MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS ON THE TETRAHEDRAL SIERPINSKI GASKET

HUA QIU, YIPENG WU AND KUI YAO

ABSTRACT. In this paper, we study the mean value property for both the harmonic functions and the functions in the domain of the Laplacian on the tetrahedral Sierpinski gasket. This paper is a continuation of the work of Strichartz and the first author [14], where the same property on p.c.f. self-similar sets with Dihedral-3 symmetry was considered.

1. INTRODUCTION

The analysis on the *post critically finite (p.c.f.) self-similar sets* has been studied extensively since Kigami's analytic construction of the fractal Laplacian on the *Sierpinski gasket* [9, 10] (see [6, 15, 17, 18, 19, 20, 21, 23] and the reference therein). Various problems have been studied including the spectral analysis of the Laplacian [3, 7, 11, 13, 16, 24], the gradient and derivatives [5, 19, 25] and the energy measures[1, 2, 4, 8], etc.

Recently, Strichartz and the first author [14] studied the *mean value property* for both the harmonic functions and some general functions in the domain of the Laplacian on p.c.f. self-similar sets with Dihedral-3 symmetry. They mainly deal with the Sierpinski gasket $\mathcal{SG}$. Let μ be the normalized Hausdorff measure on $\mathcal{SG}$ and Δ be the standard Laplacian with respect to μ. Then for each point $x \in \mathcal{SG} \setminus V_0$ (V_0 is the boundary of $\mathcal{SG}$), they proved that there is a contracting sequence of neighborhoods of x, denoted by $\{B_k(x)\}_k$, called the *mean value neighborhoods* of x, such that $\bigcap_k B_k(x) = \{x\}$ and

$$\frac{1}{\mu(B_k(x))} \int_{B_k(x)} h(y)d\mu(y) = h(x) \tag{1.1}$$

holds for every harmonic function h and $k \geq 1$. More generally, by introducing suitable constant $c_{B_k(x)}$ for each neighborhood $B_k(x)$, they showed that for any function u in the domain of the Laplacian such that Δu is a continuous,

$$\lim_{k\to\infty} \frac{1}{c_{B_k(x)}} \Big(\frac{1}{\mu(B_k(x))} \int_{B_k(x)} u(y)d\mu(y) - u(x) \Big) = \Delta u(x). \tag{1.2}$$

The proof depends strongly on the Dihedral-3 symmetry, and the proof of (1.2) for general functions is quite technical and could not be extended to other Dihedral-3

2000 *Mathematics Subject Classification.* Primary: 28A80.

Key words and phrases. mean value property, tetrahedral Sierpinski gasket, harmonic functions, Laplacian, self-similar sets.

The research of the first and the third authors were supported by the National Science Foundation of China, Grant 11471157.

p.c.f. self-similar sets. It is interesting to know to what extent these results can be extended to other p.c.f. self-similar sets.

In this paper, we continue to consider the tetrahedral Sierpinski gasket, denoted by $\mathcal{SG}^4$, which possesses fully symmetry, but not Dihedral-3 symmetry. We will prove that the analogous mean value property holds for both the harmonic functions and the general functions in the domain of the Laplacian.

Recall that a *tetrahedral Sierpinski gasket* $\mathcal{SG}^4$ is the unique nonempty compact set in $\mathbb{R}^3$ satisfying $\mathcal{SG}^4 = \bigcup_{i=0}^3 F_i(\mathcal{SG}^4)$ for an *iterated function system (IFS)* $\{F_i\}_{i=0}^3$ on $\mathbb{R}^3$ with $F_i(x) = \frac{1}{2}(x - q_i) + q_i$, where $\{q_i\}_{i=0}^3$ are the four vertices of a regular tetrahedron, see Figure 1.

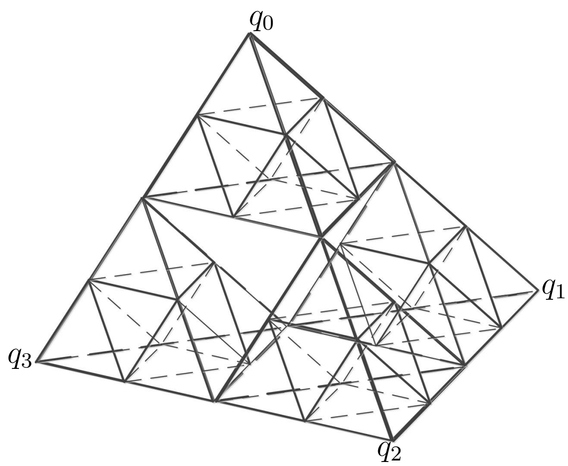

Figure 1. The tetrahedral Sierpinski gasket

We call the sets $F_i(\mathcal{SG}^4)$ the cells of level 1, and by iterating the IFS we obtain cells of higher level. For a *word* $w = w_1w_2\cdots w_m$ of *length* m with $w_i \in \{0,1,2,3\}$, let $F_w = F_{w_1} \circ F_{w_2} \circ \cdots F_{w_m}$. Call the cell $F_w(\mathcal{SG}^4)$ a *m-cell*. Denote by $V_0 = \{q_i\}_{i=0}^3$ the *boundary* of $\mathcal{SG}^4$. Inductively, write $V_m = \bigcup_{i=0}^3 F_i V_{m-1}$ and $V_* = \bigcup_{m\geq 0} V_m$.

The *standard energy form* $(\mathcal{E}, dom\mathcal{E})$ on $\mathcal{SG}^4$ is given by

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \left(\frac{3}{2}\right)^m \sum_{x\sim_m y} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big),$$

and

$$dom(\mathcal{E}) = \{u \in C(\mathcal{SG}^4) : \mathcal{E}(u) := \mathcal{E}(u,u) < \infty\},$$

where $x \sim_m y$ means $x \neq y$ and x, y belong to a same $F_w(V_0)$ for some word w of length m. Here $\frac{3}{2}$ is the reciprocal of the *renormalization factor* of the energy form.

A function h is called *harmonic* if it minimize the graph energy $\sum_{x\sim_m y} \big(h(x)-h(y)\big)^2$ for each m. It is direct to check that for any $x \in V_* \setminus V_0$,

$$h(x) = \frac{1}{6} \sum_{y\sim_m x} h(y), \tag{1.3}$$

which can be viewed as a mean value property of harmonic functions at points in $V_*\setminus V_0$. The space of harmonic functions is 4-dimensional and the values at points in V_0 may be freely assigned. There is a harmonic extension algorithm, the "$\frac{1}{3}-\frac{1}{6}$ rule" (similar to the "$\frac{1}{5}-\frac{2}{5}$ rule" in the $\mathcal{SG}$ case) for computing the values of a harmonic function at all points in V_* in terms of the boundary values. That is $h(q_{01})=\frac{1}{3}h(q_0)+\frac{1}{3}h(q_1)+\frac{1}{6}h(q_2)+\frac{1}{6}h(q_3)$ and the symmetric alternates, where q_{01} is the midpoint in the line segment joining q_0 and q_1. Any function u defined on V_m can be extended harmonically on V_*, then continuously on $\mathcal{SG}^4$. We call it an *m-piecewise harmonic function* on $\mathcal{SG}^4$.

