p-adic Laplacian in Local Fields

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Abstract: In this paper a family of multi-dimensional fractional differential operators T^α and their corresponding pseudo-differential equations over p-adic fields are investigated. The test function class $\mathcal{D}(\mathbb{Q}_p^n)$ and distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ are invariant under the actions of these operators. The p-adic Laplacian Δ_p and a fundamental solution of the Laplace equation are constructed. We study the spectral properties of the Laplacian Δ_p, and obtain an orthonormal basis of the eigen-functions of this operator in $L^2(\mathbb{Q}_p^n)$. Futhermore, the Cauchy problems for the wave and heat equations on the p-adic fields related to Δ_p are also studied.

Keywords: p-adic fields, Laplacian, pseudo-differential operators, eigen-values, Cauchy problem.

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1 Introduction

The main object of this paper is the p-adic Laplacian on $\mathbb{Q}_p^n$. To construct this operator, one need to consider the problem of how to define derivative operators on $\mathbb{Q}_p$, which is an important topic in the study of p-adic analysis[1,2]. Many mathematicians, such as J.E. Gibbs[3], P.L. Butzer[4], C.W. Onneweer[5], W.X. Zheng[6] and V. S. Vladimirov[7] paid their great attention to this topic. However, the test function class $\mathcal{D}(\mathbb{Q}_p)$ are not invariant under the actions of their definitions of derivatives. In the 90's, W.Y. Su[8,9] has given a definition of derivatives and integrals, denoted by T^s, for general locally compact Vilenkin group G, using the pseudo-differential operators, including derivatives and integrals of fractional orders. The test function class $\mathcal{D}(\mathbb{Q}_p)$, together with its distribution class $\mathcal{D}'(\mathbb{Q}_p)$ are invariant under the actions of these fractional operators. For each $s \in \mathbb{R}$, T^s is a pseudo-differential operator with the symbol $\langle\xi\rangle^s$ owing to the formula that

$$T^s f = (\langle\xi\rangle^s f^{\wedge})^{\vee},$$

where $\langle\xi\rangle = \max\{1, |\xi|_p\}$. These operators can be used to study many interesting topics in harmonic analysis[10,11], approximation theory[12−14], fractal analysis[15−18] and other scientific fields.

In [19], the convolution kernel κ_s of the pseudo-differential operator T^s is given and some important properties of κ_s are obtained which play a key role in considerations related to fractional differential operators. A fundamental solution of the pseudo-differential equation

$$P(T^s)f = g, \quad g \in \mathcal{D}'(\mathbb{Q}_p), \quad s \in \mathbb{R},$$

with respect to an unknown distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ is obtained, where P is a polynomial of finite order.

In this paper, firstly, we extend the definition of the fractional differential operators to the multi-dimensional space $\mathbb{Q}_p^n$. A family of multi-dimensional operators T^α and their corresponding pseudo-differential equations are investigated. The test function class $\mathcal{D}(\mathbb{Q}_p^n)$ and distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ are invariant under these operators. Secondly, we give the definition of the p-adic Laplacian Δ_p, analogous to that in the Euclidean case. A fundamental solution of the Laplace equation is constructed. Spectral properties of the Laplacian Δ_p are studied, and an orthonormal basis of eigen-functions of Δ_p in $L^2(\mathbb{Q}_p^n)$ is obtained. Finally, we investigate the Cauchy problems for the wave and heat equations on the p-adic fields related to Δ_p, and obtain solutions of these equations.

2 A brief review of the p-adic analysis

In this section, we make a brief review of the p-adic analysis[1–4]. Let p be a prime number. Recall that the field $\mathbb{Q}_p$ of *p-adic numbers* is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if a nonzero rational number x is represented as $x = p^r \frac{m}{n}$, where $r = \mathrm{ord}_p x \in \mathbb{Z}$, and m and n are not divisible by p, then $|x|_p = p^{-r}$. This norm satisfies the *strong triangle inequality* that $|x+y|_p \leq \max(|x|_p, |y|_p)$ for any $x, y \in \mathbb{Q}_p$.

Every element x in $\mathbb{Q}_p$ can be thought as a unique formal series

$$\sum_{i=m}^{\infty} x_i p^i, \quad 0 \leq x_i \leq p-1, \quad x_m \neq 0.$$

The set $B_0 = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is a subring of $\mathbb{Q}_p$ called the *ring of p-adic integers*. It is well known that $\mathbb{Q}_p$ is locally compact and B_0 is compact. There is a Haar measure dx on $\mathbb{Q}_p$, normalized that $\int_{B_0} dx = 1$. For any $r \in \mathbb{Z}$, denote by B_r the disc of radius p^r with center $0 \in \mathbb{Q}_p$ and by S_r its boundary:

$$B_r = \{x \in \mathbb{Q}_p : |x|_p \leq p^r\},$$

$$S_r = \{x \in \mathbb{Q}_p : |x|_p = p^r\}.$$

It is clear that $\int_{B_r} dx = p^r$ and $\int_{S_r} dx = p^r(1 - \frac{1}{p})$.

The space $\mathbb{Q}_p^n$, consisting of points $x = (x_1, x_2, \cdots, x_n)$, where $x_j \in \mathbb{Q}_p$, is a locally compact metric measure space. The p-adic norm on $\mathbb{Q}_p^n$ is defined by

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n.$$

Denote by $B_r^n = \{x \in \mathbb{Q}_p^n : |x|_p \leq p^r\}$ the ball of radius p^r with the center $0 \in \mathbb{Q}_p^n$, $r \in \mathbb{Z}$. In fact,

$$B_r^n = B_r \times B_r \times \cdots \times B_r.$$

The Haar measure dx on the field $\mathbb{Q}_p$ can be extended to a product measure $d^n x = dx_1 dx_2 \cdots dx_n$ on $\mathbb{Q}_p^n$ in the usual way.

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is called *locally-constant*, if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x), \quad \forall x' \in B_{l(x)}^n.$$

We denote by $\mathcal{E}(\mathbb{Q}_p^n)$ the linear space of locally-constant functions, $\mathcal{D}(\mathbb{Q}_p^n)$ the linear space of locally-constant functions with compact supports, on $\mathbb{Q}_p^n$, respectively, and $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p)$, $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p)$ for short. Convergence in $\mathcal{E}(\mathbb{Q}_p^n)$ is defined in the following way: $\varphi_k \to 0$ in $\mathcal{E}(\mathbb{Q}_p^n)$ as $k \to \infty$ if and only if for any compact set $E \subset \mathbb{Q}_p^n$, $\varphi_k \to 0$ uniformly in E. Convergence in $\mathcal{D}(\mathbb{Q}_p^n)$ is defined that: $\varphi_k \to 0$ in $\mathcal{D}(\mathbb{Q}_p^n)$ as $k \to \infty$ if and only if all φ_k assume constant values on cosets of a ball B_l^n and are supported in a ball B_N^n, where N, l are two numbers, not depending on k, and $\varphi_k \to 0$ uniformly. $\mathcal{D}(\mathbb{Q}_p^n)$ is called the *test function class* on $\mathbb{Q}_p^n$.

We denote by $\mathcal{D}'(\mathbb{Q}_p^n)$ the *distribution space* on $\mathcal{D}(\mathbb{Q}_p^n)$, $\mathcal{D}' = \mathcal{D}'(\mathbb{Q}_p)$. $\mathcal{D}'(\mathbb{Q}_p^n)$ is a complete topological space. Convergence in $\mathcal{D}'(\mathbb{Q}_p^n)$ is defined in the following way: $f_k \to 0$ as $k \to \infty$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ if and only if $(f_k, \varphi) \to 0$ for any $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

For a compact set E, denote by 1_E the characteristic function of E. There is a canonical δ-sequence $\delta_k^n = p^{nk} 1_{B_{-k}^n}$, and a canonical 1-sequence $\Delta_k^n = 1_{B_k^n}$, $k \in \mathbb{Z}$, in $\mathcal{D}(\mathbb{Q}_p^n)$. It is easy to check $\delta_k^n \to \delta$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ and $\Delta_k^n \to 1$ in $\mathcal{E}(\mathbb{Q}_p^n)$, as $k \to \infty$. Obviously, if we denote $\delta_k = \delta_k^1$ and $\Delta_k = \Delta_k^1$, then

$$\delta_k^n(x) = \delta_k(x_1)\delta_k(x_2)\cdots\delta_k(x_n), \quad x = (x_1, x_2, \cdots, x_n),$$

and

$$\Delta_k^n(x) = \Delta_k(x_1)\Delta_k(x_2)\cdots\Delta_k(x_n), \quad x = (x_1, x_2, \cdots, x_n).$$

The *Fourier transform* and *inverse Fourier transform* of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined by the formule

$$\varphi^\wedge(\xi) = \int_{\mathbb{Q}_p^n} \varphi(x)\chi_p(-\xi \cdot x)d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

$$\varphi^\vee(x) = \int_{\mathbb{Q}_p^n} \varphi(\xi)\chi_p(\xi \cdot x)d^n \xi, \quad x \in \mathbb{Q}_p^n,$$

where $\chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1)\chi_p(\xi_2 x_2)\cdots\chi_p(\xi_n x_n) = e^{2\pi i \sum_{j=1}^n \{\xi_j x_j\}_p}$, $\xi \cdot x$ is the scalar product of ξ and x, and the function $\chi_p(x)$ is a fixed non-trivial additive character on $\mathbb{Q}_p$ which is trivial on B_0. It is known that the Fourier transform and the inverse transform are linear isomorphisms from $\mathcal{D}(\mathbb{Q}_p^n)$ onto $\mathcal{D}(\mathbb{Q}_p^n)$. The transforms could be extended to distribution space. For each $f \in \mathcal{D}'(\mathbb{Q}_p^n)$, $f^\wedge$ and $f^\vee$ are defined by the relations

$$(f^\wedge, \varphi) = (f, \varphi^\wedge), \quad \forall \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

$$(f^\vee, \varphi) = (f, \varphi^\vee), \quad \forall \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

It is easy to see $\Delta_k^{n\wedge} = \delta_k^n$, $k \in \mathbb{Z}$.

