# Exact Hausdorff and packing measures of Cantor sets with overlaps

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**Abstract.**

Let $K$ be the attractor of a linear iterated function system (IFS) $S_j(x) = \rho_j x + b_j$, $j = 1, \cdots, m$, on the real line $\mathbb{R}$ satisfying the generalized finite type condition (whose invariant open set $\mathcal{O}$ is an interval) with an irreducible weighted incidence matrix. This condition was introduced by Lau & Ngai recently as a natural generalization of the open set condition, allowing us to include many important overlapping cases. They showed that the Hausdorff and packing dimensions of $K$ coincide and can be calculated in terms of the spectral radius of the weighted incidence matrix. Let $\alpha$ be the dimension of $K$. In this paper, we state that

$$\mathcal{H}^\alpha(K \cap J) \le |J|^\alpha$$

for all intervals $J \subset \overline{\mathcal{O}}$, and

$$\mathcal{P}^\alpha(K \cap J) \ge |J|^\alpha$$

for all intervals $J \subset \overline{\mathcal{O}}$ centered in $K$, where $\mathcal{H}^\alpha$ denotes the $\alpha$-dimensional Hausdorff measure and $\mathcal{P}^\alpha$ denotes the $\alpha$-dimensional packing measure. This result extends a recent work of Olsen where the open set condition is required. We use these inequalities to obtain some precise density theorems for the Hausdorff and packing measures of $K$. Moreover, using these densities theorems, we describe a scheme for computing $\mathcal{H}^\alpha(K)$ exactly as the minimum of a finite set of elementary functions of the parameters of the IFS. We also obtain an exact algorithm for computing $\mathcal{P}^\alpha(K)$ as the maximum of another finite set of elementary functions of the parameters of the IFS. These results extend previous ones by Ayer & Strichartz and Feng, respectively, and apply to some new classes allowing us to include Cantor sets in $\mathbb{R}$ with overlaps.

## 1 Introduction and Statement of Results

In this paper we will analyze the behavior of the Hausdorff and packing measures of self-similar sets satisfying the generalized finite type condition, which is weaker than the open set condition. In particular, we will deal with the exact calculating of the Hausdorff and packing measures for a kind of Cantor sets in $\mathbb{R}$ with overlaps.

The problem of calculating the dimension of the attractor of a self-similar iterated function system (IFS) is one of the most interesting objects in fractal geometry. During

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the past two decades there has been an enormous body of literatures investigating this problem and wide ranging generalizations thereof. See the books [2], [4], [14] and the references therein. Let $\{S_j\}_{j=1}^m$ be an IFS of contractive similitudes on $\mathbb{R}^d$ defined as

$$S_j(x) = \rho_j R_j x + b_j, \quad j = 1, \cdots, m, \tag{1.1}$$

where $0 < \rho_j < 1$ is the contraction ratio, $R_j$ is an orthogonal transformation and $b_j \in \mathbb{R}^d$, for each $j$. Let $K$ denote the *self-similar set* (or *attractor*) of the IFS, namely, $K$ is the unique non-empty compact set in $\mathbb{R}^d$ satisfying

$$K = \bigcup_{j=1}^m S_j(K).$$

A basic result (see [3]) is that the Hausdorff dimension $\dim_H K$ and the packing dimension $\dim_P K$ are always equal for the self-similar set $K$, i.e.,

$$\dim_H K = \dim_P K.$$

In general it is quite difficult to calculate this common value, except the well-known classical result (see Moran [17], Hutchinson [7]) that if the IFS satisfies the *open set condition* (OSC), i.e., there exists a non-empty bounded open set $\mathcal{O} \subset \mathbb{R}^d$ such that $\bigcup_{j=1}^m S_j(\mathcal{O}) \subset \mathcal{O}$ and $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \emptyset$ for all $i \neq j$, then the dimension of $K$ is the unique solution $\alpha$ of the equation

$$\sum_{j=1}^m \rho_j^\alpha = 1. \tag{1.2}$$

Non-overlapping or almost non-overlapping self-similar IFSs have been studied in great detail via the OSC.

In the absence of the OSC, much less is known about IFSs with overlaps. To deal with such systems, by extending a method of Lalley [9] and Rao &Wen [22], Ngai & Wang [19] formulated a weaker separation condition, the *finite type condition* (FTC), which may include many important overlapping cases, and described a method for computing the Hausdorff and packing dimensions of the attractor in terms of the spectral radius of an associated weighted incidence matrix. The FTC requires the contraction ratios of the IFS's maps to be exponentially commensurable and thus does not generalize the OSC. Recently, Lau and Ngai [10], Jin and Yau[8] independently, introduced a more general condition, the *generalized finite type condition* (GFTC), which do not need the above requirement, that extends both the OSC and the FTC. Under the GFTC, one can still compute the dimension of the attractor in terms of the spectral radius of a weighted incidence matrix.

Another central problem concerning the theory of self-similar IFSs is to estimate the Hausdorff and packing measures of self-similar sets, which is also an area of active researches [1, 6, 12, 13, 24]. In these papers and the references therein one can find some estimations of the values of the measures of some particular self-similar constructions. Since the definitions of these measures are sometimes awkward to work with, there are only very few non-trivial examples of sets whose exact measures are known. [24] is a recent review of relevant open problems in this field. So far most of these researches have been mainly addressed to the determination of the upper and lower bounds of the measures. With regard to the exact values, two papers, [1] and [6], should be mentioned.

In [1], Ayer and Strichartz considered a kind of Cantor set $K$ which is the attractor of a linear IFS $S_j(x) = \rho_j x + b_j$, $j = 1, \cdots, m$, on the real line satisfying the OSC (where the open set is the interval (0,1)). Let $\alpha$ be the dimension of $K$. They gave an algorithm for computing the Hausdorff measure $\mathcal{H}^\alpha(K)$ exactly as the minimum of a finite set of elementary functions of the parameters of the IFS by using the fact that the exact value of $\mathcal{H}^\alpha(K)$ is the inverse of the maximal density of intervals contained in $[0,1]$, with respect to the normalized measure $\lambda$ of $\mathcal{H}^\alpha$ restricted to $K$, where $\lambda = \mathcal{H}^\alpha|_K/\mathcal{H}^\alpha(K)$. It should be pointed out that if the OSC is satisfied, then $\mathcal{H}^\alpha|_K$ and $\mathcal{P}^\alpha|_K$ are proportional. Hence $\lambda$ is also equal to $\mathcal{P}^\alpha|_K/\mathcal{P}^\alpha(K)$.

On the other hand, in [6], Feng proved that the packing measure $\mathcal{P}^\alpha(K)$ is equal to the inverse of the minimal density of intervals centered in $K$ with respect to $\lambda$, which also yields an explicit formula for calculating the exact value of $\mathcal{P}^\alpha(K)$ in terms of the parameters of the IFS.

However, in these papers, one needs to work on self-similar sets under the OSC. To the best of our knowledge, there is no result concerning the exact Hausdorff or packing measures of self-similar sets without the OSC. Since the calculation of the dimension of self-similar sets under the OSC can be successfully extended to those sets satisfying the GFTC which includes many interesting overlapping cases, and in view of the above discussion, it is natural to ask whether the Hausdorff or packing measure of the Cantor set $K$ under only the GFTC can also be calculated exactly. This is the main goal of this paper.

In [1] and [6], to get the exact values of $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ of $K$ under the OSC (where the open set is the interval (0,1)), the following explicit formulae play a key role.

$$\mathcal{H}^\alpha(K)^{-1} = \sup\{\frac{\lambda(J)}{|J|^\alpha} : J \text{ is an interval with } J \subset [0,1]\}, \tag{1.3}$$

and

$$\mathcal{P}^\alpha(K)^{-1} = \inf\{\frac{\lambda(J)}{|J|^\alpha} : J \text{ is an interval centered in } K \text{ with } J \subset [0,1]\}. \tag{1.4}$$

Formula (1.3) was implicit in earlier work by Marion [12, 13] and in [1] by Ayer & Strichartz, while formula (1.4) was proved in a direct and elementary way in [6] by Feng.

Recently, Morán [18] and Olsen [20] extended the above two formulae to the higher dimensional case independently. In [18] the so-called self-similar tiling principle plays a central role in the proof. This principle says that any open subset $U$ of $K$ can be tiled by a countable set of similar copies of an arbitrarily given closed set with positive Hausdorff or packing measure while the tiling is exact in the sense that the part of $U$ which cannot be covered by the tiles is of null measure.

The proof in [20] is quite different from that in [18]. Let $K$ be a self-similar set in $\mathbb{R}^d$ as described in (1.1) with the dimension $\alpha$, under the OSC or the *strong separation condition* (SSC). Recall that in [20] Olsen performs a detailed analysis of the behavior of the Hausdorff measure $\mathcal{H}^\alpha(K \cap U)$ and the packing measure $\mathcal{P}^\alpha(K \cap B(x,r))$ of small convex Borel sets $U$ and balls $B(x,r)$. In particular, he showed that if $K$ is under the OSC, then

$$\mathcal{H}^\alpha(K \cap U) \leq |U|^\alpha$$

for each convex Borel set $U$. A dual result for the packing measure was also proved which says that

$$\mathcal{P}^\alpha(K \cap B(x,r)) \geq (2r)^\alpha$$

for each $x \in K$ and small $r > 0$ if $K$ satisfies the SSC. The latter result was generalized to the OSC case by the author recently (see [21]) in proving the continuity of the packing measure function of self-similar IFSs, which says that the above inequality actually holds for each $B(x,r)$ contained in $\mathcal{O}$ with $x \in K$, where $\mathcal{O}$ is an open set associated with the OSC, satisfying $\mathcal{O} \cap K \neq \emptyset$. (There should exist such $\mathcal{O}$ since the OSC is equivalent to the strong OSC. See [23].)

To match our question precisely we restrict our interest to the Cantor set $K$ defined before. Hence the above two formulae are rewroted in the following form, i.e.,

$$\mathcal{H}^\alpha(K \cap J) \leq |J|^\alpha \tag{1.5}$$

for all intervals $J \subset [0,1]$, and

$$\mathcal{P}^\alpha(K \cap J) \geq |J|^\alpha \tag{1.6}$$

for all intervals $J \subset [0,1]$ centered in $K$.

As an application of (1.5) and (1.6), Olsen reproved formulae (1.3) and (1.4) (he proved the general higher dimensional case) using the classical density theorems of geometric measure theory which were stated for arbitrary subsets of Euclidean space [2, 14].

Formulae (1.3) and (1.4) say that the exact values of $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ coincide with the inverses of the supremum and infimum of the densities of $\lambda$ on suitable classes of sets, respectively.

In order to calculate the exact values of the measures of $K$ satisfying the GFTC, we need to establish analogous explicit formulae of (1.3) and (1.4). Following Olsen's work, two inequalities similar to (1.5) and (1.6) are required. Recall that in proving (1.5) and (1.6), one should find optimal coverings and packings in a self-similar setting which require almost non-overlap among the various similar pieces into which the fractal decomposes. In view of this, it is therefore entirely plausible that the OSC is indispensible. In the present paper, somewhat surprisingly, we will show that the formulae (1.5) and (1.6) still hold under the assumption that the weighted incidence matrix of $K$ is irreducible where $K$ is required to satisfy only the GFTC. This leads to the following results.

**Theorem 1.1.** *Let $K$ be a Cantor set in $\mathbb{R}$ satisfying the GFTC with respect to the invariant open set $(0,1)$ with an irreducible weighted incidence matrix $A_\alpha$, where $\alpha$ is the Hausdorff dimension of $K$. Then*

$$\mathcal{H}^\alpha(K \cap J) \leq |J|^\alpha \tag{1.7}$$

*for all intervals $J \subset [0,1]$.*

**Theorem 1.2.** *Let $K$ be the Cantor set described as before. Then*

$$\mathcal{P}^\alpha(K \cap J) \geq |J|^\alpha \tag{1.8}$$

*for all intervals $J \subset [0,1]$ centered in $K$.*

We will give the detailed definition of the GFTC and the exact concept of the weighted incidence matrix $A_\alpha$ in Section 2.

The idea of establishing Theorem 1.1 and Theorem 1.2 is the following. We first observe that if the weighted incidence matrix $A_\alpha$ of $K$ is irreducible, then $K$ can be decomposed into an union of a set $K_a$ with a graph directed construction and an attractor $K_b$ of a countable infinite IFS under the OSC (see [15], [16] for further properties of the graph directed sets and the infinite IFSs, respectively). Moreover, the dimension of $K_a$ is strictly less than that of $K_b$. Hence the subset $K_a$ will have null $\alpha$-dimensional Hausdorff (packing) measure which ensures us to consider $K_b$ instead of $K$. Noticing that $K_b$ is an attractor of a countable infinite IFS satisfying the OSC, it is possible to adapt the techniques for proving (1.5) and (1.6) to establish Theorem 1.1 and Theorem 1.2.

By a similar discussion for self-similar sets under the OSC, the $\alpha$-dimensional Hausdorff measure restricted to $K_b$ and the $\alpha$-dimensional packing measure restricted to $K_b$ are also proportional. Obviously the above fact still holds if we replace $K_b$ by $K$. We still write the normalized measure of $\mathcal{H}^\alpha$ restricted to $K$ as $\lambda$, then $\lambda = \mathcal{H}^\alpha|_K/\mathcal{H}^\alpha(K) = \mathcal{P}^\alpha|_K/\mathcal{P}^\alpha(K)$. We will show the $\lambda$-measure of some special kind of sets called *islands* of $K$ can be expressed in terms of the parameters of the IFS of $K$. Then following Olsen's frame work, we use the inequalities (1.7) and (1.8) to get the following explicit formulae for $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$, analogous to (1.3) and (1.4).

**Corollary 1.3.** *Let $K$ be the Cantor set described as before. Then*

$$\mathcal{H}^\alpha(K)^{-1} = \sup\{\frac{\lambda(J)}{|J|^\alpha} : J \text{ is an interval with } J \subset [0,1]\}. \tag{1.9}$$

**Corollary 1.4.** *Let $K$ be the Cantor set described as before. Then*

$$\mathcal{P}^\alpha(K)^{-1} = \inf\{\frac{\lambda(J)}{|J|^\alpha} : J \text{ is an interval centered in } K \text{ with } J \subset [0,1]\}. \tag{1.10}$$

These corollaries extend the results in [18, 20, 21]. Following the technique frame of [1, 6], under suitable assumptions, we then give an algorithm for computing $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ exactly as the inverse of the maximal or minimal value of suitable finite sets of elementary functions of the parameters of the IFS respectively. This is possible since we could make a detailed analysis of $\lambda$, and thus a detailed analysis of the supremum in (1.9) and the infimum in (1.10) respectively. It should be mentioned here that we may allow touching islands, and indeed this case will lead to some complicated and interesting phenomena. Due to the fact that the self-similar construction of $K$ under the GFTC is much more complicated than that under the OSC, our description of the exact calculations of the two kinds of measures will need some new important notations and techniques. We will describe a big scheme for the exact computing, which is a major adaption of the techniques used in [1] and [6].

This paper is organized as follows. In Section 2, we give some notations and basic facts about the GFTC. Our description of the GFTC is slightly different but equivalent to the original version in [10]. In Section 3, we deal with the density theorems for the Hausdorff and packing measures of the Cantor sets in $\mathbb{R}$ satisfying the GFTC. Firstly, we give the proofs of Theorem 1.1 and Theorem 1.2 respectively. Secondly, we prove the formulae in Corollary 1.3 and Corollary 1.4 using the classical density theorems of geometry measure theory. Throughout this section and the following ones, we will always assume that the weighted incidence matrix of $K$ is irreducible. In Section 4, we focus on the calculation of the exact measures of the Cantor $K$ in $\mathbb{R}$ under some suitable assumptions. A scheme is provided. Section 5 collects some further discussions on this subject. We consider the possibility of dropping some assumption required in Section 4. We discuss briefly the slightly more general cases of IFSs that contain orientation reversing simlarities. We also consider the situation in higher dimensional Euclidean spaces and show why our results can not be generalized. Throughout the context, we will show some interesting and non-trivial examples.

## 2 Cantor sets in $\mathbb{R}$ under the GFTC

For convenience, we introduce a slightly different but equivalent description of the GFTC defined in [10]. We will focus the interest on the Cantor sets in the real line $\mathbb{R}$.

We will use the following notations throughout the paper. For a Borel measure $\nu$ on $\mathbb{R}$ and a Borel set $E$, we let $\nu|_E$ denote the restriction of $\nu$ to $E$. For any subset $E \subset \mathbb{R}$, we denote the *diameter* of $E$ by $|E|$. For any $x \in \mathbb{R}$, let $\mathrm{dist}(x, E)$ denote the distance between $x$ and $E$, namely, $\mathrm{dist}(x, E) = \inf\{|x - y| : y \in E\}$. If $A$ is any finite or countable set, we denote by $\sharp A$ the *cardinality* of $A$. For $E \subset \mathbb{R}^d$, $s \geq 0$ and $\delta > 0$, put $\mathcal{H}^s_\delta(E) := \inf\{\sum_i |U_i|^s\}$, where the infimum is taken over all $\delta$-*coverings* of $E$, i.e., countable collections $\{U_i\}$ of subsets of $\mathbb{R}^d$ with diameters smaller than $\delta$ such that $E \subset \bigcup_i U_i$.

Let $\{S_1, \cdots, S_m\}$ be a linear IFS of contractive similitudes on the line $\mathbb{R}$ defined by

$$S_j(x) = \rho_j x + b_j, j = 1, \cdots, m, \tag{2.1}$$

with contraction ratios satisfying $0 < |\rho_j| < 1$. Let $K$ denote its attractor. Let $\Sigma = \{1, \cdots, m\}$ and let $\Sigma_* := \bigcup_{k=0}^{\infty} \Sigma_k$ be the symbolic space representing the IFS (by convention, $\Sigma_0 = \emptyset$). For $\mathbf{i} = (i_1, \cdots, i_k) \in \Sigma_*$, $E \subset \mathbb{R}$, we use the standard notation $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_k}, \rho_{\mathbf{i}} := \rho_{i_1} \cdots \rho_{i_k}, E_{\mathbf{i}} = S_{\mathbf{i}}(E)$, with $S_\emptyset := I$, the identity map on $\mathbb{R}$, and $\rho_\emptyset := 1$. Let $\mathbf{i} = (i_1, \cdots, i_k)$ and $\mathbf{j} = (j_1, \cdots, j_{k'})$ be two indices of $\Sigma_*$. The *length* of $\mathbf{i}$ is $|\mathbf{i}| = k$. We write $\mathbf{i} \preceq \mathbf{j}$ if $\mathbf{i}$ is an initial segment of $\mathbf{j}$, and write $\mathbf{i} \not\preceq \mathbf{j}$ if $\mathbf{i}$ is not an initial segment of $\mathbf{j}$. $\mathbf{i}$ and $\mathbf{j}$ are *incomparable* if neither $\mathbf{i} \preceq \mathbf{j}$ nor $\mathbf{j} \preceq \mathbf{i}$.