Let μ be the normalized Hausdorff measure on $\mathcal{SG}^4$, write Δ the standard Laplacian associated with μ via the weak formulation

$$\mathcal{E}(u,v)=-\int v\Delta u d\mu$$

for all $v\in dom\mathcal{E}$ vanishing on V_0. The Laplacian Δ satisfies the scaling property

$$\Delta(u\circ F_w)=\frac{1}{6^m}(\Delta u)\circ F_w,$$

where $\frac{1}{6}$ is the product of the measure scaling factor $\frac{1}{4}$ and the energy renormalization factor $\frac{2}{3}$.

The Dirichlet problem for the Laplacian, i.e., the unique solution vanishing on the boundary V_0 of $-\Delta u=f$ for given continuous function f, can be solved by integrating against the *Green's function* $G(x,y)$, which is the uniform limit of $G_M(x,y)$, defined by

$$G_M(x,y)=\sum_{m=0}^{M}\sum_{z,z'\in V_{m+1}\setminus V_m}\left(\frac{2}{3}\right)^m g(z,z')\psi_z^{(m+1)}(x)\psi_{z'}^{(m+1)}(y), \tag{1.4}$$

as M goes to the infinity, where $g(z,z')=\frac{5}{36}$ when $z=z'$, $g(z,z')=\frac{1}{24}$ when $z\sim_{m+1}z'$ and $g(z,z')=\frac{1}{36}$ elsewhere (this can be calculated by finding the inverse of an appropriate matrix); and $\psi_z^{(m)}(x)$ denotes the m-piecewise harmonic function satisfying $\psi_z^{(m)}(x)=\delta_z(x)$ for $x\in V_m$.

The reader is referred to the books [12] and [22] for exact definitions and any unexplained notations.

In this paper, we will prove the mean value property for the tetrahedral Sierpinski gasket $\mathcal{SG}^4$ analogous to (1.1) and (1.2) for $\mathcal{SG}$.

Theorem 1.1. *For each x in $\mathcal{SG}^4\setminus V_0$, there exists a natural system of mean value neighborhoods $\{B_k(x)\}_k$ with $\bigcap_k B_k(x)=\{x\}$ such that for any harmonic function h and $k\geq 1$, we have*

$$\frac{1}{\mu(B_k(x))}\int_{B_k(x)}h(y)d\mu(y)=h(x).$$

For $x\in\mathcal{SG}^4\setminus V_0$ and $k\geq 1$, we introduce that

$$c_{B_k(x)}=\frac{1}{\mu(B_k(x))}\int_{B_k(x)}v(y)d\mu(y)-v(x),$$

where v is a function satisfying $\Delta v = 1$. Then

Theorem 1.2. *The coefficient $c_{B_k(x)}$ is bounded above and below by a multiple of $\frac{1}{6^k}$. Moreover, for any function $u \in dom\Delta$ with $g = \Delta u$ satisfying the Hölder condition that $|g(y)-g(z)| \leq c\gamma^k$ for all y, z belonging to a same k-cell, for some constant $0 < \gamma < 1$, $c > 0$, we have*

$$\lim_{k\to\infty} \frac{1}{c_{B_k(x)}}\Big(\frac{1}{\mu(B_k(x))}\int_{B_k(x)} u(y)d\mu(y) - u(x)\Big) = \Delta u(x). \tag{1.5}$$

The paper is organized as follows. In Section 2, we will explain how to define the mean value neighborhoods for any point x in $\mathcal{SG}^4 \backslash V_0$ and prove Theorem 1.1. In Section 3, we will deal with the mean value property for general functions in the domain of the Laplacian and then prove Theorem 1.2. The purpose of this paper is to work out the details for one specific example other than the Sierpinski gasket. We hope it will bring insights which inspire future work on a more general theory. The problem on how to extend Theorem 1.2 to other fully symmetric p.c.f. self-similar sets remains open.

2. Mean value property of harmonic functions on $\mathcal{SG}^4$

We write $M_B(u) = \frac{1}{\mu(B)}\int_B u d\mu$ for any Borel set B contained in $\mathcal{SG}^4$ and function u defined on B, for simplicity.

Lemma 2.1. *(a) Let C be any cell with boundary points p_0, p_1, p_2, p_3, and h be any harmonic function. Then $M_C(h) = \frac{1}{4}\sum_{i=0}^{3} h(p_i)$.*

(b) Let p be any point in $V_* \setminus V_0$, and C_1, C_2 be the two m-cells meeting at p. Then $M_C(h) = h(p)$.

Proof. Choose a basis $\{h_0, h_1, h_2, h_3\}$ of the harmonic functions on C by taking $h_i(p_j) = \delta_{ij}$. Noticing that $\sum_{i=0}^{3} h_i$ is identically 1 on C, $\int_C h_i d\mu = \frac{1}{4}\mu(C)$ for each i by symmetry. Thus we get (a) since any harmonic function h can be written into $h = \sum_{i=0}^{3} h(p_i)h_i$. Combing (a) for $C = C_1$ and $C = C_2$ and the formula (1.3) at p, we get (b). □

Obviously, (b) gives a trivial solution to the problem of finding mean value neighborhoods for points in $V_* \setminus V_0$. So we mainly focus on general points $x \in \mathcal{SG}^4 \setminus V_0$. Let $C_w = F_w(\mathcal{SG}^4)$ be any cell containing x, which is small enough so that C_w does not intersect V_0. Write p_0, p_1, p_2, p_3 the 4 boundary points of C_w and denote C_i the cell of the same level as C_w meeting at p_i. Let $D_w = C_w \cup \bigcup_{i=0}^{3} C_i$. See Figure 2. Similar to the $\mathcal{SG}$ case [14], we will find a mean value neighborhood B so that $C_w \subset B \subset D_w$. If we can do so, then by letting C_w shrink to x, we could get a contacting sequence of mean value neighborhoods of x. On the other hand, since mean value neighborhoods are just balls in Euclidean case, it is reasonable to require the set B as simple as possible.

Definition 2.2. Let $\boldsymbol{c} = (c_0, c_1, c_2, c_3)$ be a 4-dimensional vector with all $0 \leq c_i \leq 1$, and denote

$$B(\boldsymbol{c}) = C_w \cup \bigcup_{i=0}^{3} E_i$$

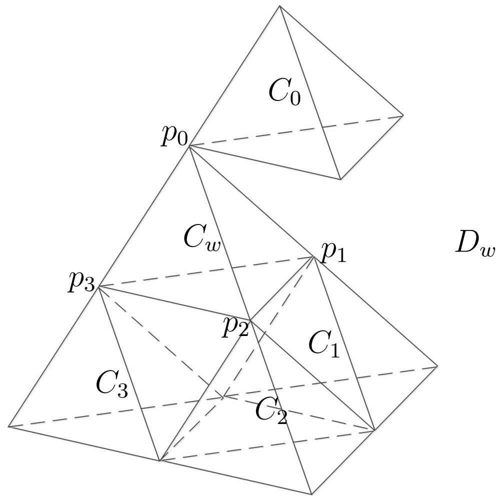

Figure 2. C_w and its neighboring cells

where each E_i is a sub-tetrahedral domain in C_i obtained by cutting C_i with a plane away from p_i symmetrically so that $\mu(E_i) = c_i\mu(C_i)$, see Figure 3. Denote by

$$\mathcal{B} = \{B(\boldsymbol{c}) : 0 \le c_i \le 1, 0 \le i \le 3\}$$

the 4-parameter family of all such sets.

