal{H}\otimes X)}&=\big\|\alpha^{-n}\|\sum_{m=0}^n A^{n-m}t_m\|_{\mathcal{H}\otimes X}\big\|_{l^p}\leq\big\|\alpha^{-n}\sum_{m=0}^n\|A^m\|\cdot\|t_{n-m}\|_{\mathcal{H}\otimes X}\big\|_{l^p}\\ &=\|\sum_{m=0}^n\alpha^{-m}\|A^m\|\alpha^{m-n}\|t_{n-m}\|_{\mathcal{H}\otimes X}\big\|_{l^p}\leq\big(\sum_{m=0}^\infty\alpha^{-m}\|A^m\|\big)\|\boldsymbol{t}\|_{l^p_\alpha(\mathcal{H}\otimes X)}\lesssim\|\boldsymbol{s}\|_{l^p_\alpha(\mathcal{H}\otimes X,A)}.\end{aligned}$$

As a consequence, we have $l^p_\alpha(\mathcal{H}\otimes X,A)=l^p_\alpha(\mathcal{H}\otimes X)$. □

Next, we aim to establish decompositions analogous to Proposition 2.9 and 2.13. The key relies on the following map $S^{L,\varphi}_{\alpha,A}$, based on $S^{L,\varphi}_\alpha$ defined in Lemma 2.8.

Definition A.2. *Let $h\in U_\lambda$ for some nonzero eigenvalue λ and $s=h\otimes x\in U_\lambda\otimes X$. Define $S^{L,\varphi}_{\alpha,A}(s)=\{S^{L,\varphi}_{\alpha,A}(s)\}_{n\geq 0}$ as the unique sequence such that $S^{L,\varphi}_{\alpha,A}(s)_0=h\otimes S^{L,\varphi}_\alpha(x)_0$ and*

$$S^{L,\varphi}_{\alpha,A}(s)_{n+1}-AS^{L,\varphi}_{\alpha,A}(s)_n=\lambda^{n+1}h\otimes\big(S^{L,\varphi}_\alpha(x)_{n+1}-S^{L,\varphi}_\alpha(x)_n\big).$$

Obviously, $S^{L,\varphi}_{\alpha,A}$ extends to be a unique map on $\mathcal{R}(A)\otimes X$.

The following lemma is an immediate consequence of lemma 2.8.

Lemma A.3. *Let $1<p<\infty$, k,φ be chosen as in Lemma 2.8, and let $0<\sigma<k$.*

(a). We have $S^{L,\varphi}_{\alpha,A}:\bar{U}_i\otimes X_{\sigma,p}\to l^p_{\alpha^{\sigma+\theta},\alpha^{-\theta}\gamma_i^{-1}}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^{\sigma+\theta})},A\big)$ for $0\leq i\leq l$ and $\theta>0$.

(b). Let $h\in U_\lambda$ with $\lambda\neq 0$ and $s=h\otimes x$. Write $h_i=(A-\lambda)^ih$. By expanding each term $S^{L,\varphi}_{\alpha,A}(s)_n=\sum_{i=0}^\infty h_i\otimes x_{i,n}$, we have

$$\lim_{n\to\infty}\lambda^{-n}x_{0,n}=x.$$

The following lemma is crucial.

Lemma A.4. *Let $\alpha^\sigma\beta<\gamma_i$, then $l^p_{\alpha^\sigma,\beta}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)},A\big)=l^p_{\alpha^\sigma,\beta}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)}\big)\oplus S^{L,\varphi}_{\alpha,A}(\bar{U}_i\otimes X_{\sigma+\frac{\log\beta\gamma_i^{-1}}{\log\alpha},p})$.*

Proof. Without loss of generality, we assume $\bar{U}_i$ is a Jordan block of A with eigenvalue λ. Thus, we can choose a basis $\{h_1,h_2,\cdots,h_m\}$ of $\bar{U}_i$ such that

$$h_m=(A-\lambda)^{m-1}h_1.$$

Of cause, $(A-\lambda)h_m=0$.