Let $\{\mathcal{M}_k\}_{k=0}^{\infty}$ be a sequence of index sets, where $\mathcal{M}_k \subset \Sigma_*$ for all $k \geq 0$ and $\mathcal{M}_0 = \Sigma_0$. We say that $\{\mathcal{M}_k\}_{k=0}^{\infty}$ is a *sequence of nested index sets* if it satisfies the following conditions:

(1) Both $\{\min\{|\mathbf{i}| : \mathbf{i} \in \mathcal{M}_k\}\}_{k=0}^{\infty}$ and $\{\max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{M}_k\}\}_{k=0}^{\infty}$ are non-decreasing and have infinity limit;

(2) For each $k \geq 0$, all $\mathbf{i}, \mathbf{j} \in \mathcal{M}_k$ are incomparable if $\mathbf{i} \neq \mathbf{j}$;

(3) For each $\mathbf{j} \in \Sigma_*$ with $|\mathbf{j}| > \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{M}_k\}$, there exists $\mathbf{i} \in \mathcal{M}_k$ such that $\mathbf{i} \preceq \mathbf{j}$;

(4) For each $\mathbf{j} \in \Sigma_*$ with $|\mathbf{j}| < \min\{|\mathbf{i}| : \mathbf{i} \in \mathcal{M}_k\}$, there exists $\mathbf{i} \in \mathcal{M}_k$ such that $\mathbf{j} \preceq \mathbf{i}$;

(5) There exists a positive integer $L$ such that for all $\mathbf{i} \in \mathcal{M}_k$ and $\mathbf{j} \in \mathcal{M}_{k+1}$ with $\mathbf{i} \preceq \mathbf{j}$, we have $|\mathbf{j}| - |\mathbf{i}| \leq L$, where $L$ is independent of $k$.

For general sequences, we allow $\mathcal{M}_k \cap \mathcal{M}_{k+1} \neq \emptyset$ and $\bigcup_{k=0}^{\infty} \mathcal{M}_k$ may be a proper subset of $\Sigma_*$. If we let $\mathcal{M}_k = \Sigma_k$ for all $k \geq 0$, we get a canonical such sequence. For $k \geq 0$, let $\Lambda_k := \{\mathbf{i} = (i_1, \cdots, i_n) \in \Sigma_* : |\rho_{\mathbf{i}}| \leq \rho_{\min}^k < |\rho_{i_1,\cdots,i_{n-1}}|\}$, where $\rho_{\min} = \min\{|\rho_j| : 1 \leq j \leq m\}$. It is easy to check that $\{\Lambda_k\}_{k=0}^{\infty}$ is also a sequence of nested index sets, which is used to define the FTC in [19].

Fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$. Note that if $\mathcal{O} \subset \mathbb{R}$ is a non-empty bounded open set which is *invariant* under $\{S_j\}_{j=1}^m$, i.e., $\bigcup_{j=1}^m \mathcal{O}_j \subset \mathcal{O}$, then $\{\bigcup_{\mathbf{i} \in \mathcal{M}_k} \mathcal{O}_{\mathbf{i}}\}_{k=0}^{\infty}$ becomes a sequence of nested subsets of $\mathbb{R}$. For each integer $k \geq 0$, let $\mathcal{V}_k$ be the set of *vertices* defined as $\mathcal{V}_k := \{(S_{\mathbf{i}}, k) : \mathbf{i} \in \mathcal{M}_k\}$. We call $(I, 0)$ the *root vertex* and let $\mathcal{V} := \bigcup_{k \geq 0} \mathcal{V}_k$. Note that if $S_{\mathbf{i}} = S_{\mathbf{j}}$ for some $\mathbf{i} \neq \mathbf{j} \in \mathcal{M}_k$, they determine the same vertex. For $\mathbf{v} = (S_{\mathbf{i}}, k) \in \mathcal{V}_k$, we introduce the convenient notation $S_{\mathbf{v}} := S_{\mathbf{i}}$ and $\rho_{\mathbf{v}} := \rho_{\mathbf{i}}$. The notation $S_{\mathbf{v}}$ allows us to refer to a vertex in $\mathcal{V}_k$ without explicitly specifying the index $\mathbf{i}$.

For simplicity, we assume that there is a following form of invariant set which will be used in the definition of the GFTC: an open interval $\mathcal{O}$ which is invariant under $\{S_j\}_{j=1}^m$. (There are examples where the GFTC holds, but not with an open interval.) Without loss of generality we take $\mathcal{O} = (0, 1)$.

For any $k \geq 0$, let $F_k = \bigcup_{\mathbf{v} \in \mathcal{V}_k} \overline{\mathcal{O}}_{\mathbf{v}}$. Notice that $\overline{\mathcal{O}}_{\mathbf{v}} \subset \overline{\mathcal{O}}$. Then we have

$$K = \bigcap_{k=0}^{\infty} F_k.$$

For each $\mathbf{v} \in \mathcal{V}_k$, $\overline{\mathcal{O}}_{\mathbf{v}}$ is a sub-interval contained in $[0, 1]$ with endpoints $S_{\mathbf{v}}(0)$ and $S_{\mathbf{v}}(1)$. We call $\overline{\mathcal{O}}_{\mathbf{v}}$ a *$k$-th generation interval* of $K$.

We call two intervals are *separate* if they have at most one common point. Otherwise, we call they are *overlapping*. From the definition of $F_k$, we see that $\bigcup_{\mathbf{v} \in \mathcal{V}_k} \mathcal{O}_{\mathbf{v}}$ consists of some separate open intervals, and each open interval is a union of the interiors of one or several $k$-th generation intervals. We call the closure of each such open interval a *$k$-th generation island*, and use $\mathcal{F}_k^0$ to denote the set of all $k$-th generation islands. Call the unique element $\overline{\mathcal{O}} = [0, 1]$ in $\mathcal{F}_0^0$ the *root island*. For the open intervals between each pair of the $k$-th generation islands, we call them *lakes*. Let $\mathcal{F}_k$ denote the collection of intervals whose endpoints are taken from those of islands in $\mathcal{F}_k^0$, and call $\mathcal{F}_k$ the *finite field* generated by $\mathcal{F}_k^0$. For each $k$-th generation island $I$, we use $V(I)$ to denote the vertices set of all $k$-th generation intervals contained in $I$, i.e.,

$$V(I) = \{\mathbf{v} \in \mathcal{V}_k : \overline{\mathcal{O}}_{\mathbf{v}} \subset I\}.$$

It is easy to verify that $I = \bigcup_{\mathbf{v} \in V(I)} \overline{\mathcal{O}}_{\mathbf{v}}$. We call each such interval $\overline{\mathcal{O}}_{\mathbf{v}}$ a *component interval* of $I$. Let $I \in \mathcal{F}_k^0$ and $I' \in \mathcal{F}_{k+1}^0$ for some $k \geq 0$. Then either $I' \subset I$ or they are separate. If it is the first case, we call $I$ a *parent* of $I'$ and $I'$ an *offspring*(or *descendant*) of $I$.

We define an equivalence relation on $\mathcal{F}^0 := \bigcup_{k\geq 0}\mathcal{F}_k^0$ to identify islands that are isomorphic in the sense that they behave the same overlap type.

**Definition 2.1.** *Two islands $I\in\mathcal{F}_k^0$ and $I'\in\mathcal{F}_{k'}^0$ are* equivalent, *denoted by $I\sim I'$, if there is a linear function $\tau$ mapping $I$ onto $I'$, such that the following conditions are satisfied:*

*(1) $\{S_{\boldsymbol{v}'}:\boldsymbol{v}'\in V(I')\}=\{\tau\circ S_{\boldsymbol{v}}:\boldsymbol{v}\in V(I)\}$;*

*(2) For any positive integer $l\geq 1$, there is an island $J\in\mathcal{F}_{k+l}^0$ contained in $I$ if and only if there is also an island $J'\in\mathcal{F}_{k'+l}^0$ contained in $I'$ where $J'=\tau(J)$.*

It is easy to see that $\sim$ is an equivalence relation. We denote the equivalence class containing $I$ by $[I]$ and call it the *overlap type* of $I$. Condition (2) says that any two islands with the same overlap type have equivalent offspring.

**Definition 2.2.** *We say that a linear IFS of contractive similitudes on $\mathbb{R}$ satisfies the generalized finite type condition (GTFC), with respect to the invariant set $\mathcal{O}=(0,1)$, if $\mathcal{O}$ is an invariant set under the IFS and there is a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$, such that $\mathcal{F}^0/\sim=\{[I]:I\in\mathcal{F}^0\}$ is a finite set.*

It is easy to see that the IFS satisfies the GTFC if and only if there exists some $k_0\geq 0$ such that none of the islands in $\mathcal{F}_{k_0+1}^0$ is of a new overlap type. Let $\mathcal{T}_1,\cdots,\mathcal{T}_q$ denote all the distinct overlap types, with $\mathcal{T}_1=[\overline{\mathcal{O}}]$.

For each $\alpha\geq 0$ we define a *weighted incidence matrix* $A_\alpha=A_\alpha(i,j)_{i,j=1}^q$ as follows. Fix $i(1\leq i\leq q)$ and an island $I\in\mathcal{F}^0$ with $[I]=\mathcal{T}_i$. Suppose that $I$ is a $k$-th generation island, let $I_1,\cdots,I_l$ be the offspring of $I$ in $\mathcal{F}_{k+1}^0$. Then we define

$$A_\alpha(i,j):=\sum\{\left(\frac{|I_s|}{|I|}\right)^\alpha:[I_s]=\mathcal{T}_j,1\leq s\leq l\}.$$

It is easy to see that the definition of $A_\alpha(i,j)$ is independent of the choice of $I$.

**Theorem 2.3**[10]. *Let $\lambda_\alpha$ be the spectral radius of the associated weighted incidence matrix $A_\alpha$. Then*

$$\dim_H K=\dim_P K=\alpha,$$

*where $\alpha$ is the unique number such that $\lambda_\alpha=1$. Moreover, $0<\mathcal{H}^\alpha(K)<\infty$.*

For the convenience of the readers, we would like to give a direct and elementary proof of this result. The proof makes use of the ideas in [10] and [15].

*Proof.* We need to define a natural probability measure $\mu$ on $K$. Let $(a_1,\cdots,a_q)^T$ be an 1-eigenvector of $A_\alpha$, normalized so that $a_1=1$. (This is possible because all overlap types are descendants of $\mathcal{T}_1$.) Here $\alpha$ is the unique number such that $\lambda_\alpha=1$. For each island $I$, where $I\in\mathcal{F}_k^0$ and $[I]=\mathcal{T}_i$ for some $k\geq 0$ and $1\leq i\leq q$, we let $\mu(I)=|I|^\alpha a_i$. Obviously, $\mu([0,1])=a_1=1$.

To show that $\mu$ is indeed a probability measure on $K$, we notice that two islands $I\in\mathcal{F}_k^0$ and $I'\in\mathcal{F}_{k'}^0$ with $k\leq k'$, are overlapping if and only if either $I=I'$ in the case $k=k'$ or $I'$ is a descendant of $I$ in the case $k<k'$. In both cases, $I'\subset I$. Now let $I\in\mathcal{F}_k^0$

and let $\mathcal{D}$ denote the set of all offspring of $I$ in $\mathcal{F}^0_{k+1}$. Then

$$\begin{aligned}\sum_{I'\in\mathcal{D}}\mu(I') &= \sum_{j=1}^{q}\sum\{\mu(I') : I'\in\mathcal{D}, [I'] = \mathcal{T}_j\} = |I|^\alpha\sum_{j=1}^{q}\sum\{\left(\frac{|I'|}{|I|}\right)^\alpha a_j : I'\in\mathcal{D}, [I'] = \mathcal{T}_j\} \\ &= |I|^\alpha\sum_{j=1}^{q}A_\alpha(i,j)a_j = |I|^\alpha a_i = \mu(I).\end{aligned}$$

It follows now from $\mu([0,1]) = 1$ that $\mu$ is indeed a probability measure on $K$.

*Lower bound.* Let $E$ be a bounded Borel subset of $\mathbb{R}$ and let $\mathcal{N}(E)$ be defined as

$$\mathcal{N}(E) := \{I\in\mathcal{F}^0 : |I| \le |E| < |\mathcal{P}(I)| \text{ and } I\cap E\neq\emptyset\},$$

where $\mathcal{P}(I)$ denotes the parent of the island $I$. It is easy to verify that for any bounded Borel set $E\subset\mathbb{R}$, $\sharp\mathcal{N}(E) \le C_0 := \max\{|\mathcal{P}(I)|/|I| : I\in\mathcal{F}^0\} + 2$. Note that $\mu(E) \le \sum_{I_j\in\mathcal{N}(E)}\mu(I_j)$. If we assume that $[I_j] = \mathcal{T}_{i_j}$, then

$$\mu(E) \le \sum_{I_j\in\mathcal{N}(E)}|I_j|^\alpha a_{i_j} \le |E|^\alpha\sum_{I_j\in\mathcal{N}(E)}a_{i_j} \le C_0\max_{1\le i\le q}a_i|E|^\alpha.$$

Thus $\mathcal{H}^\alpha(K) > 0$ and $\dim_H K \ge \alpha$(see [4]), which is the required lower bound.

*Upper bound.* To obtain the upper bound $\dim_H K \le \alpha$, we first assume that $A_\alpha$ is irreducible and thus all the $a_i$'s are positive. For each $k \ge 0$, $K\subset\bigcup_{I\in\mathcal{F}^0_k}I$ and

$$\sum_{I\in\mathcal{F}^0_k}|I|^\alpha = \sum_{i=1}^{q}\sum\{\frac{1}{a_i}|I|^\alpha a_i : I\in\mathcal{F}^0_k, [I] = \mathcal{T}_i\} \le \max_{1\le i\le q}\{\frac{1}{a_i}\}\sum_{I\in\mathcal{F}^0_k}\mu(I) = \max_{1\le i\le q}\{\frac{1}{a_i}\} < \infty.$$

Since for each $k\ge 0$, $\mathcal{F}^0_k$ is a covering of $K$, and $\lim_{k\to\infty}\max\{|I| : I\in\mathcal{F}^0_k\} = 0$, the definition of the Hausdorff measure implies that $\mathcal{H}^\alpha(K) < \infty$, and thus $\dim_H K \le \alpha$.

Now assume $A_\alpha$ is not irreducible. After an appropriate permutation we can assume that $A_\alpha$ has the form

$$A_\alpha = \begin{bmatrix} A_1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \cdots & * \\ 0 & \cdots & \cdots & A_r \end{bmatrix},$$

where each $A_i$ is either an irreducible square matrix or an $1\times 1$ zero matrix. Let $\mathcal{E} := \{A_i : A_i \text{ is non-zero}\}$, counting multiplicity. For each $A_i$, let $\mathcal{T}_{A_i}$ be the collection of overlap types corresponding to $A_i$. For each $A_i\in\mathcal{E}$ and each island $I$ with $[I]\in\mathcal{T}_{A_i}$, define a subset $K_{A_i}(I)\subset K$ as follows.

$$K_{A_i}(I) := \bigcap_{k'=0}^{\infty}\bigcup\{I'\in\mathcal{F}^0_{k+k'} : [I'], [\mathcal{P}(I')], \cdots, [\mathcal{P}^{k'}(I')]\in\mathcal{T}_{A_i}, \text{ and } \mathcal{P}^{k'}(I') = I\},$$

where $k$ is the generation of $I$, namely, $I\in\mathcal{F}^0_k$. Obviously, the proof of the irreducible case above yields $\dim_H K_{A_i}(I) \le \alpha$.

For each $A_i \in \mathcal{E}$, define $\mathcal{F}^0_{A_i} := \{I \in \mathcal{F}^0 : [I] \in \mathcal{T}_{A_i}\}$. Then it is easy to verify that $K = \bigcup_{A_i \in \mathcal{E}} \bigcup_{I \in \mathcal{F}^0_{A_i}} K_{A_i}(I)$. Hence, it follows from the countable stability of the Hausdorff dimension (see [4]) that $\dim_H K \leq \alpha$, which is the required upper bound.

We have proved that $\mathcal{H}^\alpha(K) > 0$ and $\dim_H K = \alpha$. This imply that $\mathcal{H}^\alpha(K) < \infty$ since $K$ is a self-similar set (see [4]). The proof is completed. □

We conclude this section with the following examples.

**Example 2.4.** If $\{S_j\}_{j=1}^m$ satisfies the OSC (with the open set (0,1)), then it satisfies the GFTC. The classical dimension result for $K$ is covered by the result of Theorem 2.3. See [10] for general proof where for each $k \geq 0$, $\mathcal{M}_k$ is choosen as $\Sigma_k$.

The following example taken from [10, 11] is an IFS of contractive similitudes whose contraction ratios are not exponentially commensurable. As pointed in [10], it satisfies the GFTC, but not the FTC.

**Example 2.5.** Let $\{S_j\}_{j=1}^3$ be an IFS on $\mathbb{R}$ as follows.

$$S_1(x) = \rho x, \quad S_2(x) = rx + \rho(1-r), \quad S_3(x) = rx + (1-r),$$

where $0 < \rho < 1$, $0 < r < 1$, and $\rho + 2r - \rho r \leq 1$. Then $\{S_j\}_{j=1}^3$ satisfies the GTFC with respect to $(0,1)$. The dimension $\alpha$ is the unique solution of the equation

$$\rho^\alpha + 2r^\alpha - (\rho r)^\alpha = 1.$$

*Figure 1. The first five levels of islands and the distinct overlap types in Example 2.5 with parameters $\rho = 1/3$ and $r = 1/4$.*

In this Example, there exist touching islands if and only if the contractive ratios $\rho$ and $r$ satisfy the equality $\rho + 2r - \rho r = 1$. In this case, the invariant set $K = [0,1]$.

The following example is taken from [19] which is also satisfying the FTC.

**Example 2.6.** Let $\{S_j\}_{j=1}^3$ be an IFS on $\mathbb{R}$ as follows.

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{9}x + \frac{8}{27}, \quad S_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then $\{S_j\}_{j=1}^3$ satisfies the GTFC with respect to $(0,1)$. The dimensions $\alpha$ ($\approx 0.7369$) is the logarithmic ratio of the largest root of the polynomial equation

$$x^3 - 6x^2 + 5x - 1 = 0$$

*Figure 2. The first five levels of islands and the distinct overlap types in Example 2.6.*

to 9.

The following is a non-trivial example which allows touching islands.