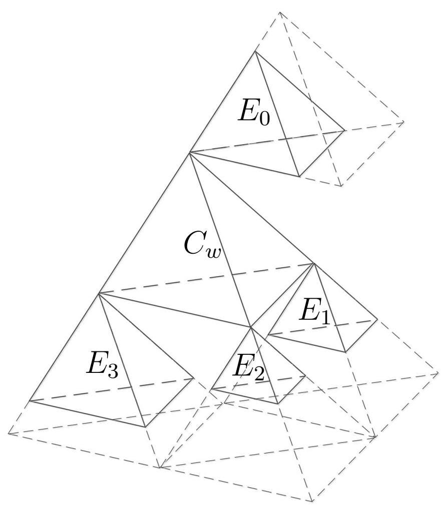

Figure 3. The relative geometry of $B(\boldsymbol{c})$ and C_w

Obviously, $B(0,0,0,0) = C_w$, $B(1,1,1,1) = D_w$ and for each set $B(\boldsymbol{c}) \in \mathcal{B}$, $C_w \subset B(\boldsymbol{c}) \subset D_w$. We have two more observations for harmonic functions defined on $B(\boldsymbol{c})$.

Firstly, for any harmonic function h, by linearity, the value $h(x)$ is a linear combination of $\{h(p_i)\}_{i=0}^3$,

$$h(x) = \sum_{i=0}^{3} \alpha_i(x)h(p_i), \tag{2.1}$$

where the coefficient vector $\boldsymbol{\alpha}(x) = \big(\alpha_0(x), \alpha_1(x), \alpha_2(x), \alpha_3(x)\big)$ depends only on the location of x in C_w. Furthermore, considering $h \equiv 1$, we have $\sum_{i=0}^{3} \alpha_i(x) = 1$ and by the maximum principle all $\alpha_i(x) \geq 0$.

Secondly, still by linearity, we have

$$M_{B(\boldsymbol{c})}(h) = \sum_{i=0}^{3} \beta_i(\boldsymbol{c}) h(p_i), \tag{2.2}$$

for some coefficient vector $\boldsymbol{\beta}(\boldsymbol{c}) = \big(\beta_0(\boldsymbol{c}), \beta_1(\boldsymbol{c}), \beta_2(\boldsymbol{c}), \beta_3(\boldsymbol{c})\big)$, which depends only on the relative geometry between $B(\boldsymbol{c})$ and C_w. Again we have $\sum_{i=0}^{3} \beta_i(\boldsymbol{c}) = 1$ by considering $h \equiv 1$. Later we will show that $\beta_i(\boldsymbol{c})$'s may not all greater than 0.

Proposition 2.3. *$\boldsymbol{\beta}(\boldsymbol{c})$ is independent on the location of C_w in $\mathcal{SG}^4$.*

Proof. Let h be a harmonic function. For $0 \leq i \leq 3$, denote by $\{p_i, r_i, s_i, t_i\}$ the boundary points of C_i. By linearity and symmetry, $\frac{1}{\mu(C_i)} \int_{E_i} h d\mu$ must be a linear combination of $\big(h(p_i), h(r_i), h(s_i), h(t_i)\big)$ such that

$$\frac{1}{\mu(C_i)} \int_{E_i} h d\mu = m_i h(p_i) + n_i \big(h(r_i) + h(s_i) + h(t_i)\big)$$

for some non-negative coefficients m_i, n_i with $m_i + 3n_i = c_i$. Notice that the coefficients m_i, n_i are independent on the location of C_i in $\mathcal{SG}^4$, and depend only on the relative position of E_i in C_i, i.e., depend only on c_i.

By using formula (1.3), we then have

$$\int_{E_i} h d\mu = \Big((m_i + 6n_i) h(p_i) - n_i \sum_{j \neq i} h(p_j)\Big) \mu(C_i), \quad 0 \leq i \leq 3. \tag{2.3}$$

Since $\mu(E_i) = c_i \mu(C_i)$ and $\mu(C_i) = \mu(C_w)$, combing the above equality with Lemma 2.1, we see that $\boldsymbol{\beta}(\boldsymbol{c})$ depends only on $\boldsymbol{c}$ and is independent of the location of C_w in $\mathcal{SG}^4$. $\square$

Define $\pi_{\boldsymbol{\alpha}}$ the range of the vector-valued function $\boldsymbol{\alpha}$ by varying x in C_w, and $\pi_{\boldsymbol{\beta}}$ the range of the vector-valued function $\boldsymbol{\beta}$ by varying $\boldsymbol{c}$ with all $0 \leq c_i \leq 1$. For $x \in C_w$, to ensure that there exists a set $B(\boldsymbol{c}) \in \mathcal{B}$ so that $B(\boldsymbol{c})$ is a mean value neighborhood of x, i.e., $M_{B(\boldsymbol{c})}(h) = h(x)$ holds for all harmonic functions, in view of formula (2.1) and (2.2), we only need to verify that $\pi_{\boldsymbol{\alpha}} \subset \pi_{\boldsymbol{\beta}}$. Let $\mathcal{S}$ denote the simplex in $\mathbb{R}^4$ defined by

$$\mathcal{S} = \{\boldsymbol{c} = (c_0, c_1, c_2, c_3) : \sum_{i=0}^{3} c_i = 1, 0 \leq c_i \leq 1\}.$$

Since it is easy to check that $\pi_{\boldsymbol{\alpha}} \subset \mathcal{S}$, it suffices to prove $\mathcal{S} \subset \pi_{\boldsymbol{\beta}}$.

Lemma 2.4. *Let M be a 4×3 matrix defined by* $M = \begin{pmatrix} 0 & 0 & 2\sqrt{2} \\ 1 & -\sqrt{3} & 0 \\ 1 & \sqrt{3} & 0 \\ -2 & 0 & 0 \end{pmatrix}$. *Then the linear transform $\boldsymbol{c} \to \boldsymbol{c}M$, still denoted by M, is homeomorphic from $\mathcal{S}$ onto $M(\mathcal{S})$, and $M(\mathcal{S})$ is a regular tetrahedron in $\mathbb{R}^3$.*

Proof. It can be directly checked since the rank of the matrix M is 3. □

Denote the boundary vertices of $M(\mathcal{S})$ by $\{P_0, P_1, P_2, P_3\}$ corresponding to $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ in $\mathcal{S}$ accordingly. By establishing a (u,v,w)-Cartesian coordinate system, we could require the coordinate of P_i to be the i-th row of M, $0 \le i \le 3$, see Figure 4.