Let $\boldsymbol{s}=\{s_n\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)},A\big)$, then we can write

$$s_n=\sum_{i=1}^m h_i\otimes x_{i,n}\text{ with }x_{i,n}\in\overline{\mathcal{D}(L^\sigma)}.$$

Clearly, we have

$$\begin{aligned}s_{n+1}-As_n&=\sum_{i=1}^m h_i\otimes x_{i,n+1}-\lambda\sum_{i=1}^m h_i\otimes x_{i,n}-\sum_{i=2}^m h_i\otimes x_{i-1,n}\\&=h_1\otimes(x_{1,n+1}-\lambda x_{1,n})+\sum_{i=2}^m h_i\otimes(x_{i,n+1}-\lambda x_{i,n}-x_{i-1,n}).\end{aligned}$$

From this identity, we see that $\{x_{1,n}\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\lambda\big)$, and for $i\geq 2$, $\{x_{i,n}\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\lambda\big)$ if $\{x_{i-1,n}\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$.

Now, according to Proposition 2.9, we have $\lim_{n\to\infty}\lambda^{-n}x_{1,n}:=x_{1,\infty}$ exists. We define $\boldsymbol{s}^{(1)}=\boldsymbol{s}-S^{L,\varphi}_{\alpha,A}(h_1\otimes x_{1,\infty})$ and write $s^{(1)}_n=\sum_{i=1}^m h_i\otimes x^{(1)}_{i,n}$ for each $s^{(1)}_n$ in $\boldsymbol{s}^{(1)}$. Then, we have $\{x^{(1)}_{1,n}\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ by applying Lemma A.3 and Proposition 2.9. Thus $\{x^{(1)}_{2,n}\}_{n\geq 0}\in l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)},\gamma\big)$. Define $x^{(1)}_{2,\infty}=\lim_{n\to\infty}\lambda^{-n}x^{(1)}_{2,n}$, write $\boldsymbol{s}^{(2)}=\boldsymbol{s}^{(1)}-S^{L,\varphi}_{\alpha,A}(h_2\otimes x^{(1)}_{2,\infty})$, and repeat this to define $\boldsymbol{s}^{(3)}\cdots$. The procedure stops when we get $\boldsymbol{s}^{(m)}$. Until now, we get

$$\boldsymbol{s}=\sum_{i=1}^m S^{L,\varphi}_{\alpha,A}(h_i\otimes x^{(i-1)}_{i,\infty})+\boldsymbol{s}^{(m)},$$

if we set $x^{(0)}_{1,\infty}=x_{1,\infty}$ for consistency. Clearly, for each $1\leq i\leq m$, we have $x^{(i-1)}_{i,\infty}\in X_{\sigma+\frac{\log\beta\gamma_i^{-1}}{\log\alpha},p}$, and $\boldsymbol{s}^{(m)}\subset l^p_{\alpha^\sigma,\beta}(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)})$. Thus we have proved that $l^p_{\alpha^\sigma,\beta}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)},A\big)=l^p_{\alpha^\sigma,\beta}\big(\bar{U}_i\otimes\overline{\mathcal{D}(L^\sigma)}\big)\oplus S^{L,\varphi}_{\alpha,A}(\bar{U}_i\otimes X_{\sigma+\frac{\log\beta\gamma_i^{-1}}{\log\alpha},p})$.

The other direction is immediate. □

Now, we have all the important ingredients for the proof of the following propositions. We omit the detailed proof.

Proposition A.5. *Let $1<p<\infty$, $k\geq 1$, and define φ and $S^{L,\varphi}_{\alpha,A}$ as in Definition A.2. Then for $0\leq\sigma<k-\frac{\log\beta}{\log\alpha}$ we have,*

$$l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)},A\big)=\begin{cases}\overline{l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A, & \text{if }\alpha^\sigma\beta\geq\gamma_0,\\ \overline{l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A\oplus\big(\bigoplus_{i'=0}^i S^{L,\varphi}_{\alpha,A}(\bar{U}_{i'}\otimes X_{\sigma+\frac{\log\beta\gamma_{i'}^{-1}}{\log\alpha},p})\big), & \text{if }\gamma_{i+1}\leq\alpha^\sigma\beta<\gamma_i,\\ \overline{l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A\oplus\big(\bigoplus_{i=0}^l S^{L,\varphi}_{\alpha,A}(\bar{U}_i\otimes X_{\sigma+\frac{\log\beta\gamma_i^{-1}}{\log\alpha},p})\big), & \text{if }\alpha^\sigma\beta<\gamma_l,\end{cases}$$

In particular, we have

$$\overline{l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A=l^p_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)\text{ if and only if }\alpha^\sigma\beta\notin\{\gamma_0,\gamma_1,\cdots,\gamma_l\}.$$