**Example 2.7.** Let $\{S_j\}_{j=1}^4$ be an IFS on $\mathbb{R}$ as follows.

$$S_1(x) = \frac{1}{4}x, \quad S_2(x) = \frac{1}{4}x + \frac{1}{4}, \quad S_3(x) = \frac{1}{4}x + \frac{3}{8}, \quad S_4(x) = \frac{1}{4}x + \frac{3}{4}.$$

Then $\{S_j\}_{j=1}^4$ satisfies the GTFC with respect to $(0, 1)$. The dimension is equal to $\alpha = \log_4(5 + \sqrt{5}) - \frac{1}{2} \approx 0.9276$.

*Figure 3. The first five levels of islands and the distinct overlap types in Example 2.7.*

*Proof.* For each $k \geq 0$, let $\mathcal{M}_k = \Sigma_k$. Denote $I_1 = [0, 1]$, $I_2 = S_2([0, 1]) \cup S_3([0, 1])$ and $I_3 = S_{22}([0, 1]) \cup S_{23}([0, 1]) \cup S_{31}([0, 1])$. It is straightforward to verify that $\mathcal{T}_1 = [I_1], \mathcal{T}_2 = [I_2]$ and $\mathcal{T}_3 = [I_3]$ are the total overlap types, with a weighted incidence matrix

$$A_\alpha = \begin{bmatrix} \frac{2}{4^\alpha} & \frac{3^\alpha}{8^\alpha} & 0 \\ \frac{2}{6^\alpha} & \frac{1}{4^\alpha} & \frac{1}{3^\alpha} \\ \frac{2}{8^\alpha} & \frac{3^\alpha}{16^\alpha} & \frac{2}{4^\alpha} \end{bmatrix}.$$

Setting the spectral radius of $A_\alpha$ equal to 1 yields the desired result. See Figure 3. □

# 3 Density Theorems

Let $K$ be a Cantor set in $\mathbb{R}$ satisfying the GFTC with respect to the invariant set $\mathcal{O} = (0,1)$. Let $\alpha$ be the number such that the spectral radius of the weighted incidence matrix $A_\alpha$ is equal to 1. Then $\alpha$ is the dimension of $K$ by Theorem 2.3. We assume that the matrix $A_\alpha$ is irreducible throughout this section and the following ones. It is easy to verify that all the examples listed in Section 2 satisfy this assumption. In this section, we analyze the local behavior of the Hausdorff measure and the packing measure of $K$.

For a matrix $A$, we use $r(A)$ to denote the spectral radius of $A$. The following basic algebraic lemma is needed.

**Lemma 3.1.** *Let $A$ be a $q \times q$ non-negative irreducible matrix with $q \geq 2$, $A'$ be the $(q-1)\times(q-1)$ sub-matrix at the right-bottom corner of $A$. Then $r(A') < r(A)$.*

*Proof.* Since $A$ is irreducible, there exists at least one positive number in the set $\{A(1,j) : 1 \leq j \leq q\} \cup \{A(i,1) : 1 \leq i \leq q\}$. We define a $q \times q$ matrix $B$ in which $B(1,j) = A(1,j)/2$ for $1 \leq j \leq q$, $B(i,1) = A(i,1)/2$ for $1 \leq i \leq q$ and $B(i,j) = A(i,j)$ for $2 \leq i,j \leq q$. Obviously, $B$ is also a non-negative irreducible matrix and $B < A$, i.e., their exists at least one coordinate $(i,j)$ such that $B(i,j) < A(i,j)$. Hence from the well-known Perron-Frobenius Theorem, it yields that $r(B) < r(A)$. Then we get the desired result since $r(A') \leq r(B)$. □

**Lemma 3.2.** S*uppose $A_\alpha$ is irreducible, then $K$ can be decomposed into a union of a set $K_a$ with a graph directed construction and an attractor $K_b$ of a countable infinite IFS. Moreover,*

$$\dim_H K_a = \dim_P K_a < \dim_H K_b = \dim_P K_b = \dim_H K = \dim_P K,$$

*and $\mathcal{H}^\alpha(K) = \mathcal{H}^\alpha(K_b), \mathcal{P}^\alpha(K) = \mathcal{P}^\alpha(K_b)$.*

*Proof.* For each $k \geq 1$, let $\mathcal{F}_k'^0$ be a subset contained in $\mathcal{F}_k^0$ as

$$\mathcal{F}_k'^0 = \{I \in \mathcal{F}_k^0 : [I] \neq \mathcal{T}_1\}.$$

Let $K_a = \bigcap_{k\geq 1} \bigcup \{I : I \in \mathcal{F}_k'^0\}$. Then it is easy to find that $K_a$ is a set with a graph directed construction. In fact, there are $q-1$ vertex sets in the construction of $K_a$ whose weighted incidence matrix is the $(q-1)\times(q-1)$ sub-matrix $A'_\alpha$ at the right-bottom corner of $A_\alpha$ for each $\alpha \geq 0$. Moreover, by a similar proof of Theorem 2.3, the Hausdorff dimension of $K_a$ is the unique $\alpha$ such that the spectral radius of $A'_\alpha$ is equal to 1 (or see a direct result in [15]). It yields from Lemma 3.1, $\dim_H K_a < \dim_H K$.

Let $I \in \mathcal{F}_k^0$ with $k \geq 1$. If none of the elements in $\{\mathcal{P}(I), \cdots, \mathcal{P}^{k-1}(I)\}$ is of $\mathcal{T}_1$ type, we call $I$ a $\mathcal{T}_1$ *type utmost island.* For each $\mathcal{T}_1$ type utmost island $I$, the vertex set $V(I)$ of $I$ consists of exactly one vertex, i.e., $\sharp V(I) = 1$. Denote the contractive similitude of the unique element in $V(I)$ as $S_I$. Then there exists a countable infinite IFS (see [16] for further properties of infinite IFS) of contractive similitudes

$$\mathcal{S} := \{S_I : I \in \mathcal{F}^0 \setminus \mathcal{F}_0^0, I \text{ is a } \mathcal{T}_1 \text{ type utmost island}\}.$$

Denote by $K_b$ the attractor of $\mathcal{S}$. From the construction of $K_a$ and $K_b$, one can easily observe that $K = K_a \cup K_b$. Hence, it follows from the stability property of the Hausdorff dimension (see [4]), $\dim_H K_b = \dim_H K$. The remaining is obvious. □

It is worth while to point out that in the above proof we could replace $\mathcal{T}_1$ by any other overlap type. With appropriate modifications, we can still prove Lemma 3.2 in a similar way. We will not go into the details here.

In order to prove Theorem 1.1 and Theorem 1.2, we need a detailed analysis of the attractor $K_b$ of a countable infinite IFS $\mathcal{S}$ described in Lemma 3.2. Now we introduce some notations for convenience. Denote the list of countable contractive similitudes in $\mathcal{S}$ as

$$\mathcal{S} = \{S'_1, \cdots, S'_j, \cdots\}$$

and $r'_j$ the contractive ratio of $S'_j$. Then for each $k \geq 1$ we will write $S'_{\mathbf{i}} = S'_{i_1} \circ \cdots \circ S'_{i_k}$ and $r'_{\mathbf{i}} = r'_{i_1} \cdots r'_{i_k}$ for all indices $\mathbf{i} = i_1 \cdots i_k$ with entries $i_j \in \mathbb{N}$. Also, for every such indice $\mathbf{i} = i_1 \cdots i_k$ we will write $|\mathbf{i}| = k$ for the length of $\mathbf{i}$. From the construction of $K_b$, we know the intervals $S'_i([0,1])$ and $S'_j([0,1])$ are separate for all $i \neq j \in \mathbb{N}$. Hence

$$\mathcal{H}^\alpha(K_b) = \sum_{i \in \mathbb{N}} \mathcal{H}^\alpha(S'_i K_b) = \sum_{i \in \mathbb{N}} r_i'^\alpha \mathcal{H}^\alpha(K_b).$$

Combing the above formula with the result in Lemma 3.2 and the fact that $0 < \mathcal{H}^\alpha(K) < \infty$ from Theorem 2.3, we have $\sum_{i \in \mathbb{N}} r_i'^\alpha = 1$. Moreover,

$$\sum_{|\mathbf{i}|=k} r_{\mathbf{i}}'^\alpha = \left( \sum_{i \in \mathbb{N}} r_i'^\alpha \right)^k = 1$$

for any integer $k \geq 1$. A useful measure $\lambda$, which is called $\alpha$-dimensional normalized Hausdorff measure, defined on $K_b$ by $\lambda(S'_{\mathbf{i}}(K_b)) = r_{\mathbf{i}}'^\alpha$, extended to Borel subsets of $K$, will be used. This is a probability measure which scales on $S'_{\mathbf{i}}(K_b)$, hence $\mathcal{H}^\alpha|_{K_b} = \mathcal{H}^\alpha|_K = \mathcal{H}^\alpha(K)\lambda$. Obviously, we also have $\mathcal{P}^\alpha|_{K_b} = \mathcal{P}^\alpha|_K = \mathcal{P}^\alpha(K)\lambda$.

We should point out that $\lambda$ is equal to the natural probability measure $\mu$ defined in the proof of Theorem 2.3. This could be easily verified by showing $\lambda$ and $\mu$ are equal on cylinder sets in $K_b$, i.e.,

$$\mu(S'_{\mathbf{i}}(K_b)) = |S'_{\mathbf{i}}([0,1])|^\alpha a_1 = r_{\mathbf{i}}'^\alpha = \lambda(S'_{\mathbf{i}}(K_b)),$$

where $a_1 = 1$ is the first element in the normalized 1-eigenvector of $A_\alpha$. Hence for each island $I \in \mathcal{F}^0$, $\lambda(I)$ can be calculated by using the parameters of the IFS of $K$.

For any interval $J \subset [0,1]$ we define the *density* of $J$ with respect to $\lambda$ as $d(J) = \frac{\lambda(J)}{|J|^\alpha}$.

Theorem 1.1 and Theorem 1.2 are used to obtain density results for Hausdorff and packing measures of $K$, namely, Corollary 1.3 and Corollary 1.4. Now we turn to the proofs.

## 3.1 Proof of Theorem 1.1 and Corollary 1.3

The formula in Theorem 1.1 is analogous to that of a general self-similar set in $\mathbb{R}^d$ which satisfies the OSC (see [20]). Using Lemma 3.2, it is now possible to adapt the techniques there to prove Theorem 1.1. We now show two more lemmas concerning the measure $\mathcal{H}^\alpha(K_b)$ of the attractor $K_b$. Theorem 1.1 will be a direct corollary of them.

**Lemma 3.3.** *Let $K_b$ be the attractor described in Lemma 3.2, $J \subset [0,1]$ be an interval and $k$ be a positive integer, then $\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) \geq \mathcal{H}^\alpha(K_b \cap J)$.*

*Proof.* Since $S'_{\mathbf{j}}K_b \subset \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}K_b = K_b$ for all $\mathbf{j}$ with $|\mathbf{j}| = k$, we have $K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J \supset \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J)$. It follows that

$$\begin{aligned}\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) &\geq \mathcal{H}^\alpha\left(\bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J)\right) = \sum_{|\mathbf{i}|=k} \mathcal{H}^\alpha\left(S'_{\mathbf{i}}(K_b \cap J)\right) \\ &= \sum_{|\mathbf{i}|=k} r'^{\alpha}_{\mathbf{i}} \mathcal{H}^\alpha(K_b \cap J) = \mathcal{H}^\alpha(K_b \cap J). \quad \square\end{aligned}$$

The following lemma is a revised version of Theorem 1.1 with $K$ replaced by $K_b$.

**Lemma 3.4.** *The attractor $K_b$ described in Lemma 3.2 satisfies*

$$\mathcal{H}^\alpha(K_b \cap J) \leq |J|^\alpha \tag{3.1}$$

*for all intervals $J \subset [0,1]$.*

*Proof.* In order to reach a contradiction, we assume that (3.1) is not satisfied, i.e., there exists a non-empty interval $J \subset [0,1]$, such that $\mathcal{H}^\alpha(K_b \cap J) > |J|^\alpha$. It follows from this we can find $0 < \kappa < 1$ with

$$(1-\kappa)\mathcal{H}^\alpha(K_b \cap J) > |J|^\alpha. \tag{3.2}$$

Next, fix $\delta > 0$ and choose a positive integer $k$ such that $|S'_{\mathbf{i}}J| \leq \delta$ for all $\mathbf{i}$ with $|\mathbf{i}| = k$. Let $\eta = \frac{1}{2}\kappa\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J)$. It follows from Lemma 3.3 and (3.2),

$$\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) \geq \mathcal{H}^\alpha(K_b \cap J) \geq \frac{|J|^\alpha}{1-\kappa} > 0, \tag{3.3}$$

which yields $\eta > 0$.

Since $\eta > 0$, we can choose a covering $\{J_i\}_i$ of $K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J$ with $|J_i| \leq \delta$ such that

$$\sum_i |J_i|^\alpha \leq \mathcal{H}^\alpha_\delta(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta \leq \mathcal{H}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta. \tag{3.4}$$

The family $\{S'_{\mathbf{i}}J\}_{|\mathbf{i}|=k} \cup \{J_i\}_i$ is clearly a $\delta$-covering of $K_b$. We therefore conclude from (3.2), (3.4) and Lemma 3.3 that

$$\begin{aligned}\mathcal{H}^\alpha_\delta(K_b) &\leq \sum_{|\mathbf{i}|=k} |S'_{\mathbf{i}}J|^\alpha + \sum_i |J_i|^\alpha \leq \sum_{|\mathbf{i}|=k} r'^{\alpha}_{\mathbf{i}}|J|^\alpha + \mathcal{H}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta \\ &= |J|^\alpha + \mathcal{H}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta \\ &\leq (1-\kappa)\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \mathcal{H}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta \\ &\leq \mathcal{H}^\alpha(K_b) - \kappa\mathcal{H}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) + \eta \\ &= \mathcal{H}^\alpha(K_b) - \eta \leq \mathcal{H}^\alpha(K_b) - \frac{1}{2}\kappa\mathcal{H}^\alpha(K_b \cap J).\end{aligned}$$

Finally, letting $\delta \to 0$ gives $\mathcal{H}^\alpha(K_b) \le \mathcal{H}^\alpha(K_b) - \frac{1}{2}\kappa\mathcal{H}^\alpha(K_b \cap J)$. Then the fact that $0 < \mathcal{H}^\alpha(K_b) < \infty$ by Lemma 3.2 and $(1/2)\kappa\mathcal{H}^\alpha(K_b \cap J) > 0$ by (3.3) provides the desired contradiction. □

*Proof of Theorem 1.1.* Indeed, it follows from Lemma 3.2 and Lemma 3.4. □

Putting $J$ equal to $[0,1]$ in Theorem 1.1 gives the upper bound for $\mathcal{H}^\alpha(K)$, namely, $\mathcal{H}^\alpha(K) \le 1$. It is a natural generalization of the same result in the OSC case, since in that case $K$ can be covered by its iterated images under the IFS. Obviously it is not always true that we have the equality. See Falconer [4] for some examples with $\mathcal{H}^\alpha(K) = 1$ satisfying the OSC. An natural question is arisen: *is there any non-trivial example with* $\mathcal{H}^\alpha(K) = 1$ *which satisfies only the GFTC?*

For a given measure $\nu$ on $\mathbb{R}$ and $s > 0$, the *upper s-dimensional convex density* of $\nu$ at $x$ is defined by

$$\mathcal{D}^{*s}(\nu, x) = \lim_{r\to 0} \sup\{\frac{\nu(J)}{|J|^s} : J \text{ is an interval and } 0 < |J| \le r, x \in J\}.$$

The *lower s-dimensional convex density* $\mathcal{D}_*^s(\nu, x)$ is defined similarly by taking the lower limit. We have the following result that if $E \subset \mathbb{R}$ and $s > 0$ with $\mathcal{H}^s(E) < \infty$, then

$$\mathcal{D}^{*s}(\mathcal{H}^s|_E, x) = 1 \text{ for } \mathcal{H}^s\text{-a.e. } x \in E. \tag{3.5}$$

The reader is referred to [2] for a proof of (3.5).

*Proof of Corollary 1.3.* From (3.5), we can pick a point $x \in K \cap (0,1)$ such that $\mathcal{D}^{*\alpha}(\mathcal{H}^\alpha|_K, x) = 1$. By the definition of $\mathcal{D}^{*\alpha}(\mathcal{H}^\alpha|_K, x)$, there exists a positive sequence $\{\delta_n\}_n$ with $\delta_n < \min\{x, 1-x\}$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\sup_{0<|J|\le\delta_n} \frac{\mathcal{H}^\alpha(K\cap J)}{|J|^\alpha} - \frac{1}{n} \le 1 \le \sup_{0<|J|\le\delta_n} \frac{\mathcal{H}^\alpha(K\cap J)}{|J|^\alpha} + \frac{1}{n}.$$

Hence there exists an interval $J_n$ with $0 < |J_n| \le \delta_n$ for each $n$ such that

$$\sup_{0<|J|\le\delta_n} \frac{\mathcal{H}^\alpha(K\cap J)}{|J|^\alpha} \le \frac{\mathcal{H}^\alpha(K\cap J_n)}{|J_n|^\alpha} + \frac{1}{n}.$$

Thus $\frac{\mathcal{H}^\alpha(K\cap J_n)}{|J_n|^\alpha} - \frac{1}{n} \le 1 \le \frac{\mathcal{H}^\alpha(K\cap J_n)}{|J_n|^\alpha} + \frac{2}{n}$, which yields that $\frac{\mathcal{H}^\alpha(K\cap J_n)}{|J_n|^\alpha} \to 1$ as $n \to \infty$. Moreover, by Theorem 1.1, for each interval $J_n$ we have $\mathcal{H}^\alpha(K\cap J_n)/|J_n|^\alpha \le 1$. Hence $\sup\{\frac{\mathcal{H}^\alpha(K\cap J)}{|J|^\alpha} : J \subset [0,1]\} = 1$. Since $\lambda = \mathcal{H}^\alpha|_K/\mathcal{H}^\alpha(K)$, (1.9) follows immediately from the above equation. □

With suitable modifications if necessary, we may generalize Theorem 1.1 and Corollary 1.3 to general self-similar sets satisfying the GFTC with irreducible weighted incidence matrix. However, in order to match the main goal of this paper and to avoid additional technical details, we will not pursue this here.

## 3.2 Proof of Theorem 1.2 and Corollary 1.4

In a manner dual to the Hausdorff measure case, the packing measure result is also analogous to that of a general self-similar set in $\mathbb{R}^d$ which satisfies the SSC or the OSC (see [20, 21]). Hence it is also possible to adapt the techniques there to prove Theorem 1.2 by using Lemma 3.2. We shall need the following two lemmas concerning $\mathcal{P}^\alpha(K_b)$. We use $P^\alpha$ to denote the premeasure of $\mathcal{P}^\alpha$. See [4] and [14] for further properties of $P^\alpha$.