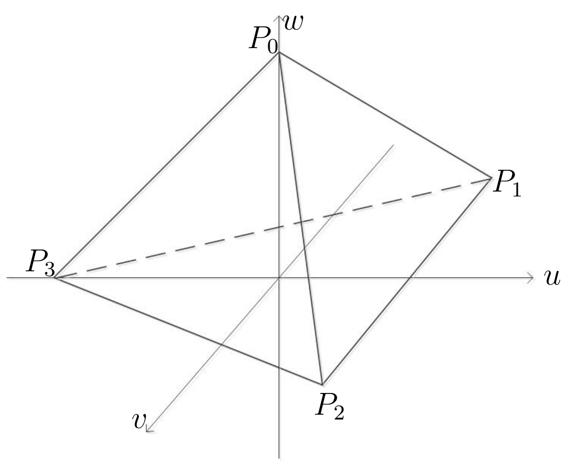

Figure 4. The regular tetrahedron $M(\mathcal{S})$

From Lemma 2.4, to prove $\mathcal{S} \subset \pi_{\boldsymbol{\beta}}$ is equivalent to prove $M(\mathcal{S}) \subset M(\pi_{\boldsymbol{\beta}})$.

Proposition 2.5. *$M(\mathcal{S}) \subset M(\pi_{\boldsymbol{\beta}})$.*

Let $\mathcal{B}_0 = \{B(\boldsymbol{c}) \in \mathcal{B} : c_0 = 0 \le c_1 \le c_2 \le c_3 \le 1\}$ and $\mathcal{B}^* = \{B(\boldsymbol{c}) \in \mathcal{B} : \prod_{i=0}^{3} c_i = 0\}$. Obviously, by symmetry, $\mathcal{B}_0$ is a $\frac{1}{24}$ part of $\mathcal{B}^*$ and $\mathcal{B}^*$ is a subfamily of $\mathcal{B}$. To prove Proposition 2.5, we will restrict to consider the range of the vector-valued function $\boldsymbol{\beta}$ over the vectors $\boldsymbol{c}$ such that $B(\boldsymbol{c}) \in \mathcal{B}_0$.

Denote O the center of the regular tetrahedron $M(\mathcal{S})$, O' the planar center of the triangle face $\Delta_{P_1P_2P_3}$ of $M(\mathcal{S})$, and T the midpoint of the line segment joining P_2 and P_3, see Figure 5. Obivously, the (u,v,w)-coordinates of O, O', T are $(0,0,\frac{\sqrt{2}}{2})$, $(0,0,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, respectively. Let $0 \le c \le 1$, write $P(c) = M\big(\boldsymbol{\beta}(0,0,0,c)\big)$, $Q(c) = M\big(\boldsymbol{\beta}(0,0,c,c)\big)$ and $R(c) = M\big(\boldsymbol{\beta}(0,c,c,c)\big)$. We need some lemmas.

Lemma 2.6. *Varying $0 \le c \le 1$, we have*

(a) the trace of $P(c)$ is a line segment joining O and P_3;

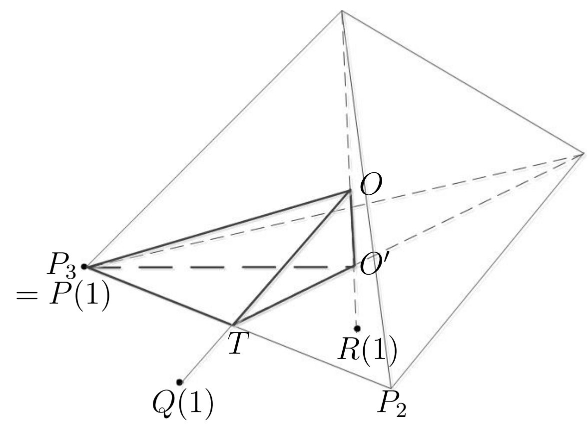

Figure 5. A $\frac{1}{24}$ part of $M(\mathcal{S})$

(b) the trace of $Q(c)$ is a line segment lying in the line l_{OT} with endpoint $Q(1)$ locating below the (u, v)-plane;

(c) the trace of $R(c)$ is a line segment lying in the line $l_{OO'}$ with endpoint $R(1)$ locating below the (u, v)-plane, see Figure 5.

Proof. (a) Let $0 \leq c \leq 1$, consider the set $B = B(0,0,0,c)$. Write $B = C_w \cup E_3$ with $\mu(E_3) = c\mu(C_w)$. For any harmonic function h, by Lemma 2.1 and the identity (2.3), we have

$$\int_{C_w} h d\mu = \frac{1}{4}\sum_{i=0}^{3} h(p_i)\mu(C_w)$$

and

$$\int_{E_3} h d\mu = \big((m+6n)h(p_3) - n\sum_{j\neq 3} h(p_j)\big)\mu(C_w),$$

where m, n are the same as that in (2.3) depending only on c.

An easy calculation yields that

$$M_B(h) = \frac{1}{1+c}\big((\frac{1}{4} - n)\sum_{j\neq 3} h(p_j) + (\frac{1}{4} + m + 6n)h(p_3)\big).$$

So $\boldsymbol{\beta}(0,0,0,c) = \frac{1}{1+c}(\frac{1}{4} - n, \frac{1}{4} - n, \frac{1}{4} - n, \frac{1}{4} + m + 6n)$. Then right multiplying by the matrix M, we get

$$P(c) = M\big(\boldsymbol{\beta}(0,0,0,c)\big) = \frac{1}{1+c}(-2m - 14n, 0, \frac{\sqrt{2}}{2} - 2\sqrt{2}n).$$

By using $m+3n = c$, it is easy to verify that the (u, v, w)-coordinate of $P(c)$ satisfies

$$\frac{u}{-2} + \sqrt{2}w = 1, \text{ and } v = 0,$$

which is exactly the equation of the line segment joining O and P_3. Then (a) follows by verifying that $P(0) = O$ and $P(1) = P_3$ and letting c vary continuously from 0 to 1.

(b) Now we consider the set $B = B(0,0,c,c)$. Write $B = C_w \cup E_2 \cup E_3$ with $\mu(E_2) = \mu(E_3) = c\mu(C_w)$. A similar calculation yields that $\boldsymbol{\beta}(0,0,c,c) = \frac{1}{1+2c}(\frac{1}{4} - 2n, \frac{1}{4} - 2n, \frac{1}{4} + m + 5n, \frac{1}{4} + m + 5n)$, where m, n are same as in (2.3) depending on c. Right multiplying by the matrix M, we get

$$Q(c) = M\big(\boldsymbol{\beta}(0,0,c,c)\big) = \frac{1}{1+2c}(-7n - m, 7\sqrt{3}n + \sqrt{3}m, \frac{\sqrt{2}}{2} - 4\sqrt{2}n).$$

By using $m + 3n = c$, it is easy to verify that the (u,v,w)-coordinate of $Q(c)$ satisfies

$$\sqrt{3}u + v = 0, \text{ and } \frac{2v}{\sqrt{3}} + \sqrt{2}w = 1,$$

which is the equation of the line l_{OT}. Furthermore, it is directly to check that $Q(0) = O$ and $Q(1) = (-\frac{1}{12}, -\frac{1}{12}, \frac{7}{12}, \frac{7}{12})M = (-\frac{2}{3}, \frac{2}{\sqrt{3}}, -\frac{\sqrt{2}}{6})$. Thus (b) follows.