For distributions $f\in\mathcal{D}'(\mathbb{Q}_p^n), g\in\mathcal{D}'(\mathbb{Q}_p^m)$, the *direct product* of them is defined by

$$(f(x)\times g(y),\varphi)=(f(x),(g(y),\varphi(x,y))),\quad \forall\varphi\in\mathcal{D}(\mathbb{Q}_p^{n+m}),$$

since any test function $\varphi\in\mathcal{D}(\mathbb{Q}_p^{n+m})$ can be represented in a finite sum of the form

$$\varphi(x,y)=\sum_k\varphi_k(x)\psi_k(y),\quad \varphi_k\in\mathcal{D}(\mathbb{Q}_p^n),\quad \psi_k\in\mathcal{D}(\mathbb{Q}_p^m).$$

Thus $f(x)\times g(y)\in\mathcal{D}'(\mathbb{Q}_p^{n+m})$. Moreover, the direct product is commutative, that is

$$f(x)\times g(y)=g(y)\times f(x).$$

Particularly, for $g=1$, the above equality implies that

$$(f(x),\int_{\mathbb{Q}_p^m}\varphi(x,y)d^my)=\int_{\mathbb{Q}_p^m}(f(x),\varphi(x,y))d^my,\quad \varphi\in\mathcal{D}(\mathbb{Q}_p^{n+m}).$$

The *convolution* $f*g$ for distributions $f,g\in\mathcal{D}'(\mathbb{Q}_p^n)$ is defined[1,2] that:

$$(f*g,\varphi)=\lim_{k\to\infty}(f(x)\times g(y),\Delta_k(x)\varphi(x+y))$$

if the limit exists for all $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$, where $f(x)\times g(y)$ is the direct product of distributions f,g. The formula

$$(f*g)^\wedge=f^\wedge g^\wedge$$

holds if the convolution $f*g$ exists. If $f,g\in\mathcal{D}'(\mathbb{Q}_p^n)$ and supp$g\subset B_N^n$ for some $N\in\mathbb{Z}$, then the convolution $f*g$ exists and

$$(f*g,\varphi)=(f(x)\times g(y),\Delta_N^n(y)\varphi(x+y)),\quad \varphi\in\mathcal{D}(\mathbb{Q}_p^n).$$

Moreover, if $g=\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$, then $f*\varphi\in\mathcal{E}(\mathbb{Q}_p^n)$ and $f*\varphi$ takes the form

$$(f*\varphi)(x)=(f(y),\varphi(x-y)),\quad x\in\mathbb{Q}_p^n.$$

3 n-dimensional pseudo-differential operator T^α

In [8, 9], W.Y. Su made a definition of derivatives and integrals, of fractional orders, for general locally compact Vilenkin group G, by using of pseudo-differential operators. The test function class $\mathcal{D}$ and the distribution class $\mathcal{D}'$ are invariant under these fractional operators.

For $\xi \in \mathbb{Q}_p$, denote $\langle\xi\rangle = \max\{1, |\xi|_p\}$. Obviously, $\langle\xi\rangle \in \mathcal{E}$. For $s \in \mathbb{R}$, T^s is defined to be a pseudo-differential operator with the symbol $\langle\xi\rangle^s$ owing to the formula that

$$T^s\varphi = (\langle\xi\rangle^s \varphi^{\wedge})^{\vee}, \quad \varphi \in \mathcal{D}.$$

It is easy to check that $T^s\varphi$ exists in $\mathcal{D}$. The definition domain of T^s can be extended to the distribution space $\mathcal{D}'$ by the relation

$$(T^s f, \varphi) = (f, T^s\varphi), \quad f \in \mathcal{D}', \quad \varphi \in \mathcal{D}.$$

So for $f \in \mathcal{D}'$, we still have

$$T^s f = (\langle\xi\rangle^s f^{\wedge})^{\vee}.$$

We call the operator T^s the *derivative operator* on $\mathcal{D}'$ of order s for $s > 0$, and the *integral operator* on $\mathcal{D}'$ of order $-s$ for $s < 0$. For $s = 0$, $T^0 f = f$ for all $f \in \mathcal{D}'$, T^0 is the identity operator.

In [19], the *convolution kernel* κ_s of the pseudo-differential operator T^s is given and some important properties of κ_s are revealed which play a key role in problems related to fractional operator T^s.

$$\kappa_s = \left(\frac{1-p^s}{1-p^{-s-1}}|x|_p^{-s-1} + \frac{p^s - 1}{p^{s+1}-1}\right)\Delta_0, \qquad \text{for } s \neq 0, -1,$$

and $\kappa_0 = \delta$, $\kappa_{-1} = (1 - \frac{1}{p})(1 - \log_p |x|_p)\Delta_0$, where $|x|_p^{-s-1}$ is a distribution[2,19] in $\mathcal{D}'(\mathbb{Q}_p)$,

$$(|x|_p^{-s-1}, \varphi) = \int_{\mathbb{Q}_p} |x|_p^{-s-1}(\varphi(x) - \varphi(0))dx, \quad \varphi \in \mathcal{D}, \quad s \neq 0.$$

The convolution kernel κ_s has the following properties:

$$\kappa_s * \kappa_t = \kappa_{s+t}, \quad \forall s, t \in \mathbb{R}.$$

Moreover, κ_s is continuous on $s \in \mathbb{R}$.

We now consider the n-dimensional case.

Firstly, we give the definition of the partial differential operator $T^s_{x_j}$ for distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$, $1 \leq j \leq n$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, it can be represented as a finite sum of the form

$$\varphi(x) = \sum_k \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots\varphi_{k_n}(x_n), \quad \varphi_{k_j} \in \mathcal{D}.$$

We define

$$T^s_{x_j}\varphi(x) = \sum_k \varphi_{k_1}(x_1)\cdots T^s\varphi_{k_j}(x_j)\cdots\varphi_{k_n}(x_n).$$

Obviously, the partial differential operator $T^s_{x_j}$ is well-defined and $T^s_{x_j}(\mathcal{D}(\mathbb{Q}_p^n)) = \mathcal{D}(\mathbb{Q}_p^n)$. We can extend the definition domain of the operator $T^s_{x_j}$ to $\mathcal{D}'(\mathbb{Q}_p^n)$ by the relation

$$(T^s_{x_j} f, \varphi) = (f, T^s_{x_j}\varphi), \quad f \in \mathcal{D}'(\mathbb{Q}_p^n), \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

And we have $T^s_{x_j}(\mathcal{D}'(\mathbb{Q}_p^n)) = \mathcal{D}'(\mathbb{Q}_p^n)$.

Secondly, we investigate the n-dimensional pseudo-differential operator T^α on $\mathcal{D}'(\mathbb{Q}_p^n)$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index, $\alpha_j \in \mathbb{R}$, with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. For $\alpha, \beta \in \mathbb{R}^n$, denote $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n)$. For $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$, denote $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. For example, for $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{Q}_p^n$, if we denote $\langle \xi \rangle = (\langle \xi_1 \rangle, \langle \xi_2 \rangle, \cdots, \langle \xi_n \rangle)$, then

$$\langle \xi \rangle^\alpha = \langle \xi_1 \rangle^{\alpha_1} \langle \xi_2 \rangle^{\alpha_2} \cdots \langle \xi_n \rangle^{\alpha_n}.$$

We write

$$\kappa_\alpha(x) = \kappa_{\alpha_1}(x_1) \times \kappa_{\alpha_2}(x_2) \times \cdots \times \kappa_{\alpha_n}(x_n),$$

where $\times$ is the direct product operation. In particular, for $\alpha = (0, 0, \cdots, 0)$,

$$\kappa_0(x) = \delta(x) = \delta(x_1) \times \delta(x_2) \times \cdots \times \delta(x_n).$$

We define the n-dimensional fractional operator T^α on the distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ by the following convolution form,

$$T^\alpha f = \kappa_\alpha * f,$$

and call κ_α the *n-dimensional convolution kernel* of T^α. In particular, for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have

$$T^\alpha \varphi(x) = (\kappa_{\alpha_1}(y_1) \times \kappa_{\alpha_2}(y_2) \times \cdots \times \kappa_{\alpha_n}(y_n), \varphi(x - y)), \quad x \in \mathbb{Q}_p^n.$$

The following are some basic properties of the pseudo-differential operators T^α and their convolution kernels κ_α.