**Lemma 3.5.** *Let $K_b$ be the attractor described in Lemma 3.2. Let $J \subset [0,1]$ be an interval centered in $K_b$ and $k$ a positive integer. Then $\mathcal{P}^\alpha(K_b \cap \bigcup_{|\boldsymbol{i}|=k} S'_{\boldsymbol{i}}J) = \mathcal{P}^\alpha(K_b \cap J) > 0$.*

*Proof.* We write $J^\circ$ as the interior of the interval $J$. First, we prove that

$$K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J^\circ = \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J^\circ). \tag{3.6}$$

Fix a point $y \in K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J^\circ$. Since $J^\circ \subset \mathcal{O} = (0,1)$, there exists an index $\mathbf{u}$ with the length $k$ such that $y \in S'_{\mathbf{u}}J^\circ \subset S'_{\mathbf{u}}\mathcal{O}$. We also have $y \in K_b = \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}K_b$ and we therefore find an index $\mathbf{v}$ with the length $k$ such that $y \in S'_{\mathbf{v}}K_b \subset S'_{\mathbf{v}}\overline{\mathcal{O}}$. Thus $y \in S'_{\mathbf{u}}\mathcal{O} \cap S'_{\mathbf{v}}\overline{\mathcal{O}}$, and therefore $\mathbf{u} = \mathbf{v}$. Hence $y \in S'_{\mathbf{u}}J^\circ \cap S'_{\mathbf{u}}K_b = S'_{\mathbf{u}}(K_b \cap J^\circ) \subset \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J^\circ)$ which yields that $K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J^\circ \subset \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J^\circ)$. The other direction is obvious.

It follows from (3.6) that

$$\begin{aligned}\mathcal{P}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) &= \mathcal{P}^\alpha\left(\bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}(K_b \cap J^\circ)\right) = \sum_{|\mathbf{i}|=k} \mathcal{P}^\alpha\left(S'_{\mathbf{i}}(K_b \cap J^\circ)\right) \\ &= \sum_{|\mathbf{i}|=k} r'^{\alpha}_{\mathbf{i}} \mathcal{P}^\alpha(K_b \cap J^\circ) = \mathcal{P}^\alpha(K_b \cap J).\end{aligned}$$

Moreover, since $J$ has its center in $K_b$, we deduce that $\mathcal{P}^\alpha(K_b \cap J) > 0$. This completes the proof of Lemma 3.5. □

**Lemma 3.6.** The attractor $K_b$ described in Lemma 3.2 satisfies

$$\mathcal{P}^\alpha(K_b \cap J) \geq |J|^\alpha \tag{3.7}$$

for all intervals $J \subset [0,1]$ centered in $K_b$.

*Proof.* In order to reach a contradiction, we assume that (3.7) is not satisfied, i.e., there exists an interval $J := [c,d] \subset [0,1]$ centered in $K_b$, such that $\mathcal{P}^\alpha(K_b \cap J) < |J|^\alpha$. Thus we can find a number $0 < \kappa < 1$ with

$$(1+\kappa)\mathcal{P}^\alpha(K_b \cap J) < |J|^\alpha. \tag{3.8}$$

Next, fix $\delta > 0$ and choose a positive integer $k$ such that $|S'_{\mathbf{i}}J| \leq \delta$ for all $\mathbf{i}$ with $|\mathbf{i}| = k$. Let $\eta = \frac{1}{2}\kappa\mathcal{P}^\alpha(K_b \cap J)$. It follows from Lemma 3.5, $\eta > 0$.

For a positive integer $n$, write $G_n = K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}\left((c - 1/n, d + 1/n)\right)$, and observe that $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ and $\bigcup_n G_n = K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J$.

If $K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J \neq \emptyset$, then there is a positive integer $n_0$ with $1/n_0 < \delta$ such that $G_{n_0} \neq \emptyset$, and

$$\mathcal{P}^\alpha(G_{n_0}) \geq \mathcal{P}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}}J) - \frac{\eta}{2}. \tag{3.9}$$

We can choose a $1/n_0$-packing $\{J_i^\circ\}_i$ of $G_{n_0}$ such that

$$\sum_i |J_i|^\alpha \geq P^\alpha_{\frac{1}{n_0}}(G_{n_0}) - \frac{\eta}{2} \geq P^\alpha(G_{n_0}) - \frac{\eta}{2} \geq \mathcal{P}^\alpha(G_{n_0}) - \frac{\eta}{2}. \tag{3.10}$$

Since $J^\circ \subset \mathcal{O}$, $S'_{\mathbf{i}}(J^\circ) \cap S'_{\mathbf{j}}(J^\circ) = \emptyset$ for all $\mathbf{i} \neq \mathbf{j}$ with lengths $k$. And for each such $\mathbf{i}$, since $(c+d)/2 \in K_b$, we have $S'_{\mathbf{i}}((c+d)/2) \in S'_{\mathbf{i}} K_b \subset K_b$. Thus the family $\{S'_{\mathbf{i}}(J^\circ)\}_{|\mathbf{i}|=k}$ is a $\delta$-packing of $K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J$ by the fact that $|S'_{\mathbf{i}} J| \leq \delta$ for all $\mathbf{i}$ with $|\mathbf{i}| = k$.

Since $\{J_i^\circ\}_i$ is also a $1/n_0$-packing of $G_{n_0}$, we conclude that $\{S'_{\mathbf{i}} J^\circ\}_{|\mathbf{i}|=k} \bigcup \{J_i^\circ\}_i$ is a $\delta$-packing of $K_b$. Using this we therefore conclude from (3.8), (3.9), (3.10), and Lemma 3.5 that

$$\begin{aligned}
P^\alpha_\delta(K_b) &\geq \sum_{|\mathbf{i}|=k} (r'_{\mathbf{i}}|J|)^\alpha + \sum_i (|J_i|)^\alpha \geq \sum_{|\mathbf{i}|=k} r'^{\alpha}_{\mathbf{i}} |J|^\alpha + \mathcal{P}^\alpha(G_{n_0}) - \frac{\eta}{2} \\
&\geq |J|^\alpha + \mathcal{P}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) - \eta \\
&\geq (1+\kappa)\mathcal{P}^\alpha(K_b \cap J) + \mathcal{P}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) - \eta \\
&= (1+\kappa)\mathcal{P}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) + \mathcal{P}^\alpha(K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) - \eta \\
&= \mathcal{P}^\alpha(K_b) + \kappa \mathcal{P}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) - \eta \\
&= \mathcal{P}^\alpha(K_b) + \frac{1}{2}\kappa \mathcal{P}^\alpha(K_b \cap J).
\end{aligned}$$

Finally, let $\delta \to 0$, we get

$$P^\alpha(K_b) \geq \mathcal{P}^\alpha(K_b) + \frac{1}{2}\kappa \mathcal{P}^\alpha(K_b \cap J). \tag{3.11}$$

In [5] it is proved that the packing premeasure $P^\alpha$ coincides with the packing measure $\mathcal{P}^\alpha$ for compact subsets with finite $P^\alpha$-measure. Thus they coincide for $K_b$, and it follows from (3.11) that $\mathcal{P}^\alpha(K_b) \geq \mathcal{P}^\alpha(K_b) + \frac{1}{2}\kappa \mathcal{P}^\alpha(K_b \cap J)$. Since $\mathcal{P}^\alpha(K_b)$ is positive and finite by Lemma 3.2, and $(1/2)\kappa \mathcal{P}^\alpha(K_b \cap J) > 0$, we get the contradiction.

On the other hand, if $K_b \setminus \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J = \emptyset$, i.e., $K_b \subset \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J$, the aforementioned string of inequalities simplifies to

$$P^\alpha_\delta(K_b) \geq \sum_{|\mathbf{i}|=k} (r'_{\mathbf{i}}|J|)^\alpha = |J|^\alpha > (1+\kappa)\mathcal{P}^\alpha(K_b \cap J) = (1+\kappa)\mathcal{P}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J).$$

Letting $\delta \to 0$ and using the fact that $\mathcal{P}^\alpha(K_b) = P^\alpha(K_b)$ gives

$$\mathcal{P}^\alpha(K_b) = P^\alpha(K_b) \geq (1+\kappa)\mathcal{P}^\alpha(K_b \cap \bigcup_{|\mathbf{i}|=k} S'_{\mathbf{i}} J) = (1+\kappa)\mathcal{P}^\alpha(K_b) > \mathcal{P}^\alpha(K_b).$$

This provides the desired contradiction. □

*Proof of Theorem 1.2.* For any $J := [c, d] \subset [0, 1]$ centered in $K$. If $J$ is centered in $K_b$, the result follows immediately from Lemma 3.2 and Lemma 3.6. If $J$ is not centered in $K_b$, i.e., $(c+d)/2 \in K_a \setminus K_b$, we claim that:

*Claim:* $K_b \cap (c, d) \neq \emptyset$ and for each positive integer $n$, there exists a point $x_n \in K_b \cap (c, d)$ such that $|x_n - (c+d)/2| \leq 1/n$.

*Proof.* Since $(c+d)/2 \in K_a$, for each $n$, there is an island $I_n \subset (c, d)$ containing the point $(c+d)/2$ with length less than $1/n$ whose overlap type is not $\mathcal{T}_1$. By the irreducible property of the weighted incidence matrix $A_\alpha$, we could find a smaller $\mathcal{T}_1$ type island $\widetilde{I}_n$ contained in $I_n$. By the construction of $K_b$, $K_b \cap \widetilde{I}_n \neq \emptyset$ which yields that $K_b \cap (c,d) \neq \emptyset$. Then fix a point $x_n$ in $K_b \cap \widetilde{I}_n$. Obviously we get $|x_n - (c+d)/2| \leq |I_n| \leq 1/n$. □

By this claim we define a sequence of intervals $\{J_n\}_n$ contained in $J$ as

$$J_n = [x_n - \min\{x_n - c, d - x_n\}, x_n + \min\{x_n - c, d - x_n\}].$$

It is not difficult to find that for each $n$ the interval $J_n$ is centered in $K_b$, and that the left endpoint of $J_n$ tends to $c$ and the right endpoint of $J_n$ tends to $d$ as $n \to \infty$. As showed in the first case, $\mathcal{P}^\alpha(K \cap J_n) \geq |J_n|^\alpha$ for all $n$. Letting $n \to \infty$, we immediately get $\mathcal{P}^\alpha(K \cap J) \geq |J|^\alpha$. □

Let $s \geq 0$, for a given measure $\nu$ on $\mathbb{R}$ and $x \in \mathbb{R}$, the *lower $s$-dimensional density* of $\nu$ at $x$ is defined as

$$\Theta_*^s(\nu, x) = \liminf_{r \to 0} \frac{\nu([x-r, x+r])}{(2r)^s}.$$

The *upper $s$-dimensional density* $\Theta^{*\alpha}(\nu, x)$ is defined similarly by taking the upper limit. We have the following result. If $E \subset \mathbb{R}$ and $s > 0$ with $0 < \mathcal{P}^s(E) < \infty$,

$$\Theta_*^s(\mathcal{P}^s|_E, x) = 1 \text{ for } \mathcal{P}^\alpha\text{-a.e.} \quad x \in E. \tag{3.12}$$

See the proof in [14]. Now we prove Corollary 1.4 on the basis of (3.12) and Theorem 1.2.

*Proof of Corollary 1.4.* Let $\mathcal{O} = (0,1)$. Since $K \cap \mathcal{O} \neq \emptyset$, we can take a point $y \in K \cap \mathcal{O}$. Choose $\delta > 0$ such that the open interval $(y-\delta, y+\delta)$ is contained in $\mathcal{O}$. Moreover, since $y \in K$, $\mathcal{P}^\alpha(K \cap (y-\delta, y+\delta)) > 0$. Hence from (3.12), there exists a point $z \in K \cap (y-\delta, y+\delta)$ with $\Theta_*^\alpha(\mathcal{P}^\alpha|_K, z) = 1$. By the definition of $\Theta_*^\alpha(\mathcal{P}^\alpha|_K, z)$, there exists a sequence $\{r_n\}_n$ with each $r_n < \delta - d(z,y)$ and $r_n \to 0$ as $n \to \infty$, such that $\lim_{n\to\infty} \mathcal{P}^\alpha(K \cap [z-r_n, z+r_n])/(2r_n)^\alpha = 1$. Notice that all intervals $[z-r_n, z+r_n]$ are contained in $(y-\delta, y+\delta) \subset \mathcal{O}$ with center $z \in K$. However, by Theorem 1.2, for each interval $J \subset [0,1]$ centered in $K$, we have $\mathcal{P}^\alpha(K \cap J)/|J|^\alpha \geq 1$. Hence we get $\inf\{\frac{\mathcal{P}^\alpha(K\cap J)}{|J|^\alpha} : J \text{ is an interval centered in } K \text{ with } J \subset [0,1]\} = 1$. Since $\lambda = \mathcal{P}^\alpha|_K/\mathcal{P}^\alpha(K)$, (1.10) follows immediately from the above equation. □

Similar to the Hausdorff measure case, with suitable modifications if necessary, we may generalize Theorem 1.2 and Corollary 1.4 to general self-similar sets in $\mathbb{R}^d$ satisfying the GFTC with irreducible weighted incidence matrix. Due to the same reason for the Hausdorff measure case, we will not pursue such generalizations here.

# 4 The maximal and minimal densities

In this section we deal with the exact computation of the Hausdorff measure and the packing measure for a special kind of Cantor sets in $\mathbb{R}$ . Let $S_j(x) = \rho_j x + b_j, j = 1, \cdots, m$, be a linear contractive IFS on the line $\mathbb{R}$ satisfying the GFTC with respect to the open set $\mathcal{O} = (0,1)$ with an irreducible weighted incidence matrix $A_\alpha$. As before, we write $K$ as its invariant set and use $\alpha$ to denote the dimension of $K$. As [1, 6], we do not allow negative $\rho$'s, namely, we assume $0 < \rho_j < 1$ for $j = 1, \cdots, m$. Moreover, without loss of generality and for convenience we assume the images $S_j([0,1])$ are in increasing order, with $S_1(0)=0$ and $S_m(1)=1$. To avoid triviality we always assume $m\geq 2$ and $\alpha<1$.

Define

$$d_{\max} := \sup\{d(J) : J \text{ is an interval with } J\subset [0,1]\}$$

the maximal density of intervals contained in $[0,1]$ and

$$d_{\min} := \inf\{d(J) : J \text{ is an interval centered in } K \text{ with } J\subset [0,1]\}$$

the minimal density of intervals centered in $K$ and contained in $[0,1]$. Then by Corollary 1.3 and Corollary 1.4,

$$\mathcal{H}^\alpha(K)=d_{\max}^{-1} \text{ and } \mathcal{P}^\alpha(K)=d_{\min}^{-1}.$$

Hence our main purpose in this section is to determine $d_{\max}$ and $d_{\min}$.

We will frequently use the notation $\mathcal{F}_k$, the *k-th generation field*, which is the finite field generated by the set $\mathcal{F}_k^0$ for $k\geq 0$. For each $k\geq 0$, let $\beta_1^{(k)},\cdots,\beta_{l_k}^{(k)}$ denote the lengths of the $k$-th generation islands in increasing order where $l_k$ denotes the cardinality of $\mathcal{F}_k^0$. Denote $\beta_{\max}^{(k)}=\max\{\beta_1^{(k)},\cdots,\beta_{l_k}^{(k)}\}$ and $\beta_{\min}^{(k)}=\min\{\beta_1^{(k)},\cdots,\beta_{l_k}^{(k)}\}$. We write the lengths of the lakes separating the $k$-th generation islands as $\gamma_1^{(k)},\cdots,\gamma_{l_k-1}^{(k)}$ in increasing order with $\gamma_{\min}^{(k)}=\min\{\gamma_1^{(k)},\cdots,\gamma_{l_k-1}^{(k)}\}$. Note that we allow $\gamma_{\min}^{(k)}=0$ in the case that touching islands exist. Denote by $\gamma_{\min}^{*(k)}$ the minimal length of non-empty lakes in $\mathcal{F}_k$. The identity

$$\beta_1^{(k)}+\cdots+\beta_{l_k}^{(k)}+\gamma_1^{(k)}+\cdots+\gamma_{l_k-1}^{(k)}=1,$$

the positivity of $\beta$'s and the non-negativity of $\gamma$'s are the only restrictions on these parameters. It should be mentioned here that $l_k\to\infty$, $\beta_{\max}^{(k)}\to 0$, $\beta_{\min}^{(k)}\to 0$, $\gamma_{\min}^{(k)}\to 0$ and $\gamma_{\min}^{*(k)}\to 0$ as $k\to\infty$.

The measure $\lambda$ will play a key role in this section. It enables us to compute the density of each interval in $\mathcal{F}_k$ as an elementary function on parameters of the IFS of $K$. Moreover, there is an obvious algorithm for finding the maximal or minimal density of intervals in $\mathcal{F}_k$.

Since $K$ has finite types of islands, there exists a smallest non-negative integer $k_0$ such that none of the islands in $\mathcal{F}_{k_0+1}^0$ is of a new overlap type. We have the following *blow-up principle* for the density of each interval $J\subset[0,1]$.

**Lemma 4.1.** *For any interval $J$, there exists another interval $J'$, not contained in a $(k_0+1)$-th generation island, with the same density.*

*Proof.* If $J$ is contained in a $(k_0+1)$-th generation island $I$, then there exists a larger island $I'\in\bigcup_{0\leq k\leq k_0}\mathcal{F}_k^0$ with the same type as $I$, i.e., $[I']=[I]$. Let $\tau$ be the linear function which maps $I$ onto $I'$, keeping the orientation. Then obviously the image $J':=\tau(J)$ has the same density as $J$. We iterate this procedure until we obtain $J'$ not lying in any $(k_0+1)$-th generation island. □

Thus we only need to consider intervals not contained in $(k_0+1)$-th generation islands.

In order to get general results for computing $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$, we need the following two additional technical assumptions. We will show that these assumptions are as general as to be satisfied by all the examples illustrated in Section 2.

**Assumption A.** For each island $I \in \mathcal{F}^0$, for any two different component intervals $\overline{\mathcal{O}}_{\mathbf{v}}$ and $\overline{\mathcal{O}}_{\mathbf{u}}$ of $I$, $\overline{\mathcal{O}}_{\mathbf{v}} \not\subseteq \overline{\mathcal{O}}_{\mathbf{u}}$ and $\overline{\mathcal{O}}_{\mathbf{u}} \not\subseteq \overline{\mathcal{O}}_{\mathbf{v}}$.

**Assumption B.** $S_1([0,1]) \cap K = S_1 K$ and $S_m([0,1]) \cap K = S_m K$.

Before we give some remarks on these assumptions, we prove that

**Proposition 4.2.** For all the examples listed in Section 2, the above two assumptions hold.

*Proof.* It is trivial for Example 2.4 and 2.7.

For Example 2.5, we only need to prove $S_1([0,1]) \cap K = S_1 K$.