(c) Consider the set $B = B(0,c,c,c)$ and write it into $B = C_w \cup \bigcup_{i=1}^{3} E_i$ with $\mu(E_i) = c\mu(C_w)$. Similar as before, we have $\boldsymbol{\beta}(0,c,c,c) = \frac{1}{1+3c}(\frac{1}{4} - 3n, \frac{1}{4} + m + 4n, \frac{1}{4} + m + 4n, \frac{1}{4} + m + 4n)$, where m, n depending on c. Right multiplying by the matrix M, we get

$$R(c) = M\big(\boldsymbol{\beta}(0,c,c,c)\big) = \frac{1}{1+3c}(0, 0, \frac{\sqrt{2}}{2} - 6\sqrt{2}n).$$

Obviously, $R(c)$ lies on the line $l_{OO'}$. It is easy to check $R(0) = O$ and $R(1) = (-\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8})M = (0, 0, -\frac{\sqrt{2}}{4})$. Then (c) follows. □

Lemma 2.7. *For fixed $0 \le c \le 1$, varying $0 \le c' \le c$, we have*

(a) the trace of $M\big(\boldsymbol{\beta}(0,c',c',c)\big)$, a continuous curve joining $P(c)$ and $R(c)$, is contained in the (u,w)-plane;

(b) the trace of $M\big(\boldsymbol{\beta}(0,c',c,c)\big)$, a continuous curve joining $Q(c)$ and $R(c)$, is contained in the plane containing the triangle $\Delta_{OO'T}$;

(c) the trace of $M\big(\boldsymbol{\beta}(0,0,c',c)\big)$, a continuous curve joining $P(c)$ and $Q(c)$, is contained in the plane containing the triangle Δ_{OP_3T}, see Figure 6.

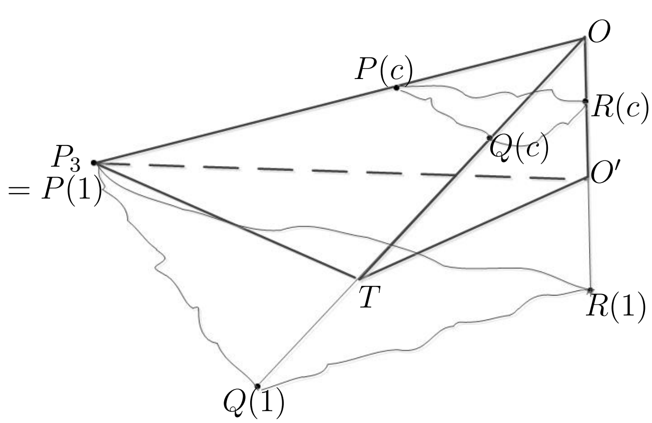

Figure 6. the traces in Lemma 2.7

Proof. (a) Consider the set $B = B(0, c', c', c)$ with $0 \le c' \le c \le 1$, and write it into $B = C_w \cup \bigcup_{i=1}^3 E_i$ with $\mu(E_1) = \mu(E_2) = c'\mu(C_w)$ and $\mu(E_3) = c\mu(C_w)$. Let m, n be associated with c and m', n' be associated with c' as in (2.3). Then an easy calculation yields that

$$\boldsymbol{\beta}(0, c', c', c) = \frac{1}{1+2c'+c}(\frac{1}{4}-2n'-n, \frac{1}{4}+m'+5n'-n, \frac{1}{4}+m'+5n'-n, \frac{1}{4}-2n'+m+6n).$$

Right multiplying the matrix M, we get

$$M\big(\boldsymbol{\beta}(0, c', c', c)\big) = \frac{1}{1+2c'+c}(2m'+14n'-2m-14n, 0, \frac{\sqrt{2}}{2}-4\sqrt{2}n'-2\sqrt{2}n).$$

Thus the v-coordinate of $M\big(\boldsymbol{\beta}(0, c', c', c)\big)$ always equals to 0, so (a) follows when varying c' continuously from 0 to c.

(b) Consider the set $B = B(0, c', c, c)$ with $0 \le c' \le c \le 1$, and write it into $B = C_w \cup \bigcup_{i=1}^3 E_i$ with $\mu(E_1) = c'\mu(C_w)$ and $\mu(E_2) = \mu(E_3) = c\mu(C_w)$. Let m, n be associated with c and m', n' be associated with c' as before. Then

$$\boldsymbol{\beta}(0, c', c, c) = \frac{1}{1+c'+2c}(\frac{1}{4}-n'-2n, \frac{1}{4}+m'+6n'-2n, \frac{1}{4}-n'+m+5n, \frac{1}{4}-n'+m+5n).$$

Right multiplying the matrix M, we get

$$M\big(\boldsymbol{\beta}(0, c', c, c)\big) = \frac{1}{1+c'+2c}\big(m'+7n'-m-7n, \sqrt{3}(-m'-7n'+m+7n), \frac{\sqrt{2}}{2}-2\sqrt{2}n'-4\sqrt{2}n\big).$$

It is directly to see that the (u, v)-coordinate of $M\big(\boldsymbol{\beta}(0, c', c, c)\big)$ satisfies $\sqrt{3}u + v = 0$. So (b) follows by varying c' continuously from 0 to c.

(c) Now we consider the set $B = B(0, 0, c', c)$ with $0 \le c' \le c \le 1$, and write it into $B = C_w \cup E_2 \cup E_3$ with $\mu(E_2) = c'\mu(C_w)$ and $\mu(E_3) = c\mu(C_w)$. Let m, n be associated with c and m', n' be associated with c' as before. Then

$$\boldsymbol{\beta}(0, 0, c', c) = \frac{1}{1+c'+c}(\frac{1}{4}-n'-n, \frac{1}{4}-n'-n, \frac{1}{4}+m'+6n'-n, \frac{1}{4}-n'+m+6n).$$

Right multiplying the matrix M, we get

$$M\big(\boldsymbol{\beta}(0, 0, c', c)\big) = \frac{1}{1+c'+c}\big(m'+7n'-2m-14n, \sqrt{3}(m'+7n'), \frac{\sqrt{2}}{2}-2\sqrt{2}n'-2\sqrt{2}n\big).$$

Noticing that the normal vector of the plane containing Δ_{OP_3T} is $\boldsymbol{n} = (\frac{1}{2\sqrt{3}}, -\frac{1}{2}, -\frac{2}{\sqrt{6}})$, it is easy to check that

$$\boldsymbol{n} \cdot \overline{P_3 M\big(\boldsymbol{\beta}(0, 0, c', c)\big)} = 0.$$

So (c) follows. □

Lemma 2.8. *For $0 \le c_1 \le c_2 \le 1$, $M\big(\boldsymbol{\beta}(0, c_1, c_2, 1)\big)$ always locates below the (u, v)-plane.*