Proposition 3.1. *Let $\alpha, \beta \in \mathbb{R}^n$. Then*

$$\kappa_\alpha * \kappa_\beta = \kappa_{\alpha+\beta}.$$

Proof.

$$\begin{aligned} \kappa_\alpha * \kappa_\beta &= (\kappa_{\alpha_1} * \kappa_{\beta_1}) \times (\kappa_{\alpha_2} * \kappa_{\beta_2}) \times \cdots \times (\kappa_{\alpha_n} * \kappa_{\beta_n}) \\ &= \kappa_{\alpha_1+\beta_1} \times \kappa_{\alpha_2+\beta_2} \times \cdots \times \kappa_{\alpha_n+\beta_n} = \kappa_{\alpha+\beta}. \quad \sharp \end{aligned}$$

Proposition 3.2. *Let $\alpha \in \mathbb{R}^n$. Then $\kappa_\alpha^\wedge = \langle \xi \rangle^\alpha$.*

Proof.

$$\kappa_\alpha^\wedge = \kappa_{\alpha_1}^\wedge \times \kappa_{\alpha_2}^\wedge \cdots \kappa_{\alpha_n}^\wedge = \langle \xi_1 \rangle^{\alpha_1} \langle \xi_2 \rangle^{\alpha_2} \cdots \langle \xi_n \rangle^{\alpha_n} = \langle \xi \rangle^\alpha. \quad \sharp$$

From the above two propositions, we obtain

Proposition 3.3. *Let $\alpha \in \mathbb{R}^n$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. Then*

$$T^\alpha f = \kappa_\alpha * f = (\langle \xi \rangle^\alpha f^\wedge)^\vee.$$

Proof. $(\kappa_\alpha * f)^\wedge = \kappa_\alpha^\wedge f^\wedge = \langle \xi \rangle^\alpha f^\wedge$. $\sharp$

Proposition 3.4. *Let $\alpha, \beta \in \mathbb{R}^n$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. Then*

$$T^{\alpha+\beta} f = T^\alpha T^\beta f = T^\beta T^\alpha f.$$

Proof. $T^{\alpha+\beta} f = \kappa_{\alpha+\beta} * f = \kappa_\alpha * \kappa_\beta * f = T^\alpha T^\beta f$. $\sharp$

Proposition 3.5. *Let $\alpha \in \mathbb{R}^n$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. Then*

$$(T^\alpha f, \varphi) = (f, T^\alpha \varphi).$$

Proof. Since $T^\alpha f = (\langle \xi \rangle^\alpha f^\wedge)^\vee$, we have

$$(T^\alpha f, \varphi) = (\langle \xi \rangle^\alpha f^\wedge, \varphi^\vee) = (f^\wedge, \langle \xi \rangle^\alpha \varphi^\vee) = (f, (\langle \xi \rangle^\alpha \varphi^\vee)^\wedge) = (f, (\langle \xi \rangle^\alpha \varphi^\wedge)^\vee) = (f, T^\alpha \varphi). \quad \sharp$$

Proposition 3.6. *$\mathcal{D}(\mathbb{Q}_p^n), \mathcal{E}(\mathbb{Q}_p^n)$ and $\mathcal{D}'(\mathbb{Q}_p^n)$ are invariant under the operators T^α.*

Proof. We only prove the $\mathcal{D}(\mathbb{Q}_p^n)$ case, since the others can be obtained by similar arguments. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have $\varphi^\wedge \in \mathcal{D}(\mathbb{Q}_p^n)$. then $\langle \xi \rangle^\alpha \varphi^\wedge \in \mathcal{D}(\mathbb{Q}_p^n)$, since $\langle \xi \rangle^\alpha \in \mathcal{E}(\mathbb{Q}_p^n)$. Thus $T^\alpha \varphi = (\langle \xi \rangle^\alpha \varphi^\wedge)^\vee \in \mathcal{D}(\mathbb{Q}_p^n)$. Hence $T^\alpha(\mathcal{D}(\mathbb{Q}_p^n)) \subset \mathcal{D}(\mathbb{Q}_p^n)$.

On the other hand, let $\psi \in \mathcal{D}(\mathbb{Q}_p^n)$, consider the equation $T^\alpha \varphi = \psi$. Let $\varphi = T^{-\alpha}\psi = \kappa_{-\alpha} * \psi$, then from Proposition 3.1,

$$T^\alpha \varphi = \kappa_\alpha * \varphi = \kappa_\alpha * \kappa_{-\alpha} * \psi = \delta * \psi = \psi.$$

Hence, $T^\alpha(\mathcal{D}(\mathbb{Q}_p^n)) \supset \mathcal{D}(\mathbb{Q}_p^n)$. $\sharp$

Theorem 3.1. *Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. Then*

$$T^\alpha = T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n},$$

where $\circ$ denotes the composition operation. Moreover, the compositions are commutable.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. It must have a finite sum form that

$$\varphi(x) = \sum_k \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots\varphi_{k_n}(x_n), \quad \varphi_{k_j} \in \mathcal{D}.$$

Then

$$\begin{aligned}
T^\alpha \varphi &= \sum_k T^\alpha \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots\varphi_{k_n}(x_n) \\
&= \sum_k (\kappa_{\alpha_1} * \varphi_{k_1})(\kappa_{\alpha_2} * \varphi_{k_2})\cdots(\kappa_{\alpha_n} * \varphi_{k_n}) \\
&= \sum_k T_{x_1}^{\alpha_1}\varphi_{k_1} T_{x_2}^{\alpha_2}\varphi_{k_2}\cdots T_{x_n}^{\alpha_n}\varphi_{k_n} \\
&= \sum_k T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n} \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots\varphi_{k_n}(x_n) \\
&= T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n} \varphi.
\end{aligned}$$

Let $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. Then for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, using Proposition 3.5, we have

$$(T^\alpha f, \varphi) = (f, T^\alpha \varphi) = (f, T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n} \varphi) = (T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n} f, \varphi). \quad \sharp$$

Taking $\alpha = (0, \cdots, 0, \alpha_j, 0, \cdots, 0)$ in the above theorem, we immediately get

Corollary 3.1. *Let $1 \le j \le n$, $\alpha_j \in \mathbb{R}$, $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. Then*

$$T_{x_j}^{\alpha_j} f = T^{(0,\cdots,0,\alpha_j,0,\cdots,0)} f.$$

Finally, we give some examples.

Example 3.1. $T^\alpha 1 = 1$.

Proof. Since $1^\wedge = \delta$, $\forall \varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have

$$(T^\alpha 1, \varphi) = (((\langle\xi\rangle^\alpha \delta)^\vee, \varphi) = (\langle\xi\rangle^\alpha \delta(\xi), \varphi^\vee(\xi)) = (\delta(\xi), \langle\xi\rangle^\alpha \varphi^\vee(\xi)) = \varphi^\vee(0) = (1, \varphi). \quad \sharp$$

Example 3.2. $T^\alpha \delta = \kappa_\alpha$.

Proof. $T^\alpha \delta = \kappa_\alpha * \delta = \kappa_\alpha$. $\sharp$

Example 3.3. *Let $\alpha = (-1, -1, \cdots, -1)$, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. Then*

$$T^\alpha \varphi(x) = T^{(-1,-1,\cdots,-1)} \varphi(x) = (1 - \frac{1}{p})^n \int_{x+B_0^n} \prod_{j=1}^n (1 - \log_p |y_j - x_j|_p) \varphi(y) d^n y.$$

Proof. Since φ can be represented as a finite sum of the form

$$\varphi = \sum_k \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots\varphi_{k_n}(x_n), \quad \varphi_{k_j} \in \mathcal{D},$$

$$\begin{aligned}
T^\alpha\varphi = \kappa_\alpha * \varphi &= \sum_k \kappa_{-1} * \varphi_{k_1}(x_1)\kappa_{-1} * \varphi_{k_2}(x_2)\cdots\kappa_{-1} * \varphi_{k_n}(x_n) \\
&= \sum_k \prod_{j=1}^n (1-\frac{1}{p})\int_{x_j+B_0}(1-\log_p|y_j-x_j|_p)\varphi_{k_j}(y_j)dy_j \\
&= (1-\frac{1}{p})^n \sum_k \int_{x+B_0^n}\prod_{j=1}^n(1-\log_p|y_j-x_j|_p)\varphi_{k_1}(y_1)\varphi_{k_2}(y_2)\cdots\varphi_{k_n}(y_n)dy_1dy_2\cdots dy_n \\
&= (1-\frac{1}{p})^n \int_{x+B_0^n}\prod_{j=1}^n(1-\log_p|y_j-x_j|_p)\varphi(y)d^ny. \quad \sharp
\end{aligned}$$

Example 3.4. *Let $\alpha \in \mathbb{R}^n$, $\eta \in \mathbb{Q}_p^n$. Then*

$$T^\alpha\chi_p(\eta\cdot x) = \langle\eta\rangle^\alpha\chi_p(\eta\cdot x).$$

Proof. Since $T_{x_j}^{\alpha_j}\chi_p(\eta_j x_j) = \langle\eta_j\rangle^{\alpha_j}\chi_p(\eta_j x_j)$[19],

$$\begin{aligned}
T^\alpha\chi_p(\eta\cdot x) &= T^\alpha\chi_p(\eta_1x_1+\eta_2x_2+\cdots+\eta_nx_n) \\
&= T_{x_1}^{\alpha_1}\chi_p(\eta_1x_1)T_{x_2}^{\alpha_2}\chi_p(\eta_2x_2)\cdots T_{x_n}^{\alpha_n}\chi_p(\eta_nx_n) \\
&= \langle\eta_1\rangle^{\alpha_1}\chi_p(\eta_1x_1)\langle\eta_2\rangle^{\alpha_2}\chi_p(\eta_2x_2)\cdots\langle\eta_n\rangle^{\alpha_n}\chi_p(\eta_nx_n) \\
&= \langle\eta\rangle^\alpha\chi_p(\eta\cdot x). \quad \sharp
\end{aligned}$$

4 the Laplacian Δ_p

Now we introduce the Laplacian Δ_p on $\mathbb{Q}_p^n$. Δ_p is an operator that

$$\Delta_p f(x) = \sum_{j=1}^n T_{x_j}^2 f(x), \quad f \in \mathcal{D}'(\mathbb{Q}_p^n).$$

If we denote by $e^j = (0,\cdots,0,1,0,\cdots,0)$, the j-th unit vector of $\mathbb{R}^n$, then

$$\Delta_p f(x) = \sum_{j=1}^n T^{2e^j} f(x).$$

Since $\sum_{j=1}^n T^{2e^j} f = (\sum_{j=1}^n \langle\xi_j\rangle^2 f^\wedge)^\vee$, the Laplacian Δ_p is a pseudo-differential operator with the symbol $\sum_{j=1}^n\langle\xi_j\rangle^2$.