By the fact $S_1([0,1]) \cap S_3 K = \emptyset$, $S_1([0,1]) \cap S_{23} K = \emptyset$ and $S_{13} = S_{21}$, we have

$$\begin{aligned} S_1([0,1]) \cap K &= (S_1([0,1]) \cap S_1 K) \cup (S_1([0,1]) \cap S_2 K) \\ &= (S_1([0,1] \cap K)) \cup (S_1([0,1]) \cap S_{13} K) \cup (S_1([0,1]) \cap S_2^2 K) \\ &= (S_1([0,1] \cap K)) \cup (S_1([0,1] \cap S_3 K)) \cup (S_1([0,1]) \cap S_2^2 K). \end{aligned}$$

For $n \geq 2$, iterate the above proceedure $n-1$ times, we get

$$\begin{aligned} S_1([0,1]) \cap K &= (S_1([0,1] \cap K)) \cup (S_1([0,1] \cap S_3 K)) \cup \\ &\quad \cdots \cup (S_1([0,1] \cap S_3^n K)) \cup (S_1([0,1]) \cap S_2^{n+1} K). \end{aligned}$$

Hence we get $S_1([0,1]) \cap K \subset S_1 K$ by the arbitrariness of $n$. The other direction is obvious.

For Example 2.6, we still only need to prove $S_1([0,1]) \cap K = S_1 K$.

By the fact $S_1([0,1]) \cap S_3 K = \emptyset$, $S_1([0,1]) \cap S_{23} K = \emptyset$ and $S_{133} = S_{21}$, we have

$$\begin{aligned} S_1([0,1]) \cap K &= (S_1([0,1]) \cap S_1 K) \cup (S_1([0,1]) \cap S_2 K) \\ &= (S_1([0,1] \cap K)) \cup (S_1([0,1]) \cap S_{133} K) \cup (S_1([0,1]) \cap S_2^2 K) \\ &= (S_1([0,1] \cap K)) \cup (S_1([0,1] \cap S_3^2 K)) \cup (S_1([0,1]) \cap S_2^2 K). \end{aligned}$$

For $n \geq 2$, iterate the above proceedure $n-1$ times, we get

$$\begin{aligned} S_1([0,1]) \cap K &= (S_1([0,1] \cap K)) \cup (S_1([0,1] \cap S_3^2 K)) \cup \\ &\quad \cdots \cup (S_1([0,1] \cap S_3^{2n} K)) \cup (S_1([0,1]) \cap S_2^{n+1} K). \end{aligned}$$

Hence we get $S_1([0,1]) \cap K \subset S_1 K$ by the arbitrariness of $n$. The other direction is obvious. □

**Remark 1.** Under Assumption B, for each positive integer $k$, we have $S_1^k([0,1]) \cap K = S_1^k K$ and $S_m^k([0,1]) \cap K = S_m^k K$.

*Proof.* We only need to prove $S_1^2([0,1]) \cap K = S_1^2 K$. Since $S_1^2([0,1]) \cap K \subset S_1([0,1]) \cap K = S_1 K$, we have $S_1^2([0,1]) \cap K = S_1^2([0,1]) \cap S_1 K = S_1(S_1([0,1]) \cap K) = S_1^2 K$. The second equality can be proved in a similar way. □

**Remark 2.** For each island $I = [c,d] \in \mathcal{F}^0$, notice that $I = \bigcup_{\mathbf{v} \in V(I)} \overline{\mathcal{O}}_{\mathbf{v}}$. By Assumption A, there is a unique vertex $\mathbf{v}_0 \in V(I)$ such that $\overline{\mathcal{O}}_{\mathbf{v}_0}$ has the same left endpoint as $I$, i.e., $S_{\mathbf{v}_0}(0) = c$. If $V(I) \setminus \{\mathbf{v}_0\} \neq \emptyset$, let $c' = \min\{S_{\mathbf{v}}(0) : \mathbf{v} \in V(I) \setminus \{\mathbf{v}_0\}\}$. Obviously $c < c' < d$. Let $k$ be the smallest positive integer such that $S_{\mathbf{v}_0} S_{\mathbf{i}_0}([0,1]) \subset [c, c')$

where $\mathbf{i}_0 = (1, \cdots, 1)$ with $|\mathbf{i}_0| = k$ (This must be done since $S_{\mathbf{v}_0} S_1^k(0)$ is always equal to c). For this $\mathbf{i}_0$, by Remark 1, we have

$$S_{\mathbf{v}_0} S_{\mathbf{i}_0}([0,1]) \cap K = S_{\mathbf{v}_0} S_{\mathbf{i}_0} K.$$

This can be verified since

$$S_{\mathbf{v}_0} S_{\mathbf{i}_0}([0,1]) \cap K = S_{\mathbf{v}_0} S_{\mathbf{i}_0}([0,1]) \cap S_{\mathbf{v}_0} K = S_{\mathbf{v}_0}(S_{\mathbf{i}_0}([0,1]) \cap K) = S_{\mathbf{v}_0} S_{\mathbf{i}_0} K.$$

Otherwise, if $V(I) \setminus \{\mathbf{v}_0\} = \emptyset$, i.e., $V(I) = \{\mathbf{v}_0\}$, we define $\mathbf{i}_0 = \emptyset$. Also, we have $S_{\mathbf{v}_0} S_{\mathbf{i}_0}([0,1]) \cap K = S_{\mathbf{v}_0} S_{\mathbf{i}_0} K$.

Similarly, In an analogous way, there is also a unique vertex $\mathbf{v}_1 \in V(I)$ such that $\overline{\mathcal{O}}_{\mathbf{v}_1}$ has the same right endpoint as $I$, i.e., $S_{\mathbf{v}_1}(1) = d$. By a similar discussion, one can find that there also exists a smallest non-negative integer $k'$ such that

$$S_{\mathbf{v}_1} S_{\mathbf{i}_1}([0,1]) \cap K = S_{\mathbf{v}_1} S_{\mathbf{i}_1} K$$

where $\mathbf{i}_1 = (m, \cdots, m)$ with the length $|\mathbf{i}_1| = k'$.

**Remark 3.** We should mention that in Remark 2 the ratios $|S_{\mathbf{v}_0}([0,1])|/|I|$, $|S_{\mathbf{v}_1}([0,1])|/|I|$ and the indices $\mathbf{i}_0, \mathbf{i}_1$ are dependent merely on the overlap type of $I$ and are independent on the choice of $I$. In order to emphasize the relation between $\mathbf{v}_0, \mathbf{v}_1, \mathbf{i}_0, \mathbf{i}_1$ and $I$, we replace $\mathbf{v}_0, \mathbf{v}_1, \mathbf{i}_0, \mathbf{i}_1$ by $\mathbf{v}_0(I), \mathbf{v}_1(I), \mathbf{i}_0(I), \mathbf{i}_1(I)$ respectively. Since the number of the overlap types is finite, we could define two positive numbers $\eta_1 \leq 1$ and $\eta_2 \leq 1$ as follows.

$$\eta_1 := \min_{I \in \mathcal{F}^0} \{ \frac{\rho_{\mathbf{v}_0(I)}}{|I|}, \frac{\rho_{\mathbf{v}_1(I)}}{|I|} \} > 0 \text{ and } \eta_2 := \min_{I \in \mathcal{F}^0} \{ \rho_{\mathbf{i}_0(I)}, \rho_{\mathbf{i}_1(I)} \} > 0.$$

Write $\eta := \eta_1 \eta_2$ which will be used later. Here $0 < \eta \leq 1$.

In the following, we will always assume Assumption A and B.

Under these assumptions, we then have the following another blow-up principle.

**Lemma 4.3.** *If $J \subset [0,1]$ is an interval of the form $[0,x]$, then there exists an interval $J' = [0,x']$ with $\rho_1 < x' \leq 1$ such that $d(J') = d(J)$; similarly, if $J$ is an interval of the form $[y,1]$, then there exists an interval $J' = [y',1]$ with $0 \leq y' < 1 - \rho_m$ such that $d(J') = d(J)$.*

*Proof.* By Assumption B, if $J \subset S_1([0,1])$, $S_1^{-1} J$ is a larger interval of the same density. We iterate this procedure until we obtain $J'$ not lying in $S_1([0,1])$. The proof of the second case is similar. □

For an island $I = [c,d] \subset [0,1]$, we introduce the following notations.

$$\underline{D}_0(I) = \inf_{0<x\leq 1} \{ d([c, c + x(d-c)]) \} \text{ and } \underline{D}_1(I) = \inf_{0<x\leq 1} \{ d([d - x(d-c), d]) \}.$$

Obviously, if $I_1$ and $I_2$ are two islands with the same overlap type, i.e., $[I_1] = [I_2]$, then

$$\underline{D}_0(I_1) = \underline{D}_0(I_2) \text{ and } \underline{D}_1(I_1) = \underline{D}_1(I_2).$$

Hence the notations $\underline{D}_0(I)$ and $\underline{D}_1(I)$ depend only on the overlap type of the island $I$. Thus we could define the following constants. For $1 \leq i \leq q$, define

$$\underline{D}_0^i := \underline{D}_0(I) \text{ and } \underline{D}_1^i := \underline{D}_1(I)$$

where $I$ is an arbitrary $\mathcal{T}_i$ type island. These notations are independent of the choice of the island $I$. The following lemma shows Assumption A and Assumption B will ensure that all $\underline{D}_0^i$'s are equal and all $\underline{D}_1^i$'s are also equal.

**Lemma 4.4.**

$$\underline{D}_0^1 = \cdots = \underline{D}_0^q \text{ and } \underline{D}_1^1 = \cdots = \underline{D}_1^q. \tag{4.1}$$

*Proof.* Fix $2 \le i \le q$. We prove $\underline{D}_0^1 = \underline{D}_0^i$. Take a $\mathcal{T}_i$ type island $I := [c, d]$. Now we turn to prove $\underline{D}_0([0,1]) = \underline{D}_0(I)$ since $[0,1]$ is of $\mathcal{T}_1$ type.

Using Remark 2, we have $S_{\mathbf{v}_0}S_{\mathbf{i}_0}([0,1]) \cap K = S_{\mathbf{v}_0}S_{\mathbf{i}_0}K$ where $\mathbf{v}_0 = \mathbf{v}_0(I)$ and $\mathbf{i}_0 = \mathbf{i}_0(I)$. Hence for any $0 < x \le 1$, since $S_{\mathbf{v}_0}S_{\mathbf{i}_0}([0,1]) \cap K$ is similar to $K$, we have

$$d([0,x]) = d([c, c + x(S_{\mathbf{v}_0}S_{\mathbf{i}_0}(1) - c)]),$$

which yields that $\underline{D}_0^i \le \underline{D}_0^1$, by the arbitrariness of $x$ and the fact that $[c, c+x(S_{\mathbf{v}_0}S_{\mathbf{i}_0}(1) - c)] \subset I$.

On the other hand, since $I = \bigcup_{\mathbf{v} \in V(I)} S_{\mathbf{v}}([0,1])$, we denote by $c = a_1 < a_2 < \cdots < a_n$ the left endpoints of all component intervals of $I$ in increasing order. (By Assumption A, it is impossible that some distinct component intervals share a common left endpoint.) For any interval $[c, z] \subset I$, choose a largest $a_i$ such that $a_i \le z$. If $a_i < z$, then

$$\begin{aligned} d([c,z]) &= \frac{\lambda([a_1,a_2]) + \cdots + \lambda([a_{i-1},a_i]) + \lambda([a_i,z])}{((a_2-a_1) + \cdots + (a_i - a_{i-1}) + (z - a_i))^\alpha} \\ &\ge \frac{\lambda([a_1,a_2]) + \cdots + \lambda([a_{i-1},a_i]) + \lambda([a_i,z])}{(a_2-a_1)^\alpha + \cdots + (a_i - a_{i-1})^\alpha + (z - a_i)^\alpha} \\ &\ge \min\{d([a_1,a_2]), \cdots, d([a_{i-1},a_i]), d([a_i,z])\} \\ &\ge \underline{D}_0^1. \end{aligned}$$

The last inequality follows from the fact that for each $\mathbf{v} \in V(I)$, $S_{\mathbf{v}}K \subset S_{\mathbf{v}}([0,1]) \cap K$, and $S_{\mathbf{v}}K$ is similar to $K$. In the case $a_i = z$, we have the same result by a similar discussion. The arbitrariness of $z$ yields that $\underline{D}_0^i \ge \underline{D}_0^1$.

Hence we have $\underline{D}_0^1 = \underline{D}_0^i$. By the arbitrariness of $2 \le i \le q$, we get the first equality in (4.1). The second equality can be proved in a similar way. □

We will frequently use the following notations.

**Notations.** Let

$$\underline{D}_0 = \inf\{d([0,x]) : 0 < x \le 1\} \text{ and } \underline{D}_1 = \inf\{d([y,1]) : 0 \le y < 1\}.$$

$$\overline{D}_0 = \sup\{d([0,x]) : 0 < x \le 1\} \text{ and } \overline{D}_1 = \sup\{d([y,1]) : 0 \le y < 1\}.$$

From Lemma 4.4, we know that $\underline{D}_0$ is the common value of $\underline{D}_0^1, \cdots, \underline{D}_0^q$, and $\underline{D}_1$ is the common value of $\underline{D}_1^1, \cdots, \underline{D}_1^q$. The following lemma characterizes $\underline{D}_0$ and $\underline{D}_1$ by the parameters $\beta$'s and $\gamma$'s.

**Lemma 4.5.** *Let $k$ be the smallest integer such that $\beta_1^{(k)} \le \rho_1$. Then*

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_k\}.$$

*Similarly, let* $k'$ *be the smallest integer such that* $\beta_{l_{k'}}^{(k')} \le \rho_m$*. Then*

$$\underline{D}_1 = \min\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_{k'}\}.$$

*Proof.* For simplicity, we only prove the first equality. By Assumption B and the blow-up principle Lemma 4.3, we only need to consider the interval $[0,x]$ with $\rho_1 < x \le 1$. Since $d([0,x])$ is a continuous function of $x$, $d([0,x])$ attains its minimum $\underline{D}_0$ at some $x_0 \in [\rho_1, 1]$. Furthermore noting that $d([0,1]) = \underline{D}_0$ whenever $d([0,\rho_1]) = \underline{D}_0$, we can assume $x_0 > \rho_1$.

It is clear that the point $x_0$ can not fall in a non-empty lake of $\mathcal{F}_k$, because then $[0,x_0]$ would not have minimal density. Therefore there exists a $k$-th generation island $I = [c,d]$ such that $x_0 \in [c,d]$. Here $c > 0$ because otherwise $x_0 \le \beta_1^{(k)} \le \rho_1$ which contradicts $x_0 > \rho_1$. Take $u = x_0 - c$. Assume that $u > 0$, then

$$d([0,x_0]) = \frac{\lambda([0,c]) + \lambda([c,x_0])}{(c+u)^\alpha} > \frac{\lambda([0,c]) + \lambda([c,x_0])}{c^\alpha + u^\alpha} \ge \min\{d([0,c]), d([c,x_0])\} \ge \underline{D}_0,$$

which contradicts the minimality of $d([0,x_0])$. Hence $u = 0$. Thus

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_k\}. \quad \square$$

We would also like to characterize $\overline{D}_0$ and $\overline{D}_1$ by the parameters $\beta$'s and $\gamma$'s. The following elementary calculus lemma by Ayer & Strichartz is useful.

**Lemma 4.6[1].** *Suppose* $0 < \alpha < 1$, $p \le p_0$, $a \ge a_0$, $\kappa > 0$ *and* $y \ge \kappa x^\alpha$*. Then*

$$0 < x \le \left(\frac{a_0\kappa}{p_0}\right)^{\frac{1}{1-\alpha}} \tag{4.2}$$

*implies*

$$\frac{p-y}{(a-x)^\alpha} < \frac{p}{a^\alpha}. \tag{4.3}$$

To make the paper self-contained, we give the proof of Lemma 4.6 as follows.

*Proof of Lemma 4.6.* Consider the function $f(x) = (p - \kappa x^\alpha)/(a-x)^\alpha$. Noting that $(p-y)/(a-x)^\alpha \le f(x)$ by the assumption $y \ge \kappa x^\alpha$. And by $p/a^\alpha = f(0)$, it suffices to show $f'(x) < 0$ on the interval $0 < x < (a_0\kappa/p_0)^{1/(1-\alpha)}$. This can be verified by a direct computation. $\square$

Using the above lemma, we have the following result concerning $\overline{D}_0$ and $\overline{D}_1$.

**Lemma 4.7.** *Let* $k$ *be the smallest integer such that* $\beta_{\max}^{(k)} \le (\rho_1 \underline{D}_1)^{1/(1-\alpha)}$*. Then*

$$\overline{D}_0 = \max\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_k\}.$$

*Similarly, let* $k'$ *be the smallest integer such that* $\beta_{\max}^{(k')} \le (\rho_m \underline{D}_0)^{1/(1-\alpha)}$*. Then*

$$\overline{D}_1 = \max\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_{k'}\}.$$

*Proof.* For simplicity, we only prove the first equality. By the blow-up principle Lemma 4.3 we can take $x_0 > \rho_1$ such that the interval $[0,x_0]$ has the maximal density (By compactness of $[\rho_1, 1]$, the maximum is attained). The point $x_0$ cannot fall in a non-empty lake of $\mathcal{F}_k$, because if so, $[0,x_0]$ would not have maximal density. Let $[0,a]$ be the smallest interval in $\mathcal{F}_k$ that contains $[0, x_0]$. Then $x_0 = a - x$ for some $x \leq \beta_{\max}^{(k)}$ because $\beta_{\max}^{(k)}$ is the length of the largest island in $\mathcal{F}_k^0$. Set $p = \lambda([0,a])$ and $y = \lambda([a-x,a])$, then $d([0,a-x]) = (p-y)/(a-x)^\alpha$ and $d([0,a]) = p/a^\alpha$. Thus the conclusion (4.3) of Lemma 4.6 would give $d([0,a-x]) < d([0,a])$ unless $x = 0$, which implies that $[0,a]$ attains the maximal density.

To complete the proof we will verify the hypothesis of Lemma 4.6 with $p_0 = 1$, $a_0 = \rho_1$ and $\kappa = \underline{D}_1$. We already know $a \geq \rho_1$, and $p \leq 1$ is trivial. To verify $y \geq \kappa x^\alpha$, we observe that $y/x^\alpha = d([a-x,a])$, and by the definition of $\underline{D}_1$, we immediately get $d([a-x,a]) \geq \underline{D}_1$. The hypothesis of Lemma 4.6 is verified, and condition (4.2) follows from $x \leq \beta_{\max}^k$ and the hypothesis $\beta_{\max}^{(k)} \leq (\rho_1 \underline{D}_1)^{1/(1-\alpha)}$. □

## 4.1 The maximal density and the Hausdorff measure

As stated before, we denote by $k_0$ the smallest non-negative integer such that none of the islands in $\mathcal{F}_{k_0+1}^0$ is of new overlap type. First, in the case that all lakes are nonempty, i.e., $\gamma_{\min}^{(k_0+1)} > 0$, we have the following result.