Proof. We need to consider the set $B = B(0, c_1, c_2, 1)$, write it into $B = C_w \cup E_1 \cup E_2 \cup C_3$ with $\mu(E_1) = c_1\mu(C_w)$, $\mu(E_2) = c_2\mu(C_w)$ and $\mu(C_3) = \mu(C_w)$. Similar as before, we can calculate that

$$\boldsymbol{\beta}(0, c_1, c_2, 1) = \frac{1}{2 + c_1 + c_2}(-n_1 - n_2, m_1 + 6n_1 - n_2, -n_1 + m_2 + 6n_2, 2 - n_1 - n_2)$$

where m_i, n_i are associated with c_i, $i = 1, 2$, respectively. Right multiplying M, we get

$$M\big(\boldsymbol{\beta}(0, c_1, c_2, 1)\big) = \frac{1}{2 + c_1 + c_2}\big(m_1+7n_1+m_2+7n_2-4, \sqrt{3}(-m_1-7n_1+m_2+7n_2), -2\sqrt{2}(n_1+n_2)\big).$$

Obviously, the w-coordinate of $M\big(\boldsymbol{\beta}(0, c_1, c_2, 1)\big)$ is always less than 0, which completes the proof. □

Proof of Proposition 2.5. By using Lemma 2.6, Lemma 2.7 and Lemma 2.8, varying the parameter c_3 between 0 and 1 continuously, it is easy to find that the range of $M\big(\boldsymbol{\beta}(\boldsymbol{c})\big)$ over the vectors $\{\boldsymbol{c} = (c_0, c_1, c_2, c_3) : c_0 = 0 \le c_1 \le c_2 \le c_3 \le 1\}$ contains the tetrahedron whose vertices are O, O', T and P_3. Then by symmetry, we have $M(\mathcal{S}) \subset \{M\big(\boldsymbol{\beta}(\boldsymbol{c})\big) : B(\boldsymbol{c}) \in \mathcal{B}^*\}$. Since $\mathcal{B}^*$ is a subfamily of $\mathcal{B}$, we then have $M(\mathcal{S}) \subset M(\pi_{\boldsymbol{\beta}})$, which completes the proof.

□

Now we have

Theorem 2.9. *For $x \in C_w$, there exists a mean value neighborhood $B \in \mathcal{B}$ of x with $C_w \subset B \subset D_w$. Moreover, if we denote by $\mathcal{B}^* = \{B(\boldsymbol{c}) \in \mathcal{B} : \prod_{i=0}^{3} c_i = 0\}$, then there exists a unique mean value neighborhood $B \in \mathcal{B}^*$.*

Proof. The existence follows readily from Lemma 2.4 and Proposition 2.5. The uniqueness follows since for and different $\boldsymbol{c}, \boldsymbol{c}'$ such that $B(\boldsymbol{c}), B(\boldsymbol{c}') \in \mathcal{B}^*$, we have obviously $M\big(\boldsymbol{\beta}(\boldsymbol{c})\big) \neq M\big(\boldsymbol{\beta}(\boldsymbol{c}')\big)$. □

In what follows, to make the mean value neighborhoods as simple as possible, we always choose them from $\mathcal{B}^*$.

Proof of Theorem 1.1. It follows by applying Theorem 2.9 to a sequence of C_w shrinking to x. □

3. Mean value property of general functions on $\mathcal{SG}^4$

In this section, we turn to consider the mean value property for more general functions, i.e., those functions in the domain of the Laplacian.

For $x \in \mathcal{SG}^4 \setminus V_0$, motivated by the $\mathcal{SG}$ case [14], for each mean value neighborhood B of x, we define

$$c_B = M_B(v) - v(x)$$

for any function v satisfying $\Delta v = 1$. We remark that the definition is independent of the particular choice of v, since any two such functions differ by a harmonic function and the equality $M_B(h) - h(x) = 0$ always holds for harmonic functions. Here we choose

$$v(\cdot) = -\int G(\cdot, y)d\mu(y)$$

which vanishes on the boundary of $\mathcal{SG}^4$, where $G(x,y)$ is the Green's function we mentioned in Section 1.

Lemma 3.1. *For $x \in \mathcal{SG}^4 \setminus V_0$, let B be a mean value neighborhood of x, then*

$$c_B = -\frac{1}{24}\sum_{m=0}^{\infty}\frac{1}{6^m}\big(M_B(\phi_m) - \phi_m(x)\big), \tag{3.1}$$

where $\phi_m = \sum_{z\in V_{m+1}\setminus V_m}\psi_z^{(m+1)}$.

Proof. Obviously, the function v is the uniform limit of $v_M = -\int G_M(\cdot, y)d\mu(y)$ where $G_M(x,y)$ is given in (1.4).

By interchanging the integral and summation, we have

$$v_M = -\sum_{m=0}^{M}\big(\frac{2}{3}\big)^m \sum_{z,z'\in V_{m+1}\setminus V_m} g(z,z')\psi_z^{(m+1)}\int \psi_{z'}^{(m+1)}(y)d\mu(y).$$

Notice that by symmetry, for each $z' \in V_{m+1}\setminus V_m$, $\int \psi_{z'}^{(m+1)}(y)d\mu(y) = \frac{2}{4^{m+2}}$. So

$$v_M = -\frac{1}{8}\sum_{m=0}^{M}\frac{1}{6^m}\sum_{z,z'\in V_{m+1}\setminus V_m} g(z,z')\psi_z^{(m+1)}.$$

Taking the value of $g(z,z')$ into the above equality, an easy calculation yields that $v_M = -\frac{1}{24}\sum_{m=0}^{M}\frac{1}{6^m}\phi_m$, and thus

$$v = -\frac{1}{24}\sum_{m=0}^{\infty}\frac{1}{6^m}\phi_m.$$

Thus by the definition of c_B, (3.1) follows, which completes the proof. □

The following lemma is obvious by scaling argument, see Lemma 4.1 in [14] for the analogous one in $\mathcal{SG}$ case.

Lemma 3.2. *Let x, x' be two points in $\mathcal{SG}^4 \setminus V_0$. Let B and B' be two k-th and k'-th mean value neighborhood of x and x' respectively. If B and B' have the same shapes (the same coefficient vector $\boldsymbol{c}$ such that $B = B(\boldsymbol{c})$ and $B' = B'(\boldsymbol{c})$), then*

$$c_B = 6^{k'-k}c_{B'}.$$

Proposition 3.3. *There exists two constant $c_\#, c^\# > 0$ such that for any $x \in \mathcal{SG}^4 \setminus V_0$ and any k, we have*

$$c_\#\frac{1}{6^k} \le c_{B_k(x)} \le c^\#\frac{1}{6^k}.$$

Proof. Assume C_w is a k-cell containing x, not intersecting V_0, $B = B_k(x)$ is the k-th mean value neighborhood of x. Then $C_w \subset B \subset D_w$. From Lemma 3.2, since c_B depends only on the relative geometry of B and C_w, as well as k, we may assume that D_w is contained in a $(k-2)$-cell in $\mathcal{SG}^4$ without loss of generality.