Example 4.1. *Let* $\psi(x) = \chi_p(p^{-1}e \cdot x)\Delta_0^n$*, with* $e = (1, 1, \cdots, 1)$*. Then* $\psi(x)$ *is an eigen-function of the Laplacian* Δ_p*,*

$$\Delta_p\psi(x) = np^2\psi(x).$$

Proof. We can easily get that,

$$\psi^\wedge(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(p^{-1}e \cdot x)\Delta_0^n \chi_p(-\xi \cdot x)d^n x = \int_{B_0^n} \chi_p(x \cdot (p^{-1}e - \xi))d^n x = 1_{p^{-1}e+B_0^n},$$

and

$$\sum_{j=1}^n \langle\xi_j\rangle^2 \psi^\wedge(\xi) = \sum_{j=1}^n \langle\xi_j\rangle^2 1_{p^{-1}e+B_0^n} = np^2 1_{p^{-1}e+B_0^n}.$$

Hence,

$$\Delta_p\psi(x) = (\sum_{j=1}^n \langle\xi_j\rangle^2 \psi^\wedge(\xi))^\vee(x) = np^2 \int_{p^{-1}e+B_0^n} \chi_p(\xi \cdot x)d^n\xi$$

$$= np^2 \int_{B_0^n} \chi_p(p^{-1}e \cdot x)\chi_p(\xi \cdot x)d^n\xi = np^2\chi_p(p^{-1}e \cdot x)\Delta_0^n = np^2\psi(x). \quad \sharp$$

Example 4.2. *Let* $\psi(x) = \chi_p(p^{-1}e \cdot x)\Delta_0^n$*,* $a \in \mathbb{Q}_p, a \neq 0, b = (b_1, b_2, \cdots, b_n) \in \mathbb{Q}_p^n$*. Then*

$$\Delta_p\psi(ax + b) = \begin{cases} np^2|a|_p^2\psi(ax + b), & for\ |a|_p > p^{-1}, \\ n\psi(ax + b), & for\ |a|_p \leq p^{-1}. \end{cases}$$

Proof. The Fourier transform of $\psi(ax + b)$ is

$$(\psi(ax + b))^\wedge(\xi) = |a|_p^{-n}\chi_p(\frac{b \cdot \xi}{a})\psi^\wedge(\frac{\xi}{a}) = |a|_p^{-n}\chi_p(\frac{b \cdot \xi}{a})1_{a(p^{-1}e+B_0^n)}.$$

Hence,

$$\Delta_p\psi(ax + b) = (\sum_{j=1}^n \langle\xi_j\rangle^2 (\psi(ax + b))^\wedge(\xi))^\vee(x)$$

$$= \int_{\mathbb{Q}_p^n} \sum_{j=1}^n \langle\xi_j\rangle^2 |a|_p^{-n}\chi_p(\frac{b \cdot \xi}{a})1_{a(p^{-1}e+B_0^n)}\chi_p(\xi \cdot x)d^n\xi$$

$$= \int_{a(p^{-1}e+B_0^n)} \sum_{j=1}^n \langle\xi_j\rangle^2 |a|_p^{-n}\chi_p((x + \frac{b}{a}) \cdot \xi)d^n\xi.$$

For $|a|_p \leq p^{-1}$, we have $a(p^{-1}e + B_0^n) \subset B_0^n$, then

$$\begin{aligned}\Delta_p \psi(ax+b) &= n \int_{a(p^{-1}e+B_0^n)} |a|_p^{-n} \chi_p((x+\frac{b}{a})\cdot\xi) d^n\xi \\ &= n \int_{B_0^n} \chi_p((x+\frac{b}{a})\cdot(ap^{-1}e + a\xi)) d^n\xi \\ &= n \int_{B_0^n} \chi_p(p^{-1}e\cdot(ax+b))\chi_p(\xi\cdot(ax+b)) d^n\xi \\ &= n\chi_p(p^{-1}e\cdot(ax+b))\Delta_0^n(ax+b) = n\psi(ax+b).\end{aligned}$$

For $|a|_p > p^{-1}$, noticing that $\forall \xi \in a(p^{-1}e + B_0^n)$, $|\xi_j|_p = p|a|_p > 1$, we have

$$\Delta_p \psi(ax+b) = \int_{a(p^{-1}e+B_0^n)} \sum_{j=1}^n |\xi_j|^2 |a|_p^{-n} \chi_p((x+\frac{b}{a})\cdot\xi) d^n\xi$$

$$= np^2|a|_p^2 \int_{a(p^{-1}e+B_0^n)} |a|_p^{-n} \chi_p((x+\frac{b}{a})\cdot\xi) d^n\xi = np^2|a|_p^2 \psi(ax+b). \quad \sharp$$

Let

$$P(x) = \sum_r a_r x^r = \sum_{r_1,r_2,\cdots,r_n} a_{r_1,r_2,\cdots,r_n} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$$

be a polynomial defined on $\mathbb{R}^n$, where $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n$ are multi-indexes and $a_r \in \mathbb{C}$ are constants. Let $\mathcal{P}$ be a pseudo-differential operator on $\mathcal{D}'(\mathbb{Q}_p^n)$, with the kernel $P(\langle\xi\rangle)$, i.e.,

$$\mathcal{P}f = (P(\langle\xi\rangle)f^\wedge)^\vee, \quad f \in \mathcal{D}'(\mathbb{Q}_p^n).$$

In particular, if $P(\langle\xi\rangle) = \sum_{j=1}^n \langle\xi_j\rangle^2$, then $\mathcal{P} = \Delta_p$.

Let us consider the equation

$$\mathcal{P}f = g, \quad g \in \mathcal{D}'(\mathbb{Q}_p^n). \tag{4.1}$$

Theorem 4.1. *If $P(x) \neq 0$ when all $x_i \geq 1$, then the equation (4.1) has a unique solution in $\mathcal{D}'(\mathbb{Q}_p^n)$ that*

$$f = (P^{-1}(\langle\xi\rangle)g^\wedge)^\vee.$$

Proof. Since $P(x) \neq 0$ when all $x_i \geq 1$, the functions $P(\langle\xi\rangle)$ and $P^{-1}(\langle\xi\rangle)$ are both belong to $\mathcal{E}(\mathbb{Q}_p^n)$. Let $f = (P^{-1}(\langle\xi\rangle)g^\wedge)^\vee$. Then

$$\mathcal{P}f = (P(\langle\xi\rangle)f^\wedge)^\vee = (P(\langle\xi\rangle)P^{-1}(\langle\xi\rangle)g^\wedge)^\vee = g.$$

For the uniqueness, we need to investigate solutions of the homogeneous equation

$$\mathcal{P}f = 0. \tag{4.2}$$

By applying to the equation (4.2) the Fourier transform, we get

$$P(\langle\xi\rangle)f^{\wedge}=0.$$

As $P(x)\neq 0$ when all $x_i\geq 1$, we have $P(\langle\xi\rangle)\neq 0$, then $f^{\wedge}=0$, so $f=0$. Thus the homogeneous equation (4.2) has only a trivial solution. ♯

A fundamental solution of (4.1) is a distribution f such that $\mathcal{P}f=\delta$.

Theorem 4.2. *The equation (4.1) has a fundamental solution*

$$f_{\mathcal{P}}(x)=(P^{-1}(\langle\xi\rangle))^{\vee},\quad \textit{i.e.},\quad \mathcal{P}f_{\mathcal{P}}=\delta.$$

Proof.

$$\mathcal{P}f_{\mathcal{P}}=(P(\langle\xi\rangle)P^{-1}(\langle\xi\rangle))^{\vee}=1^{\vee}=\delta.\quad \sharp$$

This theorem shows that the solution of equation (4.1) can be represented as

$$f=f_{\mathcal{P}}*g.$$

Corollary 4.1. *If there exists a function of finite sum $Q(x)=\sum_s b_s x^s$ defined on $\mathbb{R}^n$, $b_s\in\mathbb{C}, s\in\mathbb{R}^n$, such that $Q(x)=P^{-1}(x)$, then the fundamental solution of (4.1) is $f_{\mathcal{P}}=\sum_s b_s\kappa_s$.*

Proof. Using Proposition 3.2, we obtain

$$f_{\mathcal{P}}=(Q(\langle\xi\rangle))^{\vee}=(\sum_s b_s\langle\xi\rangle^s)^{\vee}=\sum_s b_s(\langle\xi\rangle^s)^{\vee}=\sum_s b_s\kappa_s.\quad \sharp$$

Corollary 4.2. *The Poission equation $\Delta_p f=g$, $g\in\mathcal{D}'(\mathbb{Q}_p^n)$ has a fundamental solution $f_{\Delta_p}=(\frac{1}{\langle\xi_1\rangle^2+\langle\xi_2\rangle^2+\cdots+\langle\xi_n\rangle^2})^{\vee}$, which is a distribution with support contained in B_0^n.*

Proof. Noticing that the function $P(x)=x_1^2+x_2^2+\cdots+x_n^2$, we have

$$f_{\Delta_p}=(P^{-1}(\langle\xi\rangle))^{\vee}=(\frac{1}{\langle\xi_1\rangle^2+\langle\xi_2\rangle^2+\cdots+\langle\xi_n\rangle^2})^{\vee}.$$

$\mathrm{supp} f_\alpha\subset B_0$ is a direct corollary of the fact that $P^{-1}(\langle\xi\rangle)\in\mathcal{E}$, taking constant values on cosets of B_0^n. ♯

5 Special properties of the Laplacian Δ_p

The Laplacian Δ_p is a pseudo-differential operator with the symbol $\sum_{j=1}^n \langle\xi_j\rangle^2$,

$$\Delta\psi = (\sum_{j=1}^n \langle\xi_j\rangle^2 \psi^\wedge)^\vee, \forall\psi.$$

It can be defined on those functions ψ in the Hilbert space $L^2(\mathbb{Q}_p^n)$, satisfying $\sum_{j=1}^n \langle\xi_j\rangle^2\psi^\wedge \in L^2(\mathbb{Q}_p^n)$. We denote the collection of these functions by $\mathcal{D}(\Delta_p)$, and call it the *domain of the Laplacian* Δ_p in $L^2(\mathbb{Q}_p^n)$.