**Theorem 4.8.** *Assume $\gamma_{\min}^{(k_0+1)} > 0$, and let $k \geq k_0 + 1$ be the smallest integer such that*

$$2\beta_{\max}^{(k)} \leq (\gamma_{\min}^{(k_0+1)} \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}}. \tag{4.4}$$

*Then the maximal density $d_{\max}$ is attained for an interval in $\mathcal{F}_k$.*

*Proof.* By the blow-up principle Lemma 4.1 we may focus attention to intervals containing at least one lake of $\mathcal{F}_{k_0+1}$. So we have the lower bound $\gamma_{\min}^{(k_0+1)}$ for the length of the interval, which implies by compactness that the maximal density $d_{\max}$ is attained. If $[x_1, x_2]$ is an interval of maximal density, we let $[z_1, z_2]$ be the smallest interval in $\mathcal{F}_k$ containing $[x_1, x_2]$. Write $a = z_2 - z_1$ for the length of the interval, $x = (z_2 - z_1) - (x_2 - x_1)$ for the difference of the lengths, $p = \lambda([z_1, z_2])$ and $y = \lambda([z_1, x_1]) + \lambda([x_2, z_2])$, so that $d([x_1, x_2]) = (p-y)/(a-x)^\alpha$ and $d([z_1, z_2]) = p/a^\alpha$. In a similar way in proving Lemma 4.7, we will complete the proof by applying Lemma 4.6.

We take $p_0 = 1$ and $a_0 = \gamma_{\min}^{(k_0+1)}$, so that $a \geq a_0$ and $p \leq p_0$. We choose $\kappa = \min\{\underline{D}_0, \underline{D}_1\}$. For the right side interval $[x_2, z_2]$ we have

$$\frac{\lambda([x_2, z_2])}{(z_2 - x_2)^\alpha} \geq \underline{D}_1,$$

and similarly for the left side interval $[z_1, x_1]$ we have

$$\frac{\lambda([z_1, x_1])}{(x_1 - z_1)^\alpha} \geq \underline{D}_0.$$

Thus we have

$$y \geq \min\{\underline{D}_0, \underline{D}_1\}((x_1 - z_1)^\alpha + (z_2 - x_2)^\alpha) \geq \kappa x^\alpha.$$

Thus the hypothesis of Lemma 4.6 is verified, and condition (4.2) follows from (4.4) since $x$ is the sum of two terms, $x_1 - z_1$ and $z_2 - x_2$, each being at most $\beta_{\max}^{(k)}$. □

Now we turn to discuss the case that there exist touching islands. First we can still obtain a result if we assume that $\rho_1$ and $\rho_m$ are exponentially commensurable. In the following, $\eta$ is the positive number defined in Remark 3.

**Theorem 4.9.** *Suppose there exist positive integers $n_1$ and $n_m$ such that $\rho_1^{n_1} = \rho_m^{n_m}$. Let $k$ be the smallest integer such that*

$$2\beta_{\max}^{(k)} \leq (\eta\beta_{\min}^{(k_0+1)}\rho_1^{n_1}\min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}}.$$

*Then the maximal density $d_{\max}$ is attained for an interval in $\mathcal{F}_k$.*

*Proof.* We claim that it suffices to consider intervals of length at least $\eta\beta_{\min}^{(k_0+1)}\rho_1^{n_1}$. To see this we need a variant of the blow-up principle that shows how to replace smaller intervals with larger intervals of greater density.

Start with any interval $I_0$ not contained in a $(k_0+1)$-th generation island. If it actually contains a $(k_0+1)$-th generation island, its length is at least $\beta_{\min}^{(k_0+1)}$, and we are done. If not, it begins at a point in $I$ and ends at a point in $I'$ where $I$, $I'$ are two adjacent $(k_0+1)$-th generation islands. Denote by $L$ the lake separating $I$ and $I'$. Using Remark 2, there exists a vertex $\mathbf{v} \in V(I)$ and an index $\mathbf{i} = (m, \cdots, m)$ such that $S_{\mathbf{v}}([0,1]) \subset I$ has the same right endpoint as that of $I$ and $S_{\mathbf{v}}S_{\mathbf{i}}([0,1]) \cap K = S_{\mathbf{v}}S_{\mathbf{i}}K$. Similarly, there also exists a vertex $\mathbf{v}' \in V(I')$ and an index $\mathbf{i}' = (1, \cdots, 1)$ such that $S_{\mathbf{v}'}([0,1]) \subset I'$ has the same left endpoint as that of $I'$ and $S_{\mathbf{v}'}S_{\mathbf{i}'}([0,1]) \cap K = S_{\mathbf{v}'}S_{\mathbf{i}'}K$. For simplicity, we denote $S_{\mathbf{v}}S_{\mathbf{i}}([0,1])$ and $S_{\mathbf{v}'}S_{\mathbf{i}'}([0,1])$ by $\widetilde{I}$ and $\widetilde{I}'$ respectively. (These notations will be used again in Lemma 4.11 and Theorem 4.12.) Now consider the intervals $J = S_{\mathbf{v}}S_{\mathbf{i}}S_m^{n_m}([0,1])$ and $J' = S_{\mathbf{v}'}S_{\mathbf{i}'}S_1^{n_1}([0,1])$ which lie on the extreme ends of the lake $L$ separating $\widetilde{I}$ and $\widetilde{I}'$. These intervals have length $\rho_{\mathbf{v}}\rho_{\mathbf{i}}\rho_m^{n_m}$ and $\rho_{\mathbf{v}'}\rho_{\mathbf{i}'}\rho_1^{n_1}$. Moreover, $J \cap K$ and $J' \cap K$ are similar to $K$. If the interval $I_0$ contains one of them, we are done.

Next suppose the interval begins with a point in $J$ and ends with a point in $J'$, say $I_0 = J_0 \cup L \cup J_0'$ where $J_0 = I_0 \cap J$ and $J_0' = I_0 \cap J'$ and $L$ is the lake separating $J$ and $J'$. We generate another interval $I_1 = J_1 \cup L \cup J_1'$ by blowing up $J_0$ to $J_1$ and $J_0'$ to $J_1'$ by a factor $\rho_1^{-n_1} = \rho_m^{-n_m}$. Specifically, we set $J_1 = S_{\mathbf{v}}S_{\mathbf{i}}S_m^{-n_m}(S_{\mathbf{v}}S_{\mathbf{i}})^{-1}J_0$ and $J_1' = S_{\mathbf{v}'}S_{\mathbf{i}'}S_1^{-n_1}(S_{\mathbf{v}'}S_{\mathbf{i}'})^{-1}J_0'$. Note that $S_{\mathbf{v}}S_{\mathbf{i}}S_m^{-n_m}(S_{\mathbf{v}}S_{\mathbf{i}})^{-1}$ maps $J$ onto $\widetilde{I}$ and fixes the right endpoint, while $S_{\mathbf{v}'}S_{\mathbf{i}'}S_1^{-n_1}(S_{\mathbf{v}'}S_{\mathbf{i}'})^{-1}$ maps $J'$ onto $\widetilde{I}'$ and fixes the left endpoint. So $I_1$ is an interval. We have

$$d(I_0) = \frac{\lambda(J_0) + \lambda(J_0')}{(|J_0| + |L| + |J_0'|)^\alpha},$$

while

$$d(I_1) = \frac{\rho_m^{-n_m\alpha}\lambda(J_0) + \rho_1^{-n_1\alpha}\lambda(J_0')}{(\rho_m^{-n_m}|J_0| + |L| + \rho_1^{-n_1}|J_0'|)^\alpha} = \frac{\lambda(J_0) + \lambda(J_0')}{(|J_0| + \rho_1^{n_1}|L| + |J_0'|)^\alpha}.$$

So $d(I_1) \geq d(I_0)$. By iterating this blow-up construction we eventually arrive at an interval containing either $J$ or $J'$ whose density is greater than the original interval $I_0$. Hence we only need to consider intervals containing either $J$ or $J'$ which have length at least $\min\{\rho_{\mathbf{v}}\rho_{\mathbf{i}}, \rho_{\mathbf{v}'}\rho_{\mathbf{i}'}\}\rho_1^{n_1}$. This completes the proof that it suffices to look at intervals of length at least $\eta\beta_{\min}^{(k_0+1)}\rho_1^{n_1}$.

The rest of the argument is identical to the proof of Theorem 4.8, except that we take $a_0 = \eta\beta_{\min}^{(k_0+1)}\rho_1^{n_1}$. □

We consider now the case when the contraction ratios $\rho_1$ and $\rho_m$ do not satisfy the arithmetic condition. Another elementary calculus lemma proved by Ayer & Strichartz in [1] will be needed.

**Lemma 4.10**[1]**.** *Let $a, a', q, q' > 0$, and $0 < \alpha < 1$. $F(x)$ is a function*

$$F(x) = \frac{q + q'x^{\alpha}}{(a + a'x)^{\alpha}} \tag{4.5}$$

*of positive variables. Then $F$ attains the maximal value of $\left((\frac{q}{a^{\alpha}})^{\frac{1}{1-\alpha}} + (\frac{q'}{a'^{\alpha}})^{\frac{1}{1-\alpha}}\right)^{1-\alpha}$ at the point $x_0 = (\frac{aq'}{a'q})^{\frac{1}{1-\alpha}}$. Furthermore, $F(x)$ is strictly increasing on $0 \leq x \leq x_0$ and strictly decreasing on $x > x_0$.*

*Proof.* This can be done by a directly computation of $F'(x)$. □

**Lemma 4.11.** *Suppose $I$ and $I'$ are two $(k_0+1)$-th generation touching islands, and $\rho_1$ and $\rho_m$ are not exponentially commensurable, i.e., $\rho_1^{n_1} \neq \rho_m^{n_m}$ for any positive integers $n_1$ and $n_m$. $\widetilde{I}$ and $\widetilde{I}'$ are the same as that defined in the proof of Theorem 4.9. Then the maximal density of intervals beginning in $\widetilde{I}$ and ending in $\widetilde{I}'$ is*

$$(\overline{D}_0^{\frac{1}{1-\alpha}} + \overline{D}_1^{\frac{1}{1-\alpha}})^{1-\alpha}. \tag{4.6}$$

*Proof.* Any such interval can be written as $I_0 = \widetilde{I}_0 \cup \widetilde{I}'_0$ where $\widetilde{I}_0 \subset \widetilde{I}$ ends at the right endpoint of $\widetilde{I}$, and $\widetilde{I}'_0 \subset \widetilde{I}'$ begins at the left endpoint of $\widetilde{I}'$. Let $\mathbf{v}, \mathbf{v}', \mathbf{i}, \mathbf{i}'$ be the notations used in the proof of Theorem 4.9. So $S_{\mathbf{v}}S_{\mathbf{i}}([0,1]) = \widetilde{I}$ and $S_{\mathbf{v}'}S_{\mathbf{i}'}([0,1]) = \widetilde{I}'$. For any positive integers $k$ and $k'$ we can form the interval

$$I(k,k') = S_{\mathbf{v}}S_{\mathbf{i}}S_m^k(S_{\mathbf{v}}S_{\mathbf{i}})^{-1}\widetilde{I}_0 \cup S_{\mathbf{v}'}S_{\mathbf{i}'}S_1^{k'}(S_{\mathbf{v}'}S_{\mathbf{i}'})^{-1}\widetilde{I}'_0,$$

which contracts $\widetilde{I}_0$ by a factor of $\rho_m^k$ and $\widetilde{I}'_0$ by a factor of $\rho_1^{k'}$, keeping their common endpoint fixed. Then

$$d(I(k,k')) = \frac{\rho_m^{k\alpha}\lambda(\widetilde{I}_0) + \rho_1^{k'\alpha}\lambda(\widetilde{I}'_0)}{(\rho_m^k|\widetilde{I}_0| + \rho_1^{k'}|\widetilde{I}'_0|)^{\alpha}} = \frac{\rho_m^{k\alpha}q + \rho_1^{k'\alpha}q'}{(\rho_m^k a + \rho_1^{k'}a')^{\alpha}}, \tag{4.7}$$

where $a = |\widetilde{I}_0|$, $a' = |\widetilde{I}'_0|$ and $q = \lambda(\widetilde{I}_0)$, $q' = \lambda(\widetilde{I}'_0)$. Notice that this is exactly of the form (4.5) with $x = \rho_1^{k'}\rho_m^{-k}$. By the fact that $\rho_1$ and $\rho_m$ are not exponentially commensurable, $x$ takes on a dense set of values on the positive line. Thus by Lemma 4.10, (4.7) has its maximal value

$$(d(\widetilde{I}_0)^{\frac{1}{1-\alpha}} + d(\widetilde{I}'_0)^{\frac{1}{1-\alpha}})^{1-\alpha}. \tag{4.8}$$

Since (4.8) is an increasing function of $d(\widetilde{I}_0)$ and $d(\widetilde{I}'_0)$, and by the fact that $\widetilde{I} \cap K$ and $\widetilde{I}' \cap K$ are similar to $K$, it is clear that its maximum is attained when $d(\widetilde{I}_0)$ and $d(\widetilde{I}'_0)$ assume their maxima, and these are clearly $\overline{D}_1$ and $\overline{D}_0$. Hence the maximal density of intervals beginning in $\widetilde{I}$ and ending in $\widetilde{I}'$ is $(\overline{D}_0^{1/(1-\alpha)} + \overline{D}_1^{1/(1-\alpha)})^{1-\alpha}$. □

**Theorem 4.12.** *Suppose $\gamma_{\min}^{(k_0+1)} = 0$, and $\rho_1$ and $\rho_m$ are not exponentially commensurable. Let $k$, $k_1$ and $k_2$ be the smallest integers such that*

$$2\beta_{\max}^{(k)} \leq (\eta \min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\} \cdot \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}},$$

$$\beta_{\max}^{(k_1)} \leq (\rho_1 \underline{D}_1)^{\frac{1}{1-\alpha}} \text{ and } \beta_{\max}^{(k_2)} \leq (\rho_m \underline{D}_0)^{\frac{1}{1-\alpha}}.$$

*Then the maximal density $d_{\max}$ is equal to the maximum of the finite set of values $d(I)$ as $I$ varies over all intervals in $\mathcal{F}_k$, and $(d(I_1)^{1/(1-\alpha)} + d(I_2)^{1/(1-\alpha)})^{1-\alpha}$ as $I_1$ varies over all intervals of the form $[0,x]$ in $\mathcal{F}_{k_1}$ and $I_2$ varies over all intervals of the form $[y,1]$ in $\mathcal{F}_{k_2}$.*

*Proof.* It follows from $\gamma_{\min}^{(k_0+1)} = 0$ that there exist $(k_0+1)$-th generation touching islands in $\mathcal{F}^0_{k_0+1}$. If $I$ and $I'$ are two such islands with $I$ lying on the left side of $I'$, we apply Lemma 4.11, which means we have to consider the values of (4.6). But by Lemma 4.7, $\overline{D}_0$ is attained for an interval of the form $[0,x]$ in $\mathcal{F}_{k_1}$, and $\overline{D}_1$ is attained for an interval of the form $[y,1]$ in $\mathcal{F}_{k_2}$.

For every two touching $(k_0+1)$-th generation islands $I$ and $I'$, denote $\widetilde{I}$ and $\widetilde{I}'$ the corresponding subsets of $I$ and $I'$ respectively as that discussed in Lemma 4.11. We need to consider all intervals beginning in $I \setminus \widetilde{I}$ and ending in $I'$, or beginning in $I$ and ending in $I' \setminus \widetilde{I}'$ for some touching $(k_0+1)$-th generation islands $I$ and $I'$, and all intervals that contain either a nonempty lake of $\mathcal{F}_{k_0+1}$ or a $(k_0+1)$-th generation island. In the first case, intervals have length at least $\min\{|\widetilde{I}|, |\widetilde{I}'|\}$ which always greater than $\beta_{\min}^{(k_0+1)}\eta$ by Remark 3. And in the second case, intervals have lengths bounded below by $\min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\}$. Hence the lengths of all the above two kinds of intervals are greater than $\eta \min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\}$. Then by a slightly variant of the proof of Theorem 4.8, the maximal density over all such intervals is attained by an interval in $\mathcal{F}_k$.$\square$

## 4.2 The minimal centered density and the packing measure

First, we give a lemma concerning the relation between $d_{\min}$ and $\underline{D}_0, \underline{D}_1$.

**Lemma 4.13.** $d_{\min} \leq 2^{-\alpha}\min\{\underline{D}_0, \underline{D}_1\}$.

*Proof.* It suffices to show that there exist $J_0$, $J_1$ centered in $K$, contained in $[0,1]$, such that $d(J_0) = 2^{-\alpha}\underline{D}_0$ and $d(J_1) = 2^{-\alpha}\underline{D}_1$. For simplicity, we only prove the first equality. By Lemma 4.5, let $k$ be the smallest integer such that $\beta_1^{(k)} \leq \rho_1$, then there exists $x_0$ with $[0,x_0] \subset \mathcal{F}_k$ such that $d([0,x_0]) = \underline{D}_0$. Since $\alpha < 1$, there must exist at least one non-empty lake in $\mathcal{F}_1$, i.e., there exists $1 \leq i \leq l_1 - 1$ with $\gamma_i^{(1)} > 0$, where $\gamma_i^{(1)}$ is the length of the lake separating the $i$-th and $(i+1)$-th first generation islands $I$ and $I'$. From the discussion in Remark 2, there is a unique vertex $\mathbf{v} \in V(I')$ and an index $\mathbf{i} = (1,\cdots,1)$ such that the sub-interval $S_\mathbf{v}S_\mathbf{i}([0,1]) \subset I'$ has the same left endpoint as that of $I'$ and furthermore, $S_\mathbf{v}S_\mathbf{i}([0,1]) \cap K = S_\mathbf{v}S_\mathbf{i}K$, i.e., $S_\mathbf{v}S_\mathbf{i}([0,1]) \cap K$ is similar to $K$ with a contraction ratio $\rho_\mathbf{v}\rho_\mathbf{i}$. Choose a non-negative integer $k'$ large enough so that $\rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0 < \gamma_i^{(1)}$. Hence the interval $[S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) - \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0, S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0)]$ is contained in the lake separating $I$ and $I'$. Thus $\lambda([S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) - \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0, S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0)]) = 0$. Define

$$J_0 := [S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) - \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0, S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) + \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0].$$

Since the interval $S_\mathbf{v}S_\mathbf{i}S_1^{k'}([0,1]) \cap K$ is also similar to $K$, we have

$$\begin{aligned} d(J_0) &= \frac{\lambda([S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) - \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0, S_\mathbf{v}S_\mathbf{i}S_1^{k'}(0) + \rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0])}{2^\alpha(\rho_\mathbf{v}\rho_\mathbf{i}\rho_1^{k'}x_0)^\alpha} \\ &= \frac{\lambda(S_\mathbf{v}S_\mathbf{i}S_1^{k'}([0,x_0]))}{2^\alpha|S_\mathbf{v}S_\mathbf{i}S_1^{k'}([0,x_0])|^\alpha} = 2^{-\alpha}d(S_\mathbf{v}S_\mathbf{i}S_1^{k'}([0,x_0])) \\ &= 2^{-\alpha}d([0,x_0]) = 2^{-\alpha}\underline{D}_0, \end{aligned}$$

which concludes the proof. □

**Lemma 4.14.** *Let $J \subset [0,1]$ be an interval centered in $K$ and not contained in any $(k_0+1)$-th generation island. If $|J| \le \min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\}$, then $d(J) \ge 2^{-\alpha} \min\{\underline{D}_0, \underline{D}_1\}$.*

*Proof.* For each interval $J \subset [0,1]$ centered in $K$ and not contained in any $(k_0+1)$-th generation island with $|J| \le \min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\}$, it is clear that there are only three possible cases for $J$.