Estimate of c_B from above. By Lemma 3.1, we have

$$c_B = -\frac{1}{24}\sum_{m=0}^{\infty}\frac{1}{6^m}\big(M_B(\phi_m) - \phi_m(x)\big). \tag{3.2}$$

Since when $m+1 \leq k-2$, ϕ_m is harmonic in the $(k-2)$-cell containing D_w, the first $k-2$ terms of (3.2) contribute 0 to c_B. Thus

$$c_B = -\frac{1}{24}\sum_{m=k-2}^{\infty}\frac{1}{6^m}\big(M_B(\phi_m) - \phi_m(x)\big). \tag{3.3}$$

From (3.3), we have

$$|c_B| \leq \frac{1}{24}\sum_{m=k-2}^{\infty}\frac{1}{6^m}\frac{1}{\mu(B)}\int_B |\phi_m(y) - \phi_m(x)| d\mu(y).$$

By using the maximum principle, we get

$$|c_B| \leq \frac{1}{24}\sum_{m=k-2}^{\infty}\frac{1}{6^m} = \frac{9}{5}\cdot\frac{1}{6^k}.$$

Estimate of c_B from below. By symmetry, we may assume that x is located in the $\frac{1}{4}$ region of C_w, the tetrahedron whose vertices are o, p_1, p_2, p_3, where o is the center point in the tetrahedron containing C_w, see Figure 7.

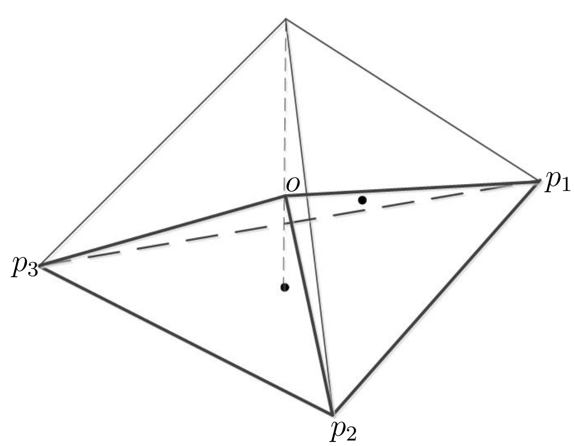

Figure 7. x **in** C_w

By Theorem 2.9 and the proof of Proposition 2.5, we can write $B = C_w \cup \bigcup_{i=1}^{3} E_i$ where each $E_i = B \cap C_i$ with $\mu(E_i) = c_i\mu(C_w)$ for some coefficients $0 \leq c_1, c_2, c_3 \leq 1$.

Let $\tilde{B} = F_0(\mathcal{SG}^4) \cup \bigcup_{i=1}^3 \tilde{E}_i$, with $\tilde{E}_i \subset F_i(\mathcal{SG}^4)$ and $\mu(\tilde{E}_i) = c_i \mu\big(F_0(\mathcal{SG}^4)\big)$, then by Lemma 3.2,

$$c_B = 6^{1-k} c_{\tilde{B}}.$$

Thus we only need to prove that $c_{\tilde{B}}$ has a positive lower bound. So for simplicity in notation, from now on, we take C_w to be $F_0(\mathcal{SG}^4)$ and write

$$B = F_0(\mathcal{SG}^4) \cup \bigcup_{i=1}^3 E_i.$$

In this setting, $p_0 = q_0$ and for $1 \le i \le 3$, $p_i = F_0 q_i$, $C_i = F_i(\mathcal{SG}^4)$ and $E_i = B \cap C_i$.

Write $v^* = \sum_{m=0}^\infty \frac{1}{6^m}\phi_m$ and $c_B^* = M_B(v^*) - v^*(x)$, then $c_B = -\frac{1}{24}c_B^*$. We only need to prove c_B^* has a negative upper bound. This can be done using the following 3 claims.

Claim 1. *$0 \le v^* \le 1$ on $\mathcal{SG}^4$ and v^* takes constant 1 on $\bigcup_{i=0}^3 F_i(\mathcal{SG}^4 \cap T_i)$, where T_i denotes the triangle whose vertices are $V_0 \setminus \{q_i\}$.*

Proof. For $M \ge 0$, write $v_M^* = \sum_{m=0}^M \frac{1}{6^m}\phi_m$. It is a $(M+1)$-piecewise harmonic function on $\mathcal{SG}^4$. We divide the points in V_{M+1} into three parts, $V_{M+1}^{(1)}, V_{M+1}^{(2)}$ and $V_{M+1}^{(3)}$, where $V_{M+1}^{(1)}$ consists of those points lying on $\bigcup_{i=0}^3 F_i(\mathcal{SG}^4 \cap T_i)$, $V_{M+1}^{(2)}$ consists of those points at distance $2^{-(M+1)}$ from $\bigcup_{i=0}^3 F_i(\mathcal{SG}^4 \cap T_i)$, and $V_{M+1}^{(3)}$ consists of the remain points. By using the "$\frac{1}{3} - \frac{1}{6}$" rule inductively, we have $v_M^* \equiv 1$ on $V_{M+1}^{(1)}$, $v_M^* \equiv 1 - \frac{1}{6^M}$ on $V_{M+1}^{(2)}$, and $v_M^* \le 1 - \frac{1}{6^M}$ on $V_{M+1}^{(3)}$. Since v_M^* goes uniformly to v^* and $V_{M+1}^{(1)}$ goes to $\bigcup_{i=0}^3 F_i(\mathcal{SG}^4 \cap T_i)$ as M goes to infinity, the claim follows. □

Claim 2. *For x contained in the tetrahedron whose vertices are o, p_1, p_2, p_3, $v^*(x) \ge \frac{215}{216}$.*

Proof. Observe that for each x in the tetrahedron whose vertices are o, p_1, p_2, p_3, it will be contained in one of the 27 4-cells lying along the face $F_0(\mathcal{SG}^4 \cap T_0)$. Then since v_3^* is harmonic in each such cell, by using the maximum principle and the proof of Claim 1, we have

$$v^*(x) \ge v_3^*(x) \ge 1 - \frac{1}{6^3}.$$

□

Claim 3. $M_B(v^*) \le \frac{39}{40}$.

Proof. It is directly to calculate that for $m \ge 0$,

$$\int_{F_0(\mathcal{SG}^4)} \phi_m d\mu = \frac{1}{4} \cdot \sharp(V_{m+1} \setminus V_m) \cdot \frac{2}{4^{m+2}} = \frac{3}{16}.$$

Thus

$$\int_{F_0(\mathcal{SG}^4)} v^* d\mu = \frac{3}{16} \sum_{m=0}^\infty \frac{1}{6^m} = \frac{9}{40}.$$

So by Claim 1, we have

$$\begin{aligned} M_B(v^*) &= \frac{1}{\mu(B)}\Big(\int_{F_0(\mathcal{SG}^4)} v^* d\mu + \sum_{i=1}^{3}\int_{E_i} v^* d\mu\Big) \\ &\leq \frac{\frac{9}{40}+\sum_{i=1}^3 \mu(E_i)}{\mu(F_0(\mathcal{SG}^4))+\sum_{i=1}^3 \mu(E_i)} \\ &= \frac{\frac{9}{10}+\sum_{i=1}^3 c_i}{1+\sum_{i=1}^3 c_i} \leq \frac{\frac{9}{10}+3}{1+3} = \frac{39}{40}, \end{aligned}$$

which completes the proof. □

Combining Claim 2 and 3, we have proved that $c_B^* \leq \frac{39}{40} - \frac{215}{216} = -\frac{11}{540}$. Hence

$$c_B \geq \frac{1}{24}\cdot\frac{11}{540} > 0.$$

This completes the proof. □

Proof of Theorem 1.2. The estimation of $c_{B_k(x)}$ follows from Proposition 3.3.