Proposition A.6. *Let $1 < p, q < \infty$, $k \geq 1$, and define φ and $S^{L,\varphi}_{\alpha,A}$ as in Definition A.2. Then for $0 < \sigma < k - \frac{\log\beta}{\log\alpha}$ we have,*

$$l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}, A\big) = \begin{cases} \overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A, & \text{if } \alpha^\sigma\beta \geq \gamma_0, \\ \overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A \oplus \big(\bigoplus_{i'=0}^{i} S^{L,\varphi}_{\alpha,A}(\bar{U}_{i'}\otimes X_{\sigma+\frac{\log\beta\gamma_{i'}^{-1}}{\log\alpha},q})\big), & \text{if } \gamma_{i+1} \leq \alpha^\sigma\beta < \gamma_i, \\ \overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A \oplus \big(\bigoplus_{i=0}^{l} S^{L,\varphi}_{\alpha,A}(\bar{U}_{i}\otimes X_{\sigma+\frac{\log\beta\gamma_{i}^{-1}}{\log\alpha},q})\big), & \text{if } \alpha^\sigma\beta < \gamma_l, \end{cases}$$

In particular, we have

$$\overline{l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big)}^A = l^{p,q}_{\alpha^\sigma,\beta}\big(\mathcal{H}\otimes\overline{\mathcal{D}(L^\sigma)}\big) \text{ if and only if } \alpha^\sigma\beta \notin \{\gamma_0,\gamma_1,\cdots,\gamma_l\}.$$

Before ending this appendix, we present another result that will be useful in Section 5. Let's recall the map $\Lambda(\alpha)L : X^{\mathbb{Z}_+} \to X^{\mathbb{Z}_+}$, with $\alpha > 0$, defined in the proof of Lemma 2.11,

$$\Lambda(\alpha)L\big(\{s_n\}_{n\geq 0}\big) = \{\alpha^n L(s_n)\}_{n\geq 0},$$

for $\{s_n\}_{n\geq 0} \in X^{\mathbb{Z}_+}$.

Lemma A.7. *Let $0 < \alpha < 1, \beta > 1$, $1 < p < \infty$ and $\sigma \geq 0$. Then*

(a). $\Lambda(\alpha)\big(1+\Lambda(\alpha)L\big)^{-1}$ is an isomorphism from $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ to $l^p_{\alpha^{\sigma+1},\beta}\big(\overline{\mathcal{D}(L^{\sigma+1})}\big)$.

(b). $\Lambda(\alpha)\big(1+\Lambda(\alpha)L\big)^{-1}$ is an isomorphism from $l^p_{\alpha^{-1}\beta}\big(\mathcal{D}(L)\big) + l^p_\beta(X)$ to $l^p_\beta\big(\mathcal{D}(L)\big)$.

Proof. (a). As in the proof of Lemma 2.11, we view $\Lambda(\alpha)L$ as a sectorial operator on $l^p_{\alpha^\sigma\beta}(X)$, then $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big) = \mathcal{D}\big(\Lambda(\alpha)L)^\sigma\big)$. So $\big(1+\Lambda(\alpha)L\big)^{-1}$ is an isomorphism from $l^p_{\alpha^\sigma,\beta}\big(\overline{\mathcal{D}(L^\sigma)}\big)$ to $l^p_{\alpha^{\sigma+1},\alpha^{-1}\beta}\big(\overline{\mathcal{D}(L^{\sigma+1})}\big)$. The claim follows immediately.

(b). First, we can see that $\Lambda(\alpha)\big(1+\Lambda(\alpha)L\big)^{-1}$ maps from $l^p_{\alpha^{-1}\beta}\big(\mathcal{D}(L)\big)$ to $l^p_\beta\big(\mathcal{D}(L)\big)$, and maps from $l^p_\beta(X)$ to $l^p_\beta\big(\mathcal{D}(L)\big)$. On the other hand, we can see that $\Lambda(\alpha^{-1})\big(1+\Lambda(\alpha)L\big)$ maps from $l^p_\beta\big(\mathcal{D}(L)\big)$ to $l^p_{\alpha^{-1}\beta}\big(\mathcal{D}(L)\big) + l^p_\beta(X)$. □

Appendix B. Distributions and harmonic functions on fractals

Definition B.1. *(a). Let $\Omega = K^l \times \tilde{K}^{d-l}$, viewed as a subspace of $\tilde{K}^d$ with the natural boundary. For each compact set E contained in the interior of Ω, we denote*

$$\mathfrak{D}_E(\Omega) = \{f \in C^\infty(\Omega) : \text{ the support of } f \text{ is contained in } E\},$$

with seminorms $\sup_{|\boldsymbol{i}|\leq l} \|\Delta^{(\boldsymbol{i})} f\|_{C(\Omega)}$, $l \in \mathbb{Z}_+$.