Lemma 5.1. $(\sum_{j=1}^n \langle\xi_j\rangle^2)^\rho \in L^2(\mathbb{Q}_p^n)$ *if and only if* $\rho < -\frac{n}{4}$.

Proof. If $\rho < -\frac{n}{4}$, then

$$\begin{aligned}
\int_{\mathbb{Q}_p^n} (\sum_{j=1}^n \langle\xi_j\rangle^2)^{2\rho} d^n\xi &= \int_{B_0^n} n^{2\rho} d^n\xi + \int_{\mathbb{Q}_p^n \setminus B_0^n} (\sum_{j=1}^n \langle\xi_j\rangle^2)^{2\rho} d^n\xi \\
&= n^{2\rho} + \sum_{r=1}^\infty \int_{|\xi|_p = p^r} (\sum_{j=1}^n \langle\xi_j\rangle^2)^{2\rho} d^n\xi \\
&\leq n^{2\rho} + \sum_{r=1}^\infty (p^{2r})^{2\rho} p^{rn} (1 - \frac{1}{p^n}) \\
&= n^{2\rho} + (1 - \frac{1}{p^n}) \sum_{r=1}^\infty p^{(n+4\rho)r} \\
&< \infty.
\end{aligned}$$

If $\rho \geq -\frac{n}{4}$, then

$$\begin{aligned}
\int_{\mathbb{Q}_p^n} (\sum_{j=1}^n \langle\xi_j\rangle^2)^{2\rho} d^n\xi &= n^{2\rho} + \sum_{r=1}^\infty \int_{|\xi|_p = p^r} (\sum_{j=1}^n \langle\xi_j\rangle^2)^{2\rho} d^n\xi \\
&\geq n^{2\rho} + \min(n^{2\rho}, 1) \sum_{r=1}^\infty (p^{2r})^{2\rho} p^{rn} (1 - \frac{1}{p^n}) \\
&= n^{2\rho} + \min(n^{2\rho}, 1)(1 - \frac{1}{p^n}) \sum_{r=1}^\infty p^{(n+4\rho)r} \\
&= \infty. \quad \sharp
\end{aligned}$$

Theorem 5.1. $\mathcal{D}(\Delta_p) \subsetneq L^2(\mathbb{Q}_p^n)$ *and* $\Delta_p(\mathcal{D}(\Delta_p)) = L^2(\mathbb{Q}_p^n)$. *Furthermore,* $\mathcal{D}(\Delta_p)$ *is dense in* $L^2(\mathbb{Q}_p^n)$.

Proof. By Lemma 5.1 and the fact that the Fourier transform is a unitary operator in $L^2(\mathbb{Q}_p^n)$, there exists a $\psi \in L^2(\mathbb{Q}_p^n)$, such that $\psi^\wedge = (\sum_{j=1}^n \langle\xi_j\rangle^2)^{-1-\frac{n}{4}} \in L^2(\mathbb{Q}_p^n)$.

Still by Lemma 5.1, we have

$$\sum_{j=1}^n \langle\xi_j\rangle^2 \psi^\wedge = (\sum_{j=1}^n \langle\xi_j\rangle^2)^{-\frac{n}{4}} \notin L^2(\mathbb{Q}_p^n).$$

Thus $\psi \in L^2(\mathbb{Q}_p^n)$, but $\psi \notin \mathcal{D}(\Delta_p)$. So $\mathcal{D}(\Delta_p) \subsetneq L^2(\mathbb{Q}_p^n)$.

Let $\varphi \in L^2(\mathbb{Q}_p^n)$. Consider the solution of the equation $\Delta_p \psi = \varphi$, i.e.,

$$\psi = ((\sum_{j=1}^n \langle\xi_j\rangle^2)^{-1} \varphi^\wedge)^\vee.$$

Then $\varphi \in L^2(\mathbb{Q}_p^n)$ and $|(\sum_{j=1}^n \langle\xi_j\rangle^2)^{-1}| \le n^{-1}$ implies that $(\sum_{j=1}^n \langle\xi_j\rangle^2)^{-1} \varphi^\wedge \in L^2(\mathbb{Q}_p^n)$, so $\psi \in L^2(\mathbb{Q}_p^n)$. Hence, $\Delta_p(\mathcal{D}(\Delta_p)) = L^2(\mathbb{Q}_p^n)$.

Noticing the fact that $\mathcal{D}(\mathbb{Q}_p^n) \subset \mathcal{D}(\Delta_p)$ and $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $L^2(\mathbb{Q}_p^n)$, we get the density of $\mathcal{D}(\Delta_p)$ in $L^2(\mathbb{Q}_p^n)$. $\sharp$

Theorem 5.2. *The Laplacian Δ_p is a non-negative self-adjoint operator on $L^2(\mathbb{Q}_p^n)$.*

Proof. Using the Parseval equality, we can easily get the following formule:

$$(\Delta_p \psi, \varphi) = \int_{\mathbb{Q}_p^n} \sum_{j=1}^n \langle\xi_j\rangle^2 \psi^\wedge(\xi) \overline{\varphi^\wedge}(\xi) d^n\xi = (\psi, \Delta_p \varphi), \quad \forall \psi, \varphi \in \mathcal{D}(\Delta_p),$$

$$\|\Delta_p \psi\|^2 = (\Delta_p \psi, \Delta_p \psi) = \int_{\mathbb{Q}_p^n} (\sum_{j=1}^n \langle\xi_j\rangle^2)^2 |\psi^\wedge(\xi)|^2 d^n\xi, \quad \psi \in \mathcal{D}(\Delta_p).$$

Here $(\cdot,\cdot)$ is the scalar product in the Hilbert space $L^2(\mathbb{Q}_p^n)$, and $\|\cdot\|$ is the L^2-norm. Moreover,

$$(\Delta_p \psi, \psi) = \int_{\mathbb{Q}_p^n} \sum_{j=1}^n \langle\xi_j\rangle^2 |\psi^\wedge(\xi)|^2 d^n\xi > 0, \quad 0 \ne \psi \in \mathcal{D}(\Delta_p). \quad \sharp$$

There is a non-negative self-adjoint operator [20] $\Delta_p^{\frac{1}{2}}$ with the symbol $(\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{1}{2}}$, associated with Δ_p. The domain of $\Delta_p^{\frac{1}{2}}$ is

$$\mathcal{D}(\Delta_p^{\frac{1}{2}}) = \{\psi \in L^2(\mathbb{Q}_p^n) : (\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{1}{2}} \psi^\wedge \in L^2(\mathbb{Q}_p^n)\}.$$

We have

$$\mathcal{D}(\Delta_p) = \{\psi : \psi \in \mathcal{D}(\Delta_p^{\frac{1}{2}}) \text{ and } \Delta_p^{\frac{1}{2}}\psi \in \mathcal{D}(\Delta_p^{\frac{1}{2}})\}.$$

Furthermore, there is a non-negative quadratic form $Q(\cdot,\cdot)$ on $L^2(\mathbb{Q}_p^n)$ with domain $\mathcal{D}(\Delta_p^{\frac{1}{2}}) \times \mathcal{D}(\Delta_p^{\frac{1}{2}})$ such that

$$Q(\psi,\varphi) = (\Delta_p^{\frac{1}{2}}\psi, \Delta_p^{\frac{1}{2}}\varphi), \quad \forall \psi,\varphi \in \mathcal{D}(\Delta_p^{\frac{1}{2}}).$$

If one define $Q_*(\psi,\varphi) = Q(\psi,\varphi) + (\psi,\varphi)$ for any $\psi,\varphi \in \mathcal{D}(\Delta^{\frac{1}{2}})$, then $(\mathcal{D}(\Delta_p^{\frac{1}{2}}), Q_*(\cdot,\cdot))$ is a Hilbert space.

Proposition 5.1. *For any $\eta \in \mathbb{Q}_p^n$, the additive character $\chi_p(\eta\cdot x)$ is an eigen-function of the Laplacian Δ_p with respect to the eigen-value $\sum_{j=1}^n \langle \eta_j\rangle^2$.*

Proof. Using Example 3.4, we have

$$\Delta_p\chi_p(\eta\cdot x) = \sum_{j=1}^n T^{2e^j}\chi_p(\eta\cdot x) = \sum_{j=1}^n \langle\eta\rangle^{2e^j}\chi_p(\eta\cdot x) = \sum_{j=1}^n \langle\eta_j\rangle^2\chi_p(\eta\cdot x). \quad \sharp$$

Let us consider the eigen-value problem in $\mathbb{Q}_p^n$,

$$\Delta_p\psi = \lambda\psi, \quad \psi \in L^2(\mathbb{Q}_p^n). \tag{5.1}$$

From Theorem 5.2, the spectrum of the operator Δ_p consists of non-negative eigen-values.