Case 1: There exist two touching $(k_0+1)$-th generation islands $I_1$ and $I_2$ with $I_1$ lying on the left side of $I_2$, and $J = J_1 \cup J_2$, where $J_1 \subset I_1$ and $J_2 \subset I_2$.

In this case, we have

$$d(J) = \frac{\lambda(J_1) + \lambda(J_2)}{(|J_1| + |J_2|)^\alpha} > \frac{\lambda(J_1) + \lambda(J_2)}{|J_1|^\alpha + |J_2|^\alpha} \ge \min\{d(J_1), d(J_2)\} \ge \min\{\underline{D}_0, \underline{D}_1\},$$

by the definition of the constants $\underline{D}_0, \underline{D}_1$. It follows that $d(J) > \min\{\underline{D}_0, \underline{D}_1\}$.

Case 2: There exist two separate $(k_0+1)$-th generation islands $I_1$ and $I_2$ with $I_1$ lying on the left side of $I_2$, and $J = J_1 \cup J_2$, where $J_1 \subset I_1$, $J_2 \subset L$, and $L$ is the lake separating $I_1$ and $I_2$.

In this case, we have $|J_1| \ge |J_2|$ since $J$ is centered in $K$. Thus

$$d(J) = \frac{\lambda(J_1)}{(|J_1| + |J_2|)^\alpha} \ge \frac{\lambda(J_1)}{2^\alpha |J_1|^\alpha} = 2^{-\alpha} d(J_1) \ge 2^{-\alpha} \underline{D}_1$$

by the definition of $\underline{D}_1$.

Case 3: There exist two separate $(k_0+1)$-th generation islands $I_1$ and $I_2$ with $I_1$ lying on the left side of $I_2$, and $J = J_1 \cup J_2$, where $J_1 \subset L$, $J_2 \subset I_2$, and $L$ is the lake separating $I_1$ and $I_2$.

In this case, we have $d(J) \ge 2^{-\alpha} \underline{D}_0$ by a discussion similar to Case 2.

Combining the above discussion completes the proof. □

For convenience, denote all $(k_0+1)$-th generation islands in increasing order by $I_1, \cdots, I_{l_{k_0+1}}$. For each $1 \le i \le l_{k_0+1}$, write $I_i = [a_i, b_i]$.

**Lemma 4.15.** *If $d_{\min} < 2^{-\alpha} \min\{\underline{D}_0, \underline{D}_1\}$, then*

$$d_{\min} = \min_{1 \le i_1 < i_2 < l_{k_0+1}} \frac{\sum_{i=i_1+1}^{i_2} \lambda([a_i, b_i])}{(a_{i_2+1} - b_{i_1} - 2dist(\frac{a_{i_2+1}+b_{i_1}}{2}, K))^\alpha}.$$

*Proof.* Since $d_{\min} < 2^{-\alpha} \min\{\underline{D}_0, \underline{D}_1\}$, by the blow-up principle Lemma 4.1 and Lemma 4.14, we only need to consider intervals with lengths greater than $\min\{\beta_{\min}^{(k_0+1)}, \gamma_{\min}^{*(k_0+1)}\}$ which are centered in $K$ and not contained in any $(k_0+1)$-th generation island.

By the compactness of $K$, there exists a such interval $J_0 = [a_0, b_0]$ such that $d_{\min} = d(J_0)$. First we prove the following statements.

(1) Either $a_0 \in \{b_i : 1 \le i \le l_{k_0+1} - 1\}$ or $a_0$ is contained in one lake;

(2) Either $b_0 \in \{a_i : 2 \le i \le l_{k_0+1}\}$ or $b_0$ is contained in one lake.

For simplicity we only prove (1). The statement (2) will follow by a similar argument. Assume that (1) is not true. Then there exists a $1 \le i \le l_{k_0+1} - 1$ such that $a_0 \in [a_i, b_i)$.

In the following we will lead to a contradiction. We first claim $(a_0+b_0)/2 > b_i$. Otherwise $(a_0+b_0)/2 \in [a_i, b_i]$, then

$$d([a_0,b_0]) = \frac{\lambda([a_0,b_0])}{(b_0-a_0)^\alpha} \geq \frac{\lambda([a_0,b_i])}{2^\alpha(b_i-a_0)^\alpha} = 2^{-\alpha}d([a_0,b_i]) \geq 2^{-\alpha}\underline{D}_1,$$

which contradicts the assumption $d_{\min} < 2^{-\alpha}\min\{\underline{D}_0, \underline{D}_1\}$. Hence it follows that

$$\begin{aligned}
d([a_0,b_0]) &= \frac{\lambda([a_0,b_i]) + \lambda([b_i, a_0+b_0-b_i]) + \lambda([a_0+b_0-b_i, b_0])}{(2(b_i-a_0)+(a_0+b_0-2b_i))^\alpha} \\
&> \frac{\lambda([a_0,b_i]) + \lambda([b_i, a_0+b_0-b_i]) + \lambda([a_0+b_0-b_i, b_0])}{2^\alpha(b_i-a_0)^\alpha+(a_0+b_0-2b_i)^\alpha} \\
&\geq \frac{\lambda([a_0,b_i]) + \lambda([b_i, a_0+b_0-b_i])}{2^\alpha(b_i-a_0)^\alpha+(a_0+b_0-2b_i)^\alpha} \\
&\geq \min\{2^{-\alpha}\frac{\lambda([a_0,b_i])}{(b_i-a_0)^\alpha}, \frac{\lambda([b_i,a_0+b_0-b_i])}{(a_0+b_0-2b_i)^\alpha}\} \\
&= \min\{2^{-\alpha}d([a_0,b_i]), d([b_i, a_0+b_0-b_i])\} \\
&\geq \min\{2^{-\alpha}\underline{D}_1, d([b_i, a_0+b_0-b_i])\}.
\end{aligned}$$

Since $[b_i, a_0+b_0-b_i] \subset [0,1]$ is an interval centered in $K$, the above inequality contradicts the fact that $d([a_0,b_0])$ attains the minimal value $d_{\min} < 2^{-\alpha}\underline{D}_1$. Thus the statement (1) is true.

By the statement (1) and (2), we have $a_0 \in [b_{i_1}, a_{i_1+1})$ and $b_0 \in (b_{i_2}, a_{i_2+1}]$ for some $1 \leq i_1 < i_2 < l_{k_0+1}$. Since $\lambda([a_0,b_0]) = \sum_{i=i_1+1}^{i_2}\lambda([a_i,b_i])$ and $d([a_0,b_0])$ attains the minimal centered density $d_{\min}$, it follows that $a_0$, $b_0$ are taken such that $(b_0-a_0)$ is the largest value under the condition $a_0 \in [b_{i_1}, a_{i_1+1})$, $b_0 \in (b_{i_2}, a_{i_2+1}]$ and $(a_0+b_0)/2 \in K$. Thus we have $b_0 - a_0 = a_{i_2+1} - b_{i_1} - 2\mathrm{dist}(\frac{a_{i_2+1}+b_{i_1}}{2}, K)$ and

$$d([a_0,b_0]) = \frac{\sum_{i=i_1+1}^{i_2}\lambda([a_i,b_i])}{(a_{i_2+1}-b_{i_1}-2\mathrm{dist}(\frac{a_{i_2+1}+b_{i_1}}{2},K))^\alpha}.$$

Therefore we complete the proof of Lemma 4.15. □

**Theorem 4.16.** Let

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_k\},$$

where $k$ is the smallest number such that $\beta_1^{(k)} \leq \rho_1$;

$$\underline{D}_1 = \min\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_{k'}\},$$

where $k'$ is the smallest number such that $\beta_{l_{k'}}^{(k')} \leq \rho_m$;

$$D = \min_{1\leq i_1<i_2<l_{k_0+1}} \frac{\sum_{i=i_1+1}^{i_2}\lambda([a_i,b_i])}{(a_{i_2+1}-b_{i_1}-2\mathrm{dist}(\frac{a_{i_2+1}+b_{i_1}}{2},K))^\alpha},$$

where $[a_i,b_i]$ is the $i$-th $(k_0+1)$-th generation island for each $1 \leq i \leq l_{k_0+1}$. Then

$$d_{\min} = \min\{2^{-\alpha}\underline{D}_0, 2^{-\alpha}\underline{D}_1, D\}.$$

*Proof.* It follows immediately from Lemma 4.5, Lemma 4.13 and Lemma 4.15. □

## 4.3 Examples on computing measures

In this subsection we will show how to compute the Hausdorff and packing measures of $K$, using the above theory. Let's look at the examples in Section 2 again.

**Example 4.17.** If $\{S_j = \rho_j x + b_j\}_{j=1}^m$ satisfies the OSC as showed in Example 2.4. We assume $0 < \rho_j < 1$ for each $j = 1, \cdots, m$. Without loss of generality, we assume that the images $S_j([0,1])$ are in increasing order, with $S_1(0) = 0$ and $S_m(1) = 1$. Then $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ can be calculated as the minimal or maximal value of a finite set of elementary functions of the parameters $\rho$'s and $b$'s. This is true since $K$ naturally satisfies all the assumptions in this section. The results have already been proved in [1] and [6] for $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$ respectively.

**Example 4.18.** Let $K$ be the invariant set of the IFS $\{S_j\}_{j=1}^3$ on $\mathbb{R}$ defined in Example 2.5. Let $\alpha$ denote the dimension of $K$.

If $\rho + 2r - \rho r = 1$, then $\alpha = 1$, $K = [0,1]$, thus $\mathcal{H}^1(K) = \mathcal{P}^1(K) = 1$. Hence we only need to consider the non-trivial case $\rho + 2r - \rho r < 1$. For $k \geq 0$, let $\mathcal{M}_k = \Sigma_k$. Denote $I_1 = [0,1]$ and $I_2 = S_1([0,1]) \cup S_2([0,1])$. It is easy to verify that $[I_1]$ and $[I_2]$, denoted respectively by $\mathcal{T}_1$ and $\mathcal{T}_2$, are the total distinct overlap types. Hence $\mathcal{F}^0/\sim = \{\mathcal{T}_1, \mathcal{T}_2\}$. For this case, $\alpha < 1$, $k_0$=1. Using Lemma 4.5, we get

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_2\} \text{ and } \underline{D}_1 = \min\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_1\}.$$

Using the definition of $\lambda$, we have $\lambda(I_1) = 1$ and $\lambda(I_2) = 1 - r^\alpha$. Then after a computation, we get the exact values

$$\underline{D}_0 = \min\{1, \frac{1-r^\alpha}{(1-r)^\alpha}, \frac{(\rho^\alpha + r^\alpha)(1-r^\alpha)}{(\rho+r)^\alpha(1-r)^\alpha}, \frac{1-r^{2\alpha}}{(1-r^2)^\alpha}\} = \frac{1-r^\alpha}{(1-r)^\alpha},$$

and

$$\underline{D}_1 = \min\{1, \frac{r^\alpha}{(1-\rho-r+\rho r)^\alpha}\} = \frac{r^\alpha}{(1-\rho-r+\rho r)^\alpha}.$$

*Hausdorff measure.* A calculation shows $\gamma_{\min}^{(k_0+1)} = \gamma_{\min}^{(2)} = (1-2r-\rho+\rho r) \cdot \min\{\rho, r\}$, and $\beta_{\max}^{(k)} = (\rho + r - \rho r) \cdot (\max\{\rho, r\})^{k-1}$ for each $k \geq 1$. Let $k \geq 2$ be the smallest integer such that $2\beta_{\max}^{(k)} \leq (\gamma_{\min}^{(2)} \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}}$, i.e.,

$$2(\rho + r - \rho r) \cdot (\max\{\rho, r\})^{k-1} \leq ((1-2r-\rho+\rho r) \cdot \min\{\rho, r\} \cdot \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}}.$$

Then by Theorem 4.8, the maximal density $d_{\max}$ is attained for an interval in $\mathcal{F}_k$. Furthermore, by Corollary 1.3, $\mathcal{H}^\alpha(K) = d_{\max}^{-1}$.

*Packing measure.* Since $k_0 = 1$, the constant $D$ in Theorem 4.16 is

$$D = \min_{1 \leq i_1 < i_2 < 5} \frac{\sum_{i=i_1+1}^{i_2} \lambda([a_i, b_i])}{(a_{i_2+1} - b_{i_1} - 2\text{dist}(\frac{a_{i_2+1}+b_{i_1}}{2}, K))^\alpha},$$

where $[a_i, b_i]$ is the $i$-th 2-th generation island for each $1 \leq i \leq 5$. A calculation yields the exact value of $D$,

$$D=\min\{\frac{r^\alpha-r^{2\alpha}}{(\rho+r-2\rho r-r^2-\rho^2+\rho^2r-2\mathrm{dist}(\frac{\rho+r-r^2+\rho^2-\rho^2r}{2},K))^\alpha},$$
$$\frac{r^\alpha}{(1-r-\rho^2-\rho r+\rho^2r-2\mathrm{dist}(\frac{1-r+\rho^2+\rho r-\rho^2r}{2},K))^\alpha},$$
$$\frac{2r^\alpha-r^{2\alpha}}{(1-r^2-\rho^2-\rho r+\rho^2r-2\mathrm{dist}(\frac{1-r^2+\rho^2+\rho r-\rho^2r}{2},K))^\alpha},$$
$$\frac{r^{2\alpha}}{(1-r-\rho-r^2+\rho r^2-2\mathrm{dist}(\frac{1-r+\rho+r^2-\rho r^2}{2},K))^\alpha},$$
$$\frac{r^\alpha}{(1-\rho-2r^2+\rho r^2-2\mathrm{dist}(\frac{1+\rho-\rho r^2}{2},K))^\alpha},$$
$$\frac{r^\alpha-r^{2\alpha}}{(1-r-\rho-r^2+\rho r-2\mathrm{dist}(\frac{1+r+\rho-r^2-\rho r}{2},K))^\alpha}\}.$$

Then by Theorem 4.16, the minimal centered density $d_{\min}=\min\{2^{-\alpha}\underline{D}_0,2^{-\alpha}\underline{D}_1,D\}$. Hence by Corollary 1.4, $\mathcal{P}^\alpha(K)=d_{\min}^{-1}$. □

**Remark.** Consider the special case where $\rho=r=1/16$ in the above example. By Example 2.5, the dimension $\alpha=\log_{16}2/(3-\sqrt{5})\approx 0.3471$. And we calculate that $\underline{D}_0=(16^\alpha-1)/15^\alpha\approx 0.6320$, $\underline{D}_1=16^\alpha/225^\alpha\approx 0.3995$, and $\gamma_{\min}^{(2)}=209/4096\approx 0.0510$. Moreover, for each $k\ge 1$, $\beta_{\max}^{(k)}=(31/256)\cdot(1/16^{k-1})$. Hence the smallest $k$ should satisfy

$$2\cdot\frac{31}{256}\cdot\frac{1}{16^{k-1}}\le(0.0510\cdot 0.3995)^{\frac{1}{1-\alpha}},$$

which yields that $k=3$. After a complicated computation using computer, we eventually get $d_{\max}=(256^\alpha-16^\alpha)/31^\alpha\approx 1.2861$, and $\mathcal{H}^\alpha(K)=31^\alpha/(256^\alpha-16^\alpha)\approx 0.7775$.

For the packing measure, a calculation using computer shows that $D=8^\alpha/225^\alpha\approx 0.3140$. Hence $d_{\min}=2^{-\alpha}\underline{D}_1=D=8^\alpha/225^\alpha\approx 0.3140$ and $\mathcal{P}^\alpha(K)=225^\alpha/8^\alpha\approx 3.1843$.

**Example 4.19.** Let $K$ be the invariant set of the IFS $\{S_j\}_{j=1}^3$ on $\mathbb{R}$ defined in Example 2.6. Then the dimension $\alpha$ of $K$ is the logarithmic ratio of the largest root of the polynomial equation $x^3-6x^2+5x-1=0$ to 9, $\alpha\approx 0.7369$.

For $k\ge 0$, let $\mathcal{M}_k=\Lambda_k$. Denote $I_1=[0,1]$, $I_2=S_{11}([0,1])\cup S_{12}([0,1])$ and $I_3=S_{13}([0,1])\cup S_2([0,1])$. It is easy to verified that $\mathcal{T}_1=[I_1],\mathcal{T}_2=[I_2]$ and $\mathcal{T}_3=[I_3]$ are the total overlap types. Using the definition of $\lambda$, we have $\lambda(I_1)=1$, $\lambda(I_2)=1/3^\alpha-1/9^\alpha$ and $\lambda(I_3)=1-2/3^\alpha+1/9^\alpha$. Using Lemma 4.5, we get

$$\underline{D}_0=\min\{d([0,x]):x>0\text{ and }[0,x]\in\mathcal{F}_1\}\text{ and }\underline{D}_1=\min\{d([y,1]):y<1\text{ and }[y,1]\in\mathcal{F}_1\}.$$

A calculation shows that $\underline{D}_0=\min\{1,\frac{3^\alpha-1}{2^\alpha},\frac{9^\alpha-1}{8^\alpha}\}=\frac{3^\alpha-1}{2^\alpha}\approx 0.7482$, and $\underline{D}_1=\min\{1,\frac{9^\alpha}{16^\alpha},\frac{81^\alpha-27^\alpha+9^\alpha}{70^\alpha}\}=\frac{9^\alpha}{16^\alpha}\approx 0.6544$.