Let $\{E_n\}_{n\geq 0}$ be an increasing sequence of compact sets contained in Ω whose union is the interior of Ω. Define $\mathfrak{D}(\Omega) = \bigcup_{n=0}^\infty \mathfrak{D}_{E_n}(\Omega)$, with the corresponding inductive limit topology.

(b). The dual space of $\mathfrak{D}(\Omega)$, denoted by $\mathfrak{D}'(\Omega)$, is called the distribution space on Ω.

(c). Define the Laplacians in the sense of distributions, i.e. for $f \in \mathfrak{D}'(\Omega)$, define $\Delta^{(i)} f$ and Δf in $\mathfrak{D}'(\Omega)$ such that

$$< \Delta^{(i)} f, \varphi > = < f, \Delta^{(i)}\varphi >, \quad < \Delta f, \varphi > = < f, \Delta\varphi >$$

holds for any $\varphi \in \mathfrak{D}(\Omega)$. In addition, $\Delta = \sum_{i=1}^d \Delta^{(i)}$.

It was shown in [32], that $\mathfrak{D}(\Omega)$ is dense in $C_0(\Omega)$ with the construction of smooth bump functions [31], which we have used several times in this paper. A related useful result is provided below, which is an important tool in Section 4.3.

Proposition B.2 (L. Rogers, R.S. Strichartz and A. Teplyaev[31]). *Let $x = \pi(\tau\dot{w}) \in V_0$. There exists $h_{j,1}, h_{j,2} \in C^\infty(K)$ for $j \geq 0$, supported in $F_\tau K$, such that*

$$\begin{cases} \Delta^i h_{j,1}(x) = \delta_{i,j}, \\ \partial_n \Delta^i h_{j,1}(x) = 0, \end{cases} \qquad \begin{cases} \Delta^i h_{j,2}(x) = 0, \\ \partial_n \Delta^i h_{j,2}(x) = \delta_{i,j}, \end{cases} \quad \forall i \geq 0.$$

Let's return to Sobolev spaces on Ω. On $\Omega = \tilde{K}^d$ and for $1 < p < \infty$, it is not hard to see that $\mathcal{D}(\Delta^{(i)}) = \big\{f \in L^p(\tilde{K}^d) : \Delta^{(i)} f \in L^p(\tilde{K}^d)\big\}, \forall 1 \leq i \leq d$. So in Proposition 3.3, we can simply say

$$H^p_{2k}(\tilde{K}^d) = \big\{f \in L^p(\tilde{K}^d) : \Delta^{(\boldsymbol{i})} f \in L^p(\tilde{K}^d), \forall \boldsymbol{i} \text{ with } |\boldsymbol{i}| \leq k\big\}.$$

Also, recall the definition of $H^p_{2k}(\Omega)$ on $\Omega = K^l \times \tilde{K}^{d-l}$ with $1 \leq l \leq d$ in Definition 3.4. Below, we provide a necessary and sufficient condition such that the same property holds.

Recall the definition of $\gamma_{i,x}$ for $x = \pi(\tau\dot{w}) \in V_0$ and $i \geq 0$, provided below Definition 4.7. It is well-known that $\gamma_{0,x} = 1$ and $\gamma_{1,x} = r_w$, with $\bar{U}_{0,x}$ and $\bar{U}_{1,x}$ being 1-dimensional spaces in $\mathcal{H}_\#$ if we assume (**C1**).

Proposition B.3. *Let $1 < p < \infty$, $k \in \mathbb{N}$ and $\Omega = K^l \times \tilde{K}^{d-l}$ with $1 \leq l \leq d$. We have*

$$H^p_{2k}(\Omega) = \big\{f \in L^p(\Omega) : \Delta^{(\boldsymbol{i})} f \in L^p(\Omega), \forall \boldsymbol{i} \textit{ with } |\boldsymbol{i}| \leq k\big\},$$

*where $\Delta^{(i)}$ is defined in distribution sense, if and only if $\gamma_{2,x} \leq r_{w_x}\mu_{w_x}^{1/p}, \forall x = \pi(\tau_x\dot{w}_x) \in V_0$ and (**C1**) holds.*

Proof. We consider the $\Omega = K$ case. It suffices to prove that $\mathcal{D}(\Delta) = \big\{f \in L^p(K) : \Delta f \in L^p(K)\big\}$ if and only if $\gamma_{2,x} \leq r_{w_x}\mu_{w_x}^{1/p}, \forall x = \pi(\tau_x\dot{w}_x) \in V_0$ and (**C1**) holds.