Let $\lambda = 0$. Then $\Delta_p\psi = 0$, which implies $\psi = 0$ from Theorem 5.1. Hence, $\lambda = 0$ is not an eigen-value of Δ_p.

Let $\lambda > 0$. Applying to the equation (5.1) the Fourier transform, we get

$$(\sum_{j=1}^n \langle\xi_j\rangle^2 - \lambda)\psi^\wedge(\xi) = 0.$$

From here we conclude that the eigen-values of the Laplacian Δ_p have the form

$$\lambda_{N_1,N_2,\cdots,N_n} = \sum_{j=1}^n p^{2N_j}, \quad N_j \in \mathbb{Z}^+, j = 1,2,\cdots,n.$$

Now we construct an orthonormal basis of eigen-functions of the Laplacian Δ_p in $L^2(\mathbb{Q}_p^n)$.

Recall that in the 1-dimensional case, an orthonormal basis of eigen-functions of T^s in $L^2(\mathbb{Q}_p)$ is given in [19].

Lemma 5.2. [19] *Let $n = 1, s \in \mathbb{R}$. The set of test functions $\{\psi_{Nk\varepsilon}(x)\}$ is an orthonormal basis of eigen-functions of T^s in $L^2(\mathbb{Q}_p)$, where*

$$\psi_{Nk\varepsilon}(x) = p^{-\frac{N}{2}}\chi_p(p^{N-1}kx)\Delta_0(p^N x - \varepsilon), \quad N \in \mathbb{Z}, \quad k = 1, 2, \cdots, p-1, \quad \varepsilon \in \mathbb{Q}_p/B_0.$$

Moreover,

$$T^s\psi_{1-N,k,\varepsilon}(x) = \begin{cases} p^{Ns}\psi_{1-N,k\varepsilon}(x), & for\ N > 0, \\ \psi_{1-N,k\varepsilon}(x), & for\ N \leq 0. \end{cases}$$

The orthonormal basis $\{\psi_{Nk\varepsilon}(x)\}$ is a p-adic wavelet basis in $L^2(\mathbb{Q}_p)$ constructed by S.V. Kozyrev[21].

For the Laplacian Δ_p, we have

Theorem 5.3. *The set of test functions $\{\prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)\}$ is an orthonormal basis of eigen-functions of the Laplacian Δ_p in $L^2(\mathbb{Q}_p^n)$, where $N_j \in \mathbb{Z}, k_j = 1, 2, \cdots, p-1, \varepsilon_j \in \mathbb{Q}_p/B_0, j = 1, 2, \cdots, n$. Moreover,*

$$\Delta_p \prod_{j=1}^n \psi_{1-N_j,k_j,\varepsilon_j}(x) = \sum_{j=1}^n p^{2\max\{0,N_j\}} \prod_{j=1}^n \psi_{1-N_j,k_j,\varepsilon_j}(x).$$

Proof. Taking $\psi(x) = \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)$, using Lemma 5.2, we have

$$\begin{aligned} \Delta_p\psi(x) = \Delta_p \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j) &= \sum_{j=1}^n T_{x_j}^2 \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j) \\ = \sum_{j=1}^n \prod_{1\leq j'\leq n, j'\neq j} &\psi_{N_{j'}k_{j'}\varepsilon_{j'}}(x_{j'})T_{x_j}^2\psi_{N_jk_j\varepsilon_j}(x_j) \\ = \sum_{j=1}^n \prod_{1\leq j'\leq n, j'\neq j} &\psi_{N_{j'}k_{j'}\varepsilon_{j'}}(x_{j'})p^{2\max\{0,1-N_j\}}\psi_{N_jk_j\varepsilon_j}(x_j) \\ = \sum_{j=1}^n p^{2\max\{0,1-N_j\}}&\psi(x). \end{aligned}$$

For the orthogonality of $\{\prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)\}$, consider the scalar product (ψ, φ) in $L^2(\mathbb{Q}_p^n)$, where $\psi(x) = \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)$ and $\varphi(x) = \prod_{j=1}^n \psi_{N'_jk'_j\varepsilon'_j}(x_j)$.

$$\begin{aligned} (\psi(x), \varphi(x)) &= (\prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j), \prod_{j=1}^n \psi_{N'_jk'_j\varepsilon'_j}(x_j)) \\ = \prod_{j=1}^n (\psi_{N_jk_j\varepsilon_j}(x_j), \psi_{N'_jk'_j\varepsilon'_j}(x_j)) &= \prod_{j=1}^n \delta_{N_jN'_j}\delta_{\varepsilon_j\varepsilon'_j}\delta_{k_jk'_j}. \end{aligned}$$

For the completeness of $\{\prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)\}$, consider the Fourier coefficient of Δ_0^n.

$$(\Delta_0^n, \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}) = \prod_{j=1}^n p^{-\frac{N_j}{2}} \int_{B_0\cap p^{-N_j}\varepsilon_j} \chi_p(-p^{N_j-1}k_jx_j)dx_j = \prod_{j=1}^n p^{-\frac{N_j}{2}} \delta_{\varepsilon_j,B_0}\gamma(N_j),$$

where γ is a function defined as $\gamma(t) = 0$ if $t \leq 0, \gamma(t) = 1$ if $t \geq 1$.

Hence,

$$\sum |(\Delta_0^n, \prod_{j=1}^n \psi_{N_jk_j\varepsilon_j})|^2 = \sum \prod_{j=1}^n p^{-N_j}\delta_{\varepsilon_j,B_0}\gamma(N_j)$$

$$= (p-1)^n \sum_{1\leq N_j<+\infty, j=1,2,\cdots,n} \prod_{j=1}^n p^{-N_j} = (p-1)^n \prod_{j=1}^n \sum_{N_j=1}^{+\infty} p^{-N_j} = 1 = \|\Delta_0\|^2.$$

Thus the Parserval equality of Δ_0^n holds, which proves the completeness of $\{\prod_{j=1}^n \psi_{N_jk_j\varepsilon_j}(x_j)\}$. ♯

6 Cauchy problem for wave equations on $\mathbb{Q}_p^n$

In this section, we consider the initial value problem

$$\begin{array}{lll} \frac{\partial^2 u}{\partial t^2} - a\Delta_p^s u = f(x,t), & x \in \mathbb{Q}_p^n, & 0 < t \leq T, \\ u(x,0) = \varphi(x), & x \in \mathbb{Q}_p^n, & \\ u_t(x,0) = \psi(x), & x \in \mathbb{Q}_p^n, & \end{array} \tag{6.1}$$

where $a \neq 0$, $s \in \mathbb{R}$, $T > 0$, the function f and the initial function φ and ψ are complex valued.

Theorem 6.1. *The homogeneous equation*

$$\begin{array}{lll} \frac{\partial^2 u}{\partial t^2} - a\Delta_p^s u = 0, & x \in \mathbb{Q}_p^n, & 0 < t \leq T, \\ u(x,0) = 0, & x \in \mathbb{Q}_p^n, & \\ u_t(x,0) = \psi(x), & x \in \mathbb{Q}_p^n, & \end{array} \tag{6.2}$$

has a fundamental solution

$$E(x,t) = \begin{cases} \left(\frac{e^{\sqrt{a}(\sum_{j=1}^n\langle\xi_j\rangle^2)^{\frac{s}{2}}t} - e^{-\sqrt{a}(\sum_{j=1}^n\langle\xi_j\rangle^2)^{\frac{s}{2}}t}}{2\sqrt{a}(\sum_{j=1}^n\langle\xi_j\rangle^2)^{\frac{s}{2}}}\right)^\vee(x), & \text{for } a > 0, \\ \left(\frac{\sin(\sqrt{-a}(\sum_{j=1}^n\langle\xi_j\rangle^2)^{\frac{s}{2}}t)}{\sqrt{-a}(\sum_{j=1}^n\langle\xi_j\rangle^2)^{\frac{s}{2}}}\right)^\vee(x), & \text{for } a < 0, \end{cases}$$

where $E(x,t) \in \mathcal{D}'(\mathbb{Q}_p^n)$ has a compact support in B_0^n for any $t \in [0,T]$. Moreover, for $\psi \in \mathcal{D}'(\mathbb{Q}_p^n)$ the equation (6.2) has a solution

$$u(x,t) = E(x,t) * \psi.$$

Proof. Let $\psi=\delta$, denote by $E(x,t)$ the fundamental solution of (6.2). Applying to (6.2) the Fourier transform, we get

$$\frac{\partial^2 E^\wedge(\xi,t)}{\partial t^2}=a(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^s E^\wedge(\xi,t),$$

$$E^\wedge(\xi,0)=0,\quad E_t^\wedge(\xi,0)=1.$$

If $a>0$, then

$$E^\wedge(\xi,t)=C_1e^{-\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t}+C_2e^{\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t},$$

where C_1 and C_2 are two constants satisfying

$$C_1+C_2=0,$$

and

$$-C_1\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}+C_2\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}=1.$$

So

$$-C_1=C_2=\frac{1}{2\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}}.$$

Hence,

$$E^\wedge(\xi,t)=\frac{e^{\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t}-e^{-\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t}}{2\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}},$$

and $\forall t\in[0,T]$, $E^\wedge(\xi,t)\in\mathcal{E}(\mathbb{Q}_p^n)$ assumes constant values on cosets of B_0^n. So

$$E(x,t)=(\frac{e^{\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t}-e^{-\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t}}{2\sqrt{a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}})^\vee(x),$$

and $\forall t\in[0,T]$, $E(x,t)\in\mathcal{D}'(\mathbb{Q}_p^n)$ with $\mathrm{supp}E(x,t)\subset B_0^n$.