*Hausdorff measure.* Since $k_0=1$, a calculation shows $\gamma_{\min}^{(k_0+1)}=\gamma_{\min}^{(2)}=\frac{7}{729}\approx 0.0096$. By Theorem 4.8, we need to find a smallest integer $k$ such that

$$2\beta_{\max}^{(k)}\le(\gamma_{\min}^{(2)}\min\{\underline{D}_0,\underline{D}_1\})^{\frac{1}{1-\alpha}}=(\frac{7}{729}\min\{\frac{3^\alpha-1}{2^\alpha},\frac{9^\alpha}{16^\alpha}\})^{\frac{1}{1-\alpha}}.$$

Noticing that $\beta_{\max}^{(k)} = 5/(27 \cdot 9^{k-1})$, the smallest $k = 10$.

Then by Theorem 4.8, the maximal density $d_{\max}$ is attained for an interval in $\mathcal{F}_{10}$. After a complicated computation by using computer, we eventually get $d_{\max} = (27^\alpha - 9^\alpha)/11^\alpha \approx 1.0756$, and $\mathcal{H}^\alpha(K) = 11^\alpha/(27^\alpha - 9^\alpha) \approx 0.9297$.

*Packing measure.* Since $k_0 = 1$, the constant $D$ in Theorem 4.16 is

$$D = \min_{1 \le i_1 < i_2 < 20} \frac{\sum_{i=i_1+1}^{i_2} \lambda([a_i, b_i])}{(a_{i_2+1} - b_{i_1} - 2\mathrm{dist}(\frac{a_{i_2+1}+b_{i_1}}{2}, K))^\alpha}.$$

where $[a_i, b_i]$ is the $i$-th 2-th generation island for each $1 \le i \le 20$. A calculation by using computer yields the exact value of $D = 9^\alpha/32^\alpha \approx 0.3927$. Hence $d_{\min} = 2^{-\alpha}\underline{D}_1 = D = 9^\alpha/32^\alpha \approx 0.3927$ and $\mathcal{P}^\alpha(K) = 32^\alpha/9^\alpha \approx 2.5467$. □

Although these algorithms applies in theory to any case considered under Assumption A and B, in practice it is useable in very few cases. Even in the following simple example.

**Example 4.20.** Let $K$ be the invariant set of the IFS $\{S_j\}_{j=1}^4$ on $\mathbb{R}$ defined in Example 2.7. The dimension $\alpha$ of $K$ equals $\log_4(5+\sqrt{5}) - \frac{1}{2} \approx 0.9276$.

We adopt the same and notations of Example 2.7, with $\mathcal{T}_1 = [I_1]$, $\mathcal{T}_2 = [I_2]$ and $\mathcal{T}_3 = [I_3]$ where $I_1 = [0,1]$, $I_2 = S_2([0,1]) \cup S_3([0,1])$ and $I_3 = S_{22}([0,1]) \cup S_{23}([0,1]) \cup S_{31}([0,1])$. Using the definition of $\lambda$, we have $\lambda(I_1) = 1$, $\lambda(I_2) = 1 - 2/4^\alpha$ and $\lambda(I_3) = 1 - 3/4^\alpha$. Using Lemma 4.5, we get

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_1\} \text{ and } \underline{D}_1 = \min\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_1\}.$$

A calculation shows that $\underline{D}_0 = \min\{1, \frac{4^\alpha - 1}{3^\alpha}\} = \frac{4^\alpha-1}{3^\alpha} \approx 0.9449$, and $\underline{D}_1 = \min\{1, \frac{2^\alpha}{3^\alpha}, \frac{4^\alpha-1}{3^\alpha}\} = \frac{2^\alpha}{3^\alpha} \approx 0.6865$.

*Hausdorff measure.* We will use Theorem 4.9 since there exist touching islands and $\rho_1 = \rho_m$. Observe that $k_0 = 2$, $n_1 = n_m = 1$ and $\eta = 1/8$. A calculation shows $\beta_{\min}^{(k_0+1)} = \beta_{\min}^{(3)} = \frac{1}{64}$. By Theorem 4.9, we need to find a smallest integer $k$ such that

$$2\beta_{\max}^{(k)} \le (\eta \beta_{\min}^{(k_0+1)} \rho_1^{n_1} \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}} = (\frac{1}{8} \cdot \frac{1}{64} \cdot \frac{1}{4} \min\{\frac{4^\alpha - 1}{3^\alpha}, \frac{2^\alpha}{3^\alpha}\})^{\frac{1}{1-\alpha}}.$$

Noticing that $\beta_{\max}^{(k)} = 1/(8 \cdot 4^{k-2}) = 1/2^{2k-1}$, the smallest $k = 81$.

Then by Theorem 4.9, the maximal density $d_{\max}$ is attained for an interval in $\mathcal{F}_{81}$. Furthermore, by Corollary 1.3, $\mathcal{H}^\alpha(K) = d_{\max}^{-1}$. However, the time involved in searching all sets in $\mathcal{F}_k$ rapidly becomes impractical. Hence our algorithm for computing $\mathcal{H}^\alpha(K)$ exceeds the computing power.

*Packing measure.* Since $k_0 = 2$, the constant $D$ in Theorem 4.16 is

$$D = \min_{1 \le i_1 < i_2 < 35} \frac{\sum_{i=i_1+1}^{i_2} \lambda([a_i, b_i])}{(a_{i_2+1} - b_{i_1} - 2\mathrm{dist}(\frac{a_{i_2+1}+b_{i_1}}{2}, K))^\alpha},$$

where $[a_i, b_i]$ is the $i$-th 3-th generation island for each $1 \le i \le 35$. A calculation by using computer yields the exact value of $D = 1/3^\alpha \approx 0.3609$. Hence $d_{\min} = 2^{-\alpha}\underline{D}_1 = D = 1/3^\alpha \approx 0.3609$ and $\mathcal{P}^\alpha(K) = 3^\alpha \approx 2.7706$. □

# 5 Further Discussions

**Are Assumption A and B necessary?**

If we permit the IFS not to satisfy Assumption A or B, things become more complicated. It seems hard to get a general formulae for $\mathcal{H}^\alpha(K)$ and $\mathcal{P}^\alpha(K)$. However, in some cases, we can still use the similar method to get the results. The following are two concrete examples.

**Example 5.1.** Consider the IFS $\{S_j\}_{j=1}^3$ as follows.

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{9}x + \frac{2}{9}, \quad S_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

If we choose $\mathcal{M}_k = \Lambda_k$ for each $k \geq 0$, then $\{S_j\}_{j=1}^3$ satisfies the GTFC with respect to the invariant set $(0, 1)$, with Assumption A not satisfied. See Figure 4. In fact, the invariant

*Figure 4. The first five levels of islands and the distinct overlap types in Example 5.1.*

set $K$ of this IFS is the classical Cantor ternary set. It is well-known that the dimension of $K$ is equal to $\log_3 2$ and $\mathcal{H}^{\log_3 2}(K) = 1$. Replacing the IFS by $\{S_1, S_3\}$ equivalently, from Theorem 4.16, one gets that $\mathcal{P}^{\log_3 2}(K) = 4^{\log_3 2}$. (See this also in [6].)

**Example 5.2.** Consider the IFS $\{S_j\}_{j=1}^3$ defined as follows.

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + \rho\frac{1-\rho}{1+\rho}, \quad S_3(x) = \rho x + 1 - \rho,$$

where $0 < \rho < 1/3$. Choose $\mathcal{M}_k = \Sigma_k$ for each $k \geq 0$, then $\{S_j\}_{j=1}^3$ satisfies the GTFC with respect to the invariant set $(0, 1)$. Observing that $\rho(1-\rho)/(1+\rho) \in S_1([0, 1]) \cap K$ and $\rho(1-\rho)/(1+\rho) \notin S_1K$, one gets that Assumption B is not satisfied. By Theorem 2.3, it is easy to verify that the dimension of the $K$ is $\alpha = \log_\rho 2/(3 + \sqrt{5})$. See Figure 5.

It is easy to verify that $[I_1]$ and $[I_2]$, denoted respectively by $\mathcal{T}_1$ and $\mathcal{T}_2$, are the all distinct overlap types, where $I_1 = [0, 1]$ and $I_2 = S_1([0, 1]) \cup S_2([0, 1])$. In this case Lemma 4.4 may not hold since Assumption B is not satisfied. Hence the original notations $\underline{D}_0$ and $\underline{D}_1$ may not be suitable. However, we redefine them by $\underline{D}_0 := \min\{\underline{D}_0^1, \underline{D}_0^2\}$, and $\underline{D}_1 := \min\{\underline{D}_1^1, \underline{D}_1^2\}$. A similar discussion as the proof of Lemma 4.4 shows that $\underline{D}_0^1 \leq \underline{D}_0^2$ and $\underline{D}_1^1 \leq \underline{D}_1^2$. Moreover, observing the distribution of the offspring of $I_2$, we can also get

$\underline{D}_1^2 \le \underline{D}_1^1$. Hence $\underline{D}_0 = \underline{D}_0^1$ and $\underline{D}_1 = \underline{D}_1^1 = \underline{D}_1^2$. By a similar argument of Lemma 4.5 (with suitable modifications), we get

$$\underline{D}_0 = \min\{d([0,x]) : x > 0 \text{ and } [0,x] \in \mathcal{F}_2\} \text{ and } \underline{D}_1 = \min\{d([y,1]) : y < 1 \text{ and } [y,1] \in \mathcal{F}_1\}.$$

Using the definition of $\lambda$, we have $\lambda(I_1) = 1$ and $\lambda(I_2) = 1 - \rho^\alpha$. Hence we can get that $d([0,x])$ attains the minimal value $\underline{D}_0 = \min\{\frac{2(1-\rho^\alpha)(1+\rho)^\alpha}{(1-\rho)^\alpha(2+\rho)^\alpha}, \frac{1-\rho^\alpha}{(1-\rho)^\alpha}\} = \frac{1-\rho^\alpha}{(1-\rho)^\alpha}$ at the point $x_0 = 1 - \rho$, and $d([y,1])$ attains the minimal value $\underline{D}_1 = \rho^\alpha \frac{(1+\rho)^\alpha}{(1-\rho)^\alpha}$ at the point $y_0 = 2\rho/(1+\rho)$.

*Hausdorff measure.* $k_0 = 1$, $\gamma_{\min}^{(2)} = \rho(1-3\rho)/(1+\rho)$, $\beta_{\max}^{(k)} = 2\rho^k/(1+\rho)$ for each $k \ge 1$. By a suitable modification of Theorem 4.8, we get $d_{\max}$ is attained for an interval in $\mathcal{F}_k$ where $k \ge 2$ is the smallest integer such that $2\beta_{\max}^{(k)} \le (\gamma_{\min}^{(2)} \min\{\underline{D}_0, \underline{D}_1\})^{\frac{1}{1-\alpha}}$, namely,

$$4\rho^k \le \left(\frac{(1-3\rho)\rho^{1+\alpha}}{(1-\rho)^\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Furthermore, by Corollary 1.3, $\mathcal{H}^\alpha(K) = d_{\max}^{-1}$.

*Packing measure.* We need a similar result of Lemma 4.13, i.e., $d_{\min} \le 2^{-\alpha} \min\{\underline{D}_0, \underline{D}_1\}$. However, at first glance, we can not prove it in general. The reason is the following. Recall the proof of Lemma 4.13, we should find two intervals $J_0$ and $J_1$ centered in $K$ with $d(J_0) = 2^{-\alpha}\underline{D}_0$ and $d(J_1) = 2^{-\alpha}\underline{D}_1$. In fact, we can define $J_1$ as that used in the proof of Lemma 4.13 since $\underline{D}_1^1 = \underline{D}_1^2$ and $S_3([0,1]) \cap K = S_3K$. (Half of Assumption B holds.) But for $J_0$, the original process is invalid since $\underline{D}_0^1$ and $\underline{D}_0^2$ may not be equal. Fortunately, a detailed verifying shows that the inequality $\underline{D}_1 \le \underline{D}_0$ holds, which ensures that it is unnecessary to find $J_0$, i.e., the existence of $J_1$ is enough. Hence Lemma 4.13 remains true. Once Lemma 4.13 is proved, Lemma 4.14, Lemma 4.15 and eventually Theorem 4.16 follow automatically, in which $\min\{\underline{D}_0, \underline{D}_1\}$ are all replaced by $\underline{D}_1$.

Hence $\mathcal{P}^\alpha(K) = d_{\min}^{-1}$ where $d_{\min} = \min\{\frac{\rho^\alpha(1+\rho)^\alpha}{2^\alpha(1-\rho)^\alpha}, D\}$, and

$$\begin{aligned} D = \min\{ & \frac{\rho^\alpha - \rho^{2\alpha}}{(\frac{2\rho-3\rho^2-\rho^3}{1+\rho} - 2\mathrm{dist}(\frac{2\rho+\rho^2-\rho^3}{2(1+\rho)}, K))^\alpha}, \frac{\rho^\alpha}{(\frac{1-3\rho^2}{1+\rho} - 2\mathrm{dist}(\frac{1+\rho^2}{2(1+\rho)}, K))^\alpha}, \\ & \frac{2\rho^\alpha - \rho^{2\alpha}}{(\frac{1+\rho-3\rho^2-\rho^3}{1+\rho} - 2\mathrm{dist}(\frac{1+\rho+\rho^2-\rho^3}{2(1+\rho)}, K))^\alpha}, \frac{\rho^{2\alpha}}{(1-2\rho-2\mathrm{dist}(\frac{1}{2}, K))^\alpha}, \\ & \frac{\rho^\alpha}{(1-\rho-\rho^2-2\mathrm{dist}(\frac{1+\rho-\rho^2}{2}, K))^\alpha}, \frac{\rho^\alpha - \rho^{2\alpha}}{(\frac{1-\rho-\rho^2-\rho^3}{1+\rho} - 2\mathrm{dist}(\frac{1+3\rho-\rho^2-\rho^3}{2(1+\rho)}, K))^\alpha}\}. \square \end{aligned}$$

**Remark.** Consider the special case where $\rho = 1/6$ in the above example. The dimension $\alpha = \log_6((3+\sqrt{5})/2) \approx 0.5371$. And we calculate that $\underline{D}_0 = (6^\alpha - 1)/5^\alpha \approx 0.6816$, $\underline{D}_1 = 7^\alpha/30^\alpha \approx 0.4576$, and $\gamma_{\min}^{(2)} = 1/14 \approx 0.0714$. Moreover, for each $k \ge 1$, $\beta_{\max}^{(k)} = (2/7) \cdot (1/6^{k-1})$. Hence the smallest $k$ should satisfy

$$2 \cdot \frac{2}{7} \cdot \frac{1}{6^{k-1}} \le (0.0714 \cdot 0.4576)^{\frac{1}{1-\alpha}},$$

which yields that $k = 5$. After a complicated computation by using computer, we eventually get $d_{\max} = 2(6^\alpha - 1)/6^\alpha \approx 1.2361$, and $\mathcal{H}^\alpha(K) = 6^\alpha/(2(6^\alpha - 1)) \approx 0.8090$.

*Figure 5. The first five levels of islands and the distinct overlap types in Example 5.2.*

For the packing measure, a calculation by using computer shows that $D = 7^\alpha/60^\alpha \approx 0.3154$. Hence $d_{\min} = 2^{-\alpha}\underline{D}_1 = D = 7^\alpha/60^\alpha \approx 0.3154$ and $\mathcal{P}^\alpha(K) = 60^\alpha/7^\alpha \approx 3.1709$.

**Can we allow negative $\rho$'s?**

Consider the IFS containing orientation reversing similarities, which simply means that we allow some of the $\rho_j$ to be negative. In some special cases, it can be proved by using a similar method that the results of Section 4 still remain true.

We again normalize by assuming that $S_j([0,1])$ are in increasing order with $S_1([0,1])$ containing 0 and $S_m([0,1])$ containing 1. For example, $\rho_1\rho_m > 0$, $\bigcup_{2\le j\le m} S_j([0,1]) \cap S_1([0,1]) = \emptyset$, $\bigcup_{1\le j\le m-1} S_j([0,1]) \cap S_m([0,1]) = \emptyset$, and $\gamma_{\min}^{(k_0+1)} > 0$. Obviously, in this case Assumption B is naturally satisfied. We can still obtain measure results following the idea of Section 4.

To illustrate this, we assume that $\rho_1$ and $\rho_m$ are both positive since otherwise we can replace the IFS with its iterated square, i.e., all compositions $S_iS_j$. Similar to Example 5.2, Lemma 4.10 may not hold. Hence the original notations $\underline{D}_0$ and $\underline{D}_1$ will be meaningless. In spite of this, the following constant defined by $\underline{D} = \min\{\underline{D}_0^1, \underline{D}_1^1\}$ will replace the important role of the original notation $\min\{\underline{D}_0, \underline{D}_1\}$ in Section 4. It is not hard to verify that $\underline{D} \le \underline{D}_0^i$ and $\underline{D} \le \underline{D}_1^i$ for all $1 \le i \le q$, where $q$ is the cardinality of all the distinct overlap types. Moreover, we could also characterize the constant $\underline{D}$ using a suitable modification of Lemma 4.5, i.e.,

$$\underline{D} = \min\{d(J) : J \in \mathcal{F}_1 \text{ and } J \text{ is of the form } [0,x] \text{ or } [y,1]\}.$$

It is not hard to verify that Theorem 4.8 for Hausdorff measure, and Lemma 4.11, Lemma 4.14, Lemma 4.15 and Theorem 4.16 for packing measure still remain true in which $\min\{\underline{D}_0, \underline{D}_1\}$ is replaced by $\underline{D}$.

**How about self-similar sets in higher dimensional Euclidean spaces?**

As mentioned earlier, with suitable modifications if necessary, the results in Section 3 may be generalized to self-similar sets in higher dimensional Euclidean spaces. How about measure results in Section 4? Our answer is: almost all the obvious generalizations of our results are false.

It is clear that the blow-up principles continue to hold. So we can also focus attention to sets not contained entirely in some $(k_0+1)$-generation island, where $k_0$ is also the smallest non-negative integer such that none of the islands in $\mathcal{F}_{k_0+1}$ is of a new overlap type. For the maximal density $d_{\max}$, since any set can be replaced by its convex hull without decreasing the density, it is reasonable to limit any searching algorithm to convex sets. However, it does not follow that the maximal density among $\mathcal{F}_k$ sets is achieved by a convex set, since the convex hull of a set in $\mathcal{F}_k$ may not belong to $\mathcal{F}_k$. To illustrate this, in [1] Ayer & Strichartz consider a concrete example, i.e, the usual Sierpinski gasket in plane. We omit it here. For the minimal centered density $d_{\min}$, one obvious obstacle is that almost all lemmas concerning $d_{\min}$ require $\alpha < 1$. Moreover, even if we were to limit attention to self-similar sets of dimension $\alpha < 1$, it is unlikely that the same results would hold. In fact, we would have to confront how to describe $\underline{D}_0$ and $\underline{D}_1$ and how to overcome the difficulty of the calculation of densities of higher dimensional sets. (It seems uncontrollable.)

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