On the one hand, let's assume $\gamma_{2,x} \leq r_{w_x}\mu_{w_x}^{1/p}, \forall x = \pi(\tau_x\dot{w}_x) \in V_0$ and (**C1**). We can show, by using Theorem 4.10, that $\mathring{H}^{p'}_2(K)$ has codimension $\#\mathcal{P}$ in $GL^{p'}(K)$, where G is the Green's operator and $p' = \frac{p}{p-1}$. As a consequence, we have $\Delta\mathring{H}^{p'}_2(K)$ has codimension $\#\mathcal{P}$ in $L^{p'}(K)$. In addition, for $f \in L^p(K)$, we have $\Delta f = 0$ if and only if $< f, \Delta\varphi >= 0, \forall \varphi \in \mathring{H}^{p'}_2(K)$, since $\Delta\mathring{H}^{p'}_2(K)$ is the closure of $\Delta\mathfrak{D}(K)$ in $L^{p'}(K)$. This shows that $\{f \in L^p(K) : \Delta f = 0\}$ is an $\#V_0 = \#\mathcal{P}$ dimensional space, thus we have $\mathcal{H}_0 = \{f \in L^p(K) : \Delta f = 0\}$. The claim follows from the equality

$$\big\{f \in L^p(K) : \Delta f \in L^p(K)\big\} = GL^p(K) \oplus \big\{f \in L^p(K) : \Delta f = 0\big\}.$$

On the other hand, if $\gamma_{2,x} > r_{w_x}\mu_{w_x}^{1/p}$ for some $x = \pi(\tau_x\dot{w}_x) \in V_0$ or (**C1**) does not hold, following a similar argument, one can see that $\mathcal{H}_0 \subsetneq \{f \in L^p(K) : \Delta f = 0\}$. □

Appendix C. Useful Facts

We collect some useful facts from the books [7, 13, 19] for convenience of readers.

1. Sectorial operators and semigroups. Let's first briefly introduce the definition of sectorial operators, which can be found in [19]. In the following, X always denotes a (non-trivial) Banach space and L a single-valued operator on X. For $0\leq\theta<\pi$, let

$$S_\theta := \begin{cases} \{z\in\mathbb{C}: z\neq 0 \text{ and } |\arg z|<\theta\} & \text{if } \theta\in(0,\pi),\\ (0,\infty) & \text{if } \theta=0.\end{cases}$$

Definition C.1. *An operator L on X is called sectorial of angle θ if*
1) $\sigma(L)\subset\overline{S_\theta}$;
2) $\sup_{\lambda\in\mathbb{C}\setminus\overline{S}_{\theta'}}\|\lambda(\lambda+L)^{-1}\|<\infty$ for all $\theta'\in(\theta,\pi)$.

A useful result concerning powers of sectorial operators is given as follows (Proposition 3.1.2 in [19]).

Lemma C.2. *Let L be a sectorial operator of angle θ, then L^σ is a sectorial operator of angle $\sigma\theta$ for $0<\sigma<\pi/\theta$.*

A wide class of sectorial operators come from semigroups. In particular, let $-L$ be the generator of a bounded single-valued semigroup $\{T(t)\}_{t\geq 0}$, then L is a sectorial operator of angle $\frac{\pi}{2}$, due to the identity

$$(\lambda+L)^{-1}=\int_0^\infty e^{-\lambda t}T(t)dt,\quad \forall\lambda\in\mathbb{C} \text{ with } \mathrm{Re}\lambda>0.$$

Definition C.3. *For $\theta\in(0,\frac{\pi}{2}]$, a map $T:S_\theta\to\mathcal{L}(X)$ is called a bounded holomorphic (degenerate) semigroup of angle θ if it has the following properties:*
1) The semigroup law $T(\lambda)T(\mu)=T(\lambda+\mu)$ holds for all $\lambda,\mu\in S_\theta$.
2) The map $T:S_\theta\to\mathcal{L}(X)$ is holomorphic.
3) The map T satisfies $\sup_{\lambda\in S_{\theta'}}\|T(\lambda)\|<\infty$ for any $0<\theta'<\theta$.