If $a<0$, then

$$E^\wedge(\xi,t)=\frac{\sin(\sqrt{-a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t)}{\sqrt{-a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}},$$

and also $\forall t\in[0,T]$, $E^\wedge(\xi,t)\in\mathcal{E}(\mathbb{Q}_p^n)$ assumes constant values on cosets of B_0^n. So

$$E(x,t)=(\frac{\sin(\sqrt{-a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}t)}{\sqrt{-a}(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^{\frac{s}{2}}})^\vee(x),$$

and we have $\forall t\in[0,T]$, $E(x,t)\in\mathcal{D}'(\mathbb{Q}_p^n)$ with $\mathrm{supp}E(x,t)\subset B_0^n$.

Hence $\forall t \in [0, T]$, $E(x,t) * \psi(x)$ exists. Let $u(x,t) = E(x,t) * \psi(x)$. Then we have

$$\begin{aligned}
(\frac{\partial^2}{\partial t} - a\Delta_p^s)u(x,t) &= (\frac{\partial^2}{\partial t} - a\Delta_p^s)(E(x,t) * \psi(x)) \\
&= ((\frac{\partial}{\partial t} - a\Delta_p^s)E(x,t)) * \psi(x) \\
&= 0 * \psi(x) = 0,
\end{aligned}$$

and

$$u_t(x,0) = E_t(x,0) * \psi(x) = \delta(x) * \psi(x) = \psi(x). \quad \sharp$$

Hence, $u(x,t) = E(x,t) * \psi(x)$ is a solution of (6.2). $\sharp$

For the function $f(x,t)$ defined on $\mathbb{Q}_p^n \times [0,T]$, we say that $f \in \mathcal{E}(\mathbb{Q}_p^n)$ *uniformly with respect to* t, if its exponent of local constancy do not depend on t.

Lemma 6.1. *Let $\omega(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to t, and ω is continuous on t. Then*

$$\Delta_p^s \int_0^t \omega(x,\tau)d\tau = \int_0^t \Delta_p^s \omega(x,\tau)d\tau.$$

Proof. It is easy to check that

$$\int_0^t \omega(x,\tau)d\tau \in \mathcal{E}(\mathbb{Q}_p^n) \text{ and } \int_0^t \Delta_p^s \omega(x,\tau)d\tau \in \mathcal{E}(\mathbb{Q}_p^n).$$

Then for any $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have

$$\begin{aligned}
(\Delta_p^s \int_0^t \omega(x,\tau)d\tau, \phi(x)) &= (\int_0^t \omega(x,\tau)d\tau, \Delta_p^s \phi(x)) \\
&= \int_{\mathbb{Q}_p^n} d^n x \int_0^t \omega(x,\tau)\Delta_p^s \phi(x)d\tau.
\end{aligned}$$

Using Fubini Theorem, we get

$$\begin{aligned}
(\Delta_p^s \int_0^t \omega(x,\tau)d\tau, \phi(x)) &= \int_0^t d\tau \int_{\mathbb{Q}_p^n} \omega(x,\tau)\Delta_p^s \phi(x)d^n x \\
&= \int_0^t d\tau \int_{\mathbb{Q}_p^n} \Delta_p^s \omega(x,\tau)\phi(x)d^n x \\
&= \int_{\mathbb{Q}_p^n} d^n x \int_0^t \Delta_p^s \omega(x,\tau)\phi(x)d\tau \\
&= (\int_0^t \Delta_p^s \omega(x,\tau)d\tau, \phi(x)).
\end{aligned}$$

Hence,

$$\Delta_p^s \int_0^t \omega(x,\tau)d\tau = \int_0^t \Delta_p^s \omega(x,\tau)d\tau. \quad \sharp$$

Theorem 6.2. *Denote by M_ψ the solution of the homogenous equation (6.2), $\varphi, \psi \in \mathcal{E}(\mathbb{Q}_p^n)$, $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $f(x,t) \in C[0,T]$. Then the inhomogeneous equation (6.1) has a solution $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$, uniformly with respect to $t \in [0,T]$, $u(x,t) \in C^2[0,T]$, with*

$$u = M_\psi + \frac{\partial}{\partial t} M_\varphi(x,t) + \int_0^t M_{f_\tau}(x, t-\tau)d\tau.$$

Proof. A solution of (6.1) is given by

$$u = u_1 + u_2 + u_3,$$

where u_2 is the solution of (6.2), and u_1, u_3 are solutions of the following two equations, respectively.

$$\begin{array}{lll} \frac{\partial^2 u}{\partial t^2} - a\Delta_p^s u = 0, & x \in \mathbb{Q}_p^n, & 0 < t \leq T, \\ u(x,0) = \varphi(x), & x \in \mathbb{Q}_p^n, & \\ u_t(x,0) = 0, & x \in \mathbb{Q}_p^n, & \end{array} \tag{6.3}$$

and

$$\begin{array}{lll} \frac{\partial^2 u}{\partial t^2} - a\Delta_p^s u = f(x,t), & x \in \mathbb{Q}_p^n, & 0 < t \leq T, \\ u(x,0) = 0, & x \in \mathbb{Q}_p^n, & \\ u_t(x,0) = 0, & x \in \mathbb{Q}_p^n. & \end{array} \tag{6.4}$$

Let $u_1 = \frac{\partial}{\partial t} M_\varphi$. Then

$$\frac{\partial^2 u_1}{\partial t^2} - a\Delta_p^s u_1 = \frac{\partial}{\partial t}(\frac{\partial^2 M_\varphi}{\partial t^2} - a\Delta_p^s M_\varphi) = 0,$$

$$u_1(x,0) = \frac{\partial}{\partial t} M_\varphi(x,t)|_{t=0} = \varphi(x),$$

$$u_{1t}(x,0) = \frac{\partial^2}{\partial t^2} M_\varphi(x,t)|_{t=0} = a\Delta_p^s M_\varphi(x,t)|_{t=0} = 0.$$

So $u_1 = \frac{\partial}{\partial t} M_\varphi$ solves the equation (6.3).

Let $f_\tau = f(x,\tau)$, $u_3 = \int_0^t M_{f_\tau}(x, t-\tau)d\tau$. It is easy to check that $u_3(x,0) = 0$.

Since $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$ and $f(x,t) \in C[0,T]$, we have that $\forall t \in [0,T]$, $M_{f_\tau}(x,t-\tau) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to τ and is continuous on τ. Hence, $u_3(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to t. So we have

$$\frac{\partial u_3}{\partial t} = M_{f_\tau}(x,t-\tau)|_{\tau=t} + \int_0^t \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t} d\tau = \int_0^t \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t} d\tau.$$

Then

$$u_{3t}(x,0) = \frac{\partial u_3}{\partial t}|_{t=0} = 0.$$

Using Lemma 6.1, we get

$$\begin{aligned} \frac{\partial^2 u_3}{\partial t^2} &= \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t}|_{\tau=t} + \int_0^t \frac{\partial^2 M_{f_\tau}(x,t-\tau)}{\partial t^2} d\tau \\ &= f(x,t) + a\int_0^t \Delta_p^s M_{f_\tau}(x,t-\tau) d\tau \\ &= f(x,t) + a\Delta_p^s u_3. \end{aligned}$$

So $u_3 = \int_0^t M_{f_\tau}(x,t-\tau)d\tau$ is a solution of (6.4).

Since $\varphi, \psi \in \mathcal{E}(\mathbb{Q}_p^n)$, it is obvious that $u_1, u_2 \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to t. Hence, $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$. ♯

Lemma 6.2. *If $a < 0$, $s > n$, then the fundamental solution $E(x,t)$ is a continuous function supported in B_0^n for any $0 < t \le T$.*

Proof. If $a < 0$, $s > n$, then for any $0 < t \le T$,

$$\begin{aligned} \int_{\mathbb{Q}_p^n} |E^\wedge(\xi,t)| d^n\xi &= \int_{\mathbb{Q}_p^n} \left|\frac{\sin(\sqrt{-a}(\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{s}{2}} t)}{\sqrt{-a}(\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{s}{2}}}\right| d^n\xi \\ &\le \frac{1}{\sqrt{-a}} \int_{\mathbb{Q}_p^n} \frac{1}{(\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{s}{2}}} d^n\xi \\ &= \frac{1}{\sqrt{-a} n^{\frac{s}{2}}} + \frac{1}{\sqrt{-a}} \sum_{r=1}^{+\infty} \int_{|\xi|=p^r} \frac{1}{(\sum_{j=1}^n \langle\xi_j\rangle^2)^{\frac{s}{2}}} d^n\xi \\ &\le \frac{1}{\sqrt{-a} n^{\frac{s}{2}}} + \frac{1}{\sqrt{-a}} (1 - \frac{1}{p^n}) \sum_{r=1}^{+\infty} p^{(n-s)r} \\ &< \infty. \end{aligned}$$

So $\forall t \in (0,T]$, $E^\wedge(\xi,t) \in L^1(\mathbb{Q}_p^n)$. Hence $E(x,t)$ is a continuous function supported in B_0^n for any $0 < t \le T$. ♯