For $x \in \mathcal{SG}^4 \setminus V_0$ and C_w a k-cell containing x, not intersecting V_0. Then $C_w \subset B_k(x) \subset D_w$. Let $u \in dom\Delta$ satisfy the Hölder condition. We write

$$u = h^{(k)} + \big(\Delta u(x)\big)v + r^{(k)}$$

where $h^{(k)}$ is harmonic in C_w and $h^{(k)} + \big(\Delta u(x)\big)v$ assumes the same boundary values as u at the boundary of C_w.

We first prove that the remainder $r^{(k)}$ satisfies $r^{(k)} = O\big((\frac{\gamma}{6})^k\big)$ on $B_k(x)$. In fact, it is easy to check that $\Delta r^{(k)}(\cdot) = \Delta u(\cdot) - \Delta u(x)$ and the value of $r^{(k)}$ vanishes at the boundary of C_w. So $r^{(k)}$ is given by the integral of $\Delta u(\cdot) - \Delta u(x)$ against a scaled Green's function on C_w. Noticing that the scaling factor is $(\frac{1}{6})^k$ and $|\Delta u(\cdot) - \Delta u(x)| \leq c\gamma^k$ on C_w, we then get $r^{(k)} = O\big((\frac{\gamma}{6})^k\big)$ on C_w, and thus on $B_k(x)$.

Now we come to prove (1.5). Since $M_{B_k(x)}(h^{(k)}) - h^{(k)}(x) = 0$ and $M_{B_k(x)}(v) - v(x) = c_{B_k(x)}$, we have

$$\frac{1}{c_{B_k(x)}}\big(M_{B_k(x)}(u) - u(x)\big) - \Delta u(x) = \frac{1}{c_{B_k(x)}}\big(M_{B_k(x)}(r^{(k)}) - r^{(k)}(x)\big).$$

Noticing that $r^{(k)} = O\big((\frac{\gamma}{6})^k\big)$ and by the estimation of $c_{B_k(x)}$ in Proposition 3.3, we then have

$$\frac{1}{c_{B_k(x)}}\big(M_{B_k(x)}(u) - u(x)\big) - \Delta u(x) = O(\gamma^k).$$

Hence by letting $k \to \infty$, we finally get (1.5). □

References

[1] J. Azzam, M.A. Hall and R.S. Strichartz, Conformal energy, conformal Laplacian, and energy measures on the Sierpinski gasket, *Trans. Amer. Math. Soc.*, 360 (2008), 2089-2131.
[2] R. Bell, C.W. Ho and R.S. Strichartz, Energy measures of harmonic functions on the Sierpinski gasket, *Indina U. Math. J.*, 63:3(2013), 831-868.
[3] M.T. Barlow and J. Kigami, Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets, *J. London Math. Soc.*, 56 (1997), 320-332.
[4] O. Ben-Bassat, R.S. Strichartz and A. Teplyaev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals, *J. Funct. Anal.*, 166(1999), 197-217.
[5] S.P. Cao and H. Qiu, Some properties of the derivatives on Sierpinski gasket type fractals, to appear in *Constr. Approx.*
[6] K. Dalrymple, R.S. Strichartz and J.P. Vinson, Fractal differential equations on the Sierpinski gasket, *J. Fourier Anal. Appl.*, 5(1999), 203-284.
[7] M. Fukushima and T. Shima, On a spectral analysis for the Sierpinski gasket, *Potential Anal.*, 1 (1992),1-35.
[8] M. Hino, Some properties of energy measures on Sierpinski gasket type fractals, *J. Fractal Geom.*, 3(2016), 245-263.
[9] J. Kigami, A harmonic calculus on the Sierpinski spaces, *Japan J. Appl. Math.*, 6:2(1989), 259-290.
[10] J. Kigami, Harmonic calculus on p.c.f. self-similar sets, *Trans. Amer. Math. Soc.*, 335:2(1993), 721-755.
[11] J. Kigami, Distribution of localized eigenvalues of Laplacian on p.c.f. self-similar sets, *J. Funct. Anal.*, 128 (1998), 170-198.
[12] J. Kigami, Analysis on fractals, in *Cambridge Tracts in Mathematics*, Vol. 143 (Cambridge University Press, 2001).
[13] J. Kigami and M.L. Lapidus, Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals, *Comm. Math. Phys.*, 158 (1993), 93-125.
[14] H. Qiu and R.S. Strichartz, Mean value properties of harmonic functions on Sierpinski gasket type fractals, *J. Fourier Anal. Appl.*, 19(2013), 943–966.
[15] L.G. Rogers and R.S. Strichartz, Distribution theory on p.c.f. fractals, *J. Anal. Math.*, 112 (2010), 137-191.
[16] T. Shima, On eigenvalue problems for Laplacians on p.c.f. self-similar sets, *Japan J. Indust. Appl. Math.*, 13(1996), 1-23.
[17] R.S. Strichartz, Fractals in large, *Canad. J. Math.*, 50 (1998), 638-657.
[18] R.S. Strichartz, Some properties of Laplacians on fractals, *J. Funct. Anal.*, 164 (1999), 181-208.
[19] R.S. Strichartz, Taylor approximations on Sierpinski gasket type fractals, *J. Funct. Anal.*, 174(2000), 76-127.
[20] R.S. Strichartz, Function spaces on fractals, *J. Funct. Anal.*, 198:1(2003), 43-83.
[21] R.S. Strichartz, Solvability for differential equations on fractals, *J. Anal. Math.*, 96(2005), 247-267.
[22] R.S. Strichartz, Differential Equations on Fractals: A Tutorial (Princeton University Press, 2006).
[23] R.S. Strichartz and M. Usher, Splines on fractals, *Math. Proc. Cambridge Philos. Soc.*, 129(2000), 331-360.
[24] A. Teplyaev, Spectral analysis on infinite Sierpinski gaskets, *J. Funct. Anal.*, 159(1998), 537-567.
[25] A. Teplyaev, Gradients on fractals, *J. Funct. Anal.*, 174 (2000), 128-154.

Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, P.R. China

E-mail address: huaqiu@nju.edu.cn

Institute of Science, PLA University of Science and Technology, Nanjing 211101, P. R. China

E-mail address: ypwupla@163.com

Institute of Science, PLA University of Science and Technology, Nanjing 211101, P. R. China

E-mail address: flyyaok@sina.com