The following well-known result shows the relationship between bounded holomorphic semigroups and sectorial operators.

Proposition C.4. *There is a one to one correspondence between (single-valued) sectorial operators L of angle $\theta\in[0,\pi/2)$ and bounded (single-valued) holomorphic semigroups T on $S_{\pi/2-\theta}$, given by the relations*

$$T(\lambda)=e^{-\lambda L},\quad (\lambda+L)^{-1}=\int_0^\infty e^{-\lambda t}T(t)dt.$$

Readers can read Proposition 3.4.4 in book [19] for details, where the original proposition deals with mutli-valued operators as well.

A wide class of examples of holomorphic semigroups are given by symmetric Markov semigroups $\{P_t\}_{t\geq 0}$ on $L^2(\Omega)$. In particular, Theorem 1.4.2 in [13] shows that any symmetric Markov semigroup $\{P_t\}_{t\geq 0}$ extends to a bounded holomorphic semigroup $\{P_\lambda\}_{\lambda\in S_{\theta_p}}$ on $L^p(\Omega)$ for $1<p<\infty$, with $\theta_p=\frac{\pi}{2}(1-|\frac{2}{p}-1|)$. For stronger results, we will need better estimates of the heat kernel.

2. Heat kernel estimate. Let's now return to the specific setting of p.c.f. self-similar sets. The self-similar Dirichlet form $(\mathcal{E},dom\mathcal{E})$ was constructed by both a probabilistic approach [3, 4, 5, 15, 29, 28] and an analytic approach [24, 25]. The sub-Gaussian heat kernel estimates of the associated Markov semigroup $\{P_t\}_{t\geq 0}$ are due to Hambly and Kumagai in [20], Kumagai and Sturm in [27]. In particular, on the double cover $\tilde{K}$ or K with Neumann boundary condition, we have the upper bound

$$p_t(x,y) \le \frac{c}{t^{\alpha/\beta}} \exp\big(-C(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big), \text{ for } 0<t<1.$$

In the above formula, α is the Hausdorff dimension and β is known as the walk-dimension of K. In addition, it has been shown that $\beta \ge 2$ for more general settings. See [17].

The sub-Gaussian upper bound can be generalized to long time estimate if we subtracting its projection onto constant functions, i.e.

$$\big|p_t(x,y) - \mu^{-1/2}\big| \le \frac{c}{t^{\alpha/\beta}} \exp\big(-C(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big), \text{ for } 0<t<\infty,$$

where μ is the total measure. In fact, for t large, $p_t(x,y) - \mu^{-1/2}$ has exponential decay over t controlled by the first non-zero eigenvalue of the Laplacian.

Following a similar proof of Lemma 3.4.6 and Theorem 3.4.8 in [13], one can extend the estimate to a half space of the complex plane,

$$\big|p_\lambda(x,y) - \mu^{-1/2}\big| \le \frac{c'}{(r\cos\theta)^{\alpha/\beta}} \exp\big(-C'(\frac{d(x,y)^\beta}{r})^{\frac{1}{\beta-1}}\cos\theta\big), \tag{C.1}$$

for any $\lambda = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This in particular, by applying Proposition C.4, implies the following result.

Lemma C.5. *The heat semigroup* $\{P_\lambda\}_{\lambda\in S_{\pi/2}}$ *is a holomorphic semigroup, so* $L = -\Delta$ *is sectorial of angle* 0.

Stronger results are established in [], where the Calderon-Zygmund operators are studied, using (C.1). We will not state the general results here, but instead an important consequence.

Lemma C.6. $\{(1-\Delta)^{it}\}_{t\in\mathbb{R}}$ *is a* C_0*-group of bounded operators from* $L^p(\tilde{K}) \to L^p(\tilde{K})$ *for* $1<p<\infty$.

The lemmas enables us to apply complex interpolation. Proposition 3.1 at the beginning of Section 3 is another useful consequence of [23].

3. Retract. Lastly, we would like to mention the concept of retracts, see Section 6.4 in [7] for more details. Let's write $\bar{X} = (X_1, X_2)$ and $\bar{Z} = (Z_1, Z_2)$ for interpolation couples of Banach spaces, and write $\bar{X}_\theta$ and $\bar{Z}_\theta$ for the corresponding interpolation spaces respectively, given by the same interpolation functor $_\theta$ (real, complex, or more general interpolation functors).