Theorem 6.3. *If $a < 0$, $s > n$, $\varphi, \psi \in \mathcal{E}(\mathbb{Q}_p^n)$, $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $f(x,t) \in C[0,T]$, then the equation (6.1) has a solution $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $u(x,t) \in C^2[0,T]$, with*

$$u(x,t) = \int_{\mathbb{Q}_p^n} E(x-\eta,t)\psi(\eta)d^n\eta + \int_{\mathbb{Q}_p^n} \frac{\partial}{\partial t} E(x-\eta,t)\varphi(\eta)d^n\eta + \int_0^t d\tau \int_{\mathbb{Q}_p^n} E(x-\eta,t-\tau)f(\eta,\tau)d^n\eta.$$

Proof. Using Theorem 6.2 and Lemma 6.2, we have

$$\begin{aligned} u(x,t) &= M_\psi + \frac{\partial}{\partial t} M_\varphi(x,t) + \int_0^t M_{f_\tau}(x,t-\tau)d\tau \\ &= E(x,t) * \psi + \frac{\partial}{\partial t} E(x,t) * \varphi + \int_0^t E(\cdot,t) * f_\tau(x,t-\tau)d\tau \\ &= \int_{\mathbb{Q}_p^n} E(x-\eta,t)\psi(\eta)d^n\eta + \int_{\mathbb{Q}_p^n} \frac{\partial}{\partial t} E(x-\eta,t)\varphi(\eta)d^n\eta \\ &\quad + \int_0^t d\tau \int_{\mathbb{Q}_p^n} E(x-\eta,t-\tau)f(\eta,\tau)d^n\eta. \quad \sharp \end{aligned}$$

7 Cauchy problem for heat equations on $\mathbb{Q}_p^n$

In this section, we consider another initial value problem

$$\begin{array}{ll} \frac{\partial u}{\partial t} - a\Delta_p^s u = f(x,t), & x \in \mathbb{Q}_p^n, \quad 0 < t \le T, \\ u(x,0) = \varphi(x), & x \in \mathbb{Q}_p^n, \end{array} \tag{7.1}$$

where $a \neq 0$, $s \in \mathbb{R}$, $T > 0$, the function f and the initial function φ are complex valued.

Theorem 7.1. *The homogeneous equation*

$$\begin{array}{ll} \frac{\partial u}{\partial t} - a\Delta_p^s u = 0, & x \in \mathbb{Q}_p^n, \quad 0 < t \le T, \\ u(x,0) = \varphi(x), & x \in \mathbb{Q}_p^n, \end{array} \tag{7.2}$$

has a fundamental solution

$$F(x,t) = (e^{a(\sum_{j=1}^n \langle \xi_j \rangle^2)^s t})^\vee(x),$$

where $F(x,t) \in \mathcal{D}'(\mathbb{Q}_p^n)$ has a compact support in B_0^n, for any $t \in [0,T]$. Moreover, for $\varphi \in \mathcal{D}'(\mathbb{Q}_p^n)$ the equation (7.2) has a solution

$$u(x,t) = F(x,t) * \varphi.$$

Proof. Let $\varphi = \delta$, denote by $F(x,t)$ the fundamental solution of (7.2). Applying to (7.2) the Fourier transform, we get

$$\frac{\partial F^{\wedge}(\xi,t)}{\partial t} = a(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^s F^{\wedge}(\xi,t), \quad F^{\wedge}(\xi,0)=1.$$

Thus,

$$F^{\wedge}(\xi,t) = e^{a(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^s t},$$

and $\forall t \in [0,T]$, $F^{\wedge}(\xi,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ assumes constant values on cosets of B_0^n. So

$$F(x,t) = (e^{a(\sum_{j=1}^{n}\langle\xi_j\rangle^2)^s t})^{\vee}(x),$$

and $\forall t \in [0,T]$, $F(x,t) \in \mathcal{D}'(\mathbb{Q}_p^n)$ with $\text{supp}F(x,t) \subset B_0^n$.

Hence $\forall t \in [0,T]$, $F(x,t) * \varphi(x)$ exists. Let $u(x,t) = F(x,t) * \varphi(x)$. Then

$$\begin{aligned}(\frac{\partial}{\partial t} - a\Delta_p^s)u(x,t) &= (\frac{\partial}{\partial t} - a\Delta_p^s)(F(x,t) * \varphi(x)) \\ &= ((\frac{\partial}{\partial t} - a\Delta_p^s)F(x,t)) * \varphi(x) \\ &= 0 * \varphi(x) = 0,\end{aligned}$$

and

$$u(x,0) = F(x,0) * \varphi(x) = \delta(x) * \varphi(x) = \varphi(x).$$

Hence, $u(x,t) = F(x,t) * \varphi(x)$ is a solution of (7.2). ♯

Theorem 7.2. *If we denote by W_φ the solution of the homogenous equation (7.2), $\varphi \in \mathcal{E}(\mathbb{Q}_p^n)$, $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $f(x,t) \in C[0,T]$, then the inhomogeneous equation (7.1) has a solution $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $u(x,t) \in C^1[0,T]$, with*

$$u = W_\varphi + \int_0^t W_{f_\tau}(x,t-\tau)d\tau.$$

Proof. A solution of (7.1) is given by

$$u = u_1 + u_2,$$

where u_1 is the solution of (7.2), and u_2 is the solution of the following equation.

$$\begin{cases}\frac{\partial u}{\partial t} - a\Delta_p^s u = f(x,t), & x \in \mathbb{Q}_p^n, \quad 0 < t \leq T, \\ u(x,0) = 0, & x \in \mathbb{Q}_p^n.\end{cases} \tag{7.3}$$

Let $f_\tau = f(x,\tau)$, $u_2 = \int_0^t W_{f_\tau}(x,t-\tau)d\tau$. It is easy to get $u_2(x,0)=0$.

Since $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$ and $f(x,t) \in C[0,T]$, we have $\forall t \in [0,T]$, $W_{f_\tau}(x,t-\tau) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to τ and is continuous on τ. Hence, $u_2(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to t. Using Lemma 6.1, we have

$$\begin{aligned}\frac{\partial u_2}{\partial t} &= W_{f_\tau}(x,t-\tau)|_{\tau=t} + \int_0^t \frac{\partial W_{f_\tau}(x,t-\tau)}{\partial t} d\tau \\ &= f(x,t) + \int_0^t a\Delta_p^s W_{f_\tau}(x,t-\tau)d\tau \\ &= f(x,t) + a\Delta_p^s u_2.\end{aligned}$$

So $u_2 = \int_0^t W_{f_\tau}(x,t-\tau)d\tau$ solves the equation (6.3).

Since $\varphi \in \mathcal{E}(\mathbb{Q}_p^n)$ and $\mathrm{supp}F(x,t) \in B_0^n$, we get that $u_1(x,t) = F(x,t) * \varphi(x) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to t. Hence, $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$. ♯

Lemma 7.1. *If $a < 0$, $s > 0$, then the fundamental solution $F(x,t)$ is a non-negative continuous function supported in B_0^n for any $0 < t \le T$.*

Proof. If $a < 0$, $s > 0$, then for any $0 < t \le T$,

$$\begin{aligned}\int_{\mathbb{Q}_p^n} F^\wedge(\xi,t)d^n\xi &= \int_{\mathbb{Q}_p^n} e^{a(\sum_{j=1}^n \langle\xi_j\rangle^2)^s t} d^n\xi \\ &= \int_{B_0^n} e^{an^s t} d^n\xi + \int_{\mathbb{Q}_p^n \setminus B_0^n} e^{a(\sum_{j=1}^n \langle\xi_j\rangle^2)^s t} d^n\xi \\ &= e^{an^s t} + \sum_{r=1}^{+\infty} \int_{|\xi|=p^r} e^{a(\sum_{j=1}^n \langle\xi_j\rangle^2)^s t} d^n\xi \\ &\le e^{an^s t} + \sum_{r=1}^{+\infty} \int_{|\xi|=p^r} e^{ap^{2rs}t} d^n\xi \\ &= e^{an^s t} + (1-\frac{1}{p^n}) \sum_{r=1}^{+\infty} e^{ap^{2rs}t} p^{nr} \\ &< \infty.\end{aligned}$$

So $\forall t \in (0,T]$, $F^\wedge(\xi,t) \in L^1(\mathbb{Q}_p^n)$, and hence $F(x,t)$ is a continuous function supported in B_0^n for any $0 < t \le T$. For the non-negative property of $F(x,t)$, one can verify it by a direct calculation. ♯

Theorem 7.3. *If $a < 0$, $s > 0$, $\varphi \in \mathcal{E}(\mathbb{Q}_p^n)$, $f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0,T]$, $f(x,t) \in C[0,T]$, then the equation (7.1) has a solution $u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n)$*

uniformly with respect to $t \in [0,T]$, $u(x,t) \in C^1[0,T]$, *with*

$$u(x,t) = \int_{\mathbb{Q}_p^n} F(x-\eta,t)\varphi(\eta)d^n\eta + \int_0^t d\tau \int_{\mathbb{Q}_p^n} F(x-\eta,t-\tau)f(\eta,\tau)d^n\eta.$$

Proof. From Theorem 7.2 and Lemma 7.2, we have

$$\begin{aligned} u(x,t) &= W_\varphi + \int_0^t W_{f_\tau}(x,t-\tau)d\tau \\ &= F(x,t) * \varphi + \int_0^t F(\cdot,t) * f_\tau(x,t-\tau)d\tau \\ &= \int_{\mathbb{Q}_p^n} F(x-\eta,t)\varphi(\eta)d^n\eta + \int_0^t d\tau \int_{\mathbb{Q}_p^n} F(x-\eta,t-\tau)f(\eta,\tau)d^n\eta. \quad \sharp \end{aligned}$$

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