Definition C.7. *We say* X *is a retract of* Z *if there is a bounded map* $R: Z \to X$ *and a bounded map* $E: X \to Z$ *such that* $RE = id$ *is the identity map on* X.

For convenience, we call R *the restriction map and* E *the extension map, and we write* $Z \curvearrowright X$ *from time to time.*

More generally, for two classes of spaces $\{X_i\}_{i\in I}$ and $\{Z_i\}_{i\in I}$, we say $\{X_i\}_{i\in I}$ is a retract of $\{Z_i\}_{i\in I}$, with restriction map R and extension map E, if each X_i is a retract of Z_i with $R: Z_i \to X_i$, $E: X_i \to Z_i$ and $RE = id$ on X_i.

The following lemma (Theorem 6.4.2 in [7]) is an easy consequence of the definition of interpolation functors.

Lemma C.8. *Assume $\bar{X}$ is a retract of $\bar{Z}$ with the restriction map R and the extension map E. Then $\bar{X}_\theta$ is a retract of $\bar{Z}_\theta$, with the same R and E. In particular, $\bar{X}_\theta = R\bar{Z}_\theta$.*

In our situation, we will use some variants of the above lemma.

Lemma C.9. *Assume $\bar{X}$ is a retract of $\bar{Z}$ with the restriction map R and the extension map E.*

(a). Define $T_i = EX_i$ and $K_i = \{z \in Z_i : Rz = 0\}$ for $i = 1, 2$. Then

$$Z_i = T_i \oplus K_i, \text{ and } \bar{Z}_\theta = \bar{T}_\theta \oplus \bar{K}_\theta.$$

(b). Let $\bar{Y} = (Y_0, Y_1)$ be an interpolation couple such that $Y_i \subset X_i$. Define $\dot{T}_i = EY_i$ with norm induced from Y_i, and Banach spaces

$$\dot{Z}_i = \{z \in Z_i : Rz \in Y_i\} = \dot{T}_i \oplus K_i \subset Z_i$$

for $i = 1, 2$. Then

$$\bar{\dot{Z}}_\theta = \bar{\dot{T}}_\theta \oplus \bar{K}_\theta = E\bar{Y}_\theta \oplus \bar{K}_\theta = \{z \in \bar{Z}_\theta : Rz \in \bar{Y}_\theta\}.$$

The proof of part (a) is straightforward using basic property of interpolation functors, and (b) is clearly using part (a).

References

1. P. Alonso-Ruiz, F. Baudoin, L. Chen, L.G. Rogers, N. Shanmugalingam and A. Teplyaev, *Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities*, arXiv: 1811.04267.
2. P. Alonso-Ruiz, F. Baudoin, L. Chen, L.G. Rogers, N. Shanmugalingam and A. Teplyaev, *Besov class via heat semigroup on Dirichlet spaces II: BV functions and Gaussian heat kernel estimates*, arXiv: 1811.11010.
3. M.T. Barlow and R.F. Bass, *The construction of Brownian motion on the Sierpinski carpet*, Ann. Inst. H. Henri Poincaré Probab. Statist. 25 (1989), no. 3, 225–257.
4. M.T. Barlow and E.A. Perkins, *Brownian motion on the Sierpiński gasket,* Probab. Theory Related Fields 79 (1988), no. 4, 543–623.
5. M.T. Barlow, R.F. Bass, T. Kumagai and A. Teplyaev, *Uniqueness of Brownian motion on Sierpinski carpets*, J. Eur. Math. Soc. 12 (2010), no. 3, 655–701.
6. O. Ben-Bassat, R.S. Strichartz and A. Teplyaev, *What is not in the domain of the Laplacian on Sierpinski gasket type fractals*, J. Funct. Anal. 166 (1999), no. 2, 197–217.
7. J. Bergh and J. Löfström, *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.
8. J. Cao and A. Grigor'yan, *Heat kernels and Besov spaces associated with second order divergence form elliptic operators,* to appear in J. Fourier Anal. Appl.
9. J. Cao and A. Grigor'yan, *Heat kernels and Besov spaces on metric measure spaces,* preprint.
10. S. Cao and H. Qiu, *Sobolev spaces on p.c.f. self-similar sets: critical orders and atomic decompositions*, arXiv: 1904.00342.
11. S. Cao and H. Qiu, *Sobolev spaces on p.c.f. self-similar sets: boundary behavior and interpolation theorems*, arXiv: 2002.05888.
12. S. Cao and H. Qiu, *Some properties of the derivatives on Sierpinski gasket type fractals,* Constr. Approx. 46 (2017), no. 2, 319–347.
13. E.B. Davies, *Heat kernels and spectral theory.* Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1990.
14. A. Gogatishvili, P. Koskela and N. Shanmugalingam, *Interpolation properties of Besov spaces defined on metric spaces,* Math. Nachr. 283 (2010), no. 2, 215–231.
15. S. Goldstein, *Random walks and diffusions on fractals*, Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), 121–129, IMA Vol. Math. Appl., 8, Springer, New York, 1987.

16. A. Grigor'yan, *Heat kernels and function theory on metric measure spaces,* Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 143–172, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.
17. A. Grigor'yan, J. Hu and K. Lau, *Heat kernels on metric measure spaces and an application to semilinear elliptic equations,* Trans. Amer. Math. Soc. 355 (2003), no. 5, 2065–2095.
18. A. Grigor'yan and L. Liu, *Heat kernel and Lipschitz-Besov spaces,* Forum Math. 27 (2015), no. 6, 3567–3613.
19. M. Haase, *The functional calculus for sectorial operators.* Operator Theory: Advances and Applications, 169. Birkäuser Verlag, Basel, 2006.
20. B.M. Hambly and T. Kumagai, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proc. London Math. Soc. (3) 78 (1999), no. 2, 431–458.
21. M. Hinz, D. Koch and M. Meinert, *Sobolev spaces and calculus of variations on fractals*, arXiv: 1805.04456.
22. J. Hu and M. Zähle, *Potential spaces on fractals.* Studia Math. 170 (2005), no. 3, 259–281.
23. M. Ionescu, L.G. Rogers and R.S. Strichartz, *Pseudo-differential operators on fractals and other metric measure spaces,* Rev. Mat. Iberoam. 29 (2013), no. 4, 1159–1190.
24. J. Kigami, *A harmonic calculus on the Sierpinski spaces,* Japan J. Appl. Math. 6 (1989), no. 2, 259–290.
25. J. Kigami, *A harmonic calculus on p.c.f. self-similar sets,* Trans. Amer. Math. Soc. 335 (1993), no. 2, 721–755.
26. J. Kigami, *Analysis on Fractals.* Cambridge Tracts in Mathematics, 143. Cambridge University Press, Cambridge, 2001.
27. T. Kumagai and K.T. Sturm, *Construction of diffusion processes on fractals, d-sets, and general metric measure spaces,* J. Math. Kyoto Univ. 45 (2005), no. 2, 307–327.
28. S. Kusuoka and X.Y. Zhou, *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields 93 (1992), no. 2, 169–196.
29. T. Lindstrøm, *Brownian motion on nested fractals,* Mem. Amer. Math. Soc. 83 (1990), no. 420, iv+128 pp.
30. J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications.* Vol. I. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.
31. L.G. Rogers, R.S. Strichartz and A. Teplyaev, *Smooth bumps, a Borel theorem and partitions of smooth functions on p.c.f. fractals,* Trans. Amer. Math. Soc. 361 (2009), no. 4, 1765–1790.
32. L.G. Rogers and R.S. Strichartz, *Distribution theory on p.c.f. fractals,* J. Anal. Math. 112 (2010), 137–191.
33. R.S. Strichartz, *Taylor approximations on Sierpinski gasket type fractals,* J. Funct. Anal. 174 (2000), no. 1, 76–127.
34. R.S. Strichartz, *Function spaces on fractals,* J. Funct. Anal. 198 (2003), no. 1, 43–83.
35. R.S. Strichartz, *Analysis on products of fractals,* Trans. Amer. Math. Soc. 357 (2005), no. 2, 571–615.
36. R.S. Strichartz, *Differential Equations on Fractals: A Tutorial.* Princeton University Press, Princeton, NJ, 2006.
37. A. Teplyaev, *Gradients on fractals,* J. Funct. Anal. 174 (2000), no. 1, 128–154.
38. H. Triebel, *Interpolation theory, function spaces, differential operators.* Second edition. Johann Ambrosius Barth, Heidelberg, 1995.

Department of Mathematics, Cornell University, Ithaca 14853, USA
Email address: sc2873@cornell.edu

Department of Mathematics, Nanjing University, Nanjing 210093, China
Email address: huaqiu@nju.edu.cn