p-ENERGIES ON P.C.F. SELF-SIMILAR SETS

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Abstract. We study p-energies on post-critically finite (p.c.f.) self-similar sets for $1 < p < \infty$, as limits of discrete p-energies on approximation graphs, extending the construction of Dirichlet forms, the $p = 2$ setting. By suitably enlarging the choices of discrete p-energies, and employing the energy averaging method developed by Kusuoka-Zhou, we prove the existence of symmetric p-energies on affine nested fractals, and extend Sabot's celebrated criterion for existence and non-existence of Dirichlet forms on p.c.f. self-similar sets to the $1 < p < \infty$ setting.

Contents

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1. Introduction

In this paper, we consider the problem of constructing p-energy forms ($p \in (1,\infty)$) on post-critically finite (p.c.f.) self-similar sets. A typical example is on the unit interval $[0,1]$, for each $f \in W^{1,p}$, the p-energy of f is defined as

$$\mathcal{E}_p(f) = \int_0^1 |\nabla f(x)|^p dx,$$

which can be approximated by discrete energies,

$$\mathcal{E}_p(f) = \lim_{k\to\infty} 2^{(p-1)k} \sum_{l=1}^{2^k} \left| f\left(\frac{l}{2^k}\right) - f\left(\frac{l-1}{2^k}\right)\right|^p.$$

For $p = 2$, on p.c.f. self-similar sets, the same construction was introduced by Kigami [15, 16] to construct Dirichlet forms. In this paper, for $1 < p < \infty$, on p.c.f. self-similar sets, we will define the p-energy forms in a similar manner, by taking the limit of the averaged discrete energies on first n level graphs approximating to the fractal, inspired by Kusuoka-Zhou's method [21].

In history, the energies ($p = 2$ case) on p.c.f. self-similar sets were motivated by the study of Brownian motions on fractals, with pioneering works of Kusuoka [20], Goldstein [10] and Barlow-Perkins [5] on the Sierpinski gasket, and of Lindstrøm [22] on nested fractals. By introducing the tool of Dirichlet forms [16], all the constructions can be done in a purely analytic way (see books [1, 17, 31]). There have been deep studies on the existence of such forms [11, 23, 24, 25, 27, 28, 29]. In particular, a famous test was introduced in the celebrated work of Sabot [29] (also contributes to the uniqueness under some reasonable assumptions).

On the other hand, for $p \neq 2$, on a general p.c.f. self-similar set K (with associated iterated function system $\{F_i\}_{i=1}^N$, $N \geq 2$), we can not expect a direct extension of Kigami's approach, since there is no eigenforms of the renormalization map acting only on p-energies of the (standard) form

$$\sum_{x,y\in V_0} c_{x,y}|f(x) - f(y)|^p.$$

Here V_0 is the boundary of K which consists of finitely many points, the renormalization map refers to the process of defining a new energy form on V_0 out of an old one, by first defining a form on V_1 (the first level iteration of V_0 under $\{F_i\}_{i=1}^N$) in a self-similar way, and then tracing it back on V_0. This was first observed in the explorative work [14] of Herman, Peirone and Strichartz on the Sierpinski gasket (and more generally, on weakly completely symmetric fractals), and the problem was resolved by considering the renormalization map on a larger class which includes certain non-standard p-energies. In this work, we will admit the same idea of [14] and show the existence of an eigenform on affine nested fractals, and generalize Sabot's test to $p \neq 2$, which greatly extends the results of [14].

To construct a natural p-energy form on a fractal K, we admit the construction of Kusuoka-Zhou [21]. As long as we show the existence of an eigenform of the renormalization map with scaling factors $\{r_i\}_{i=1}^N$, we see that

$$\frac{1}{n}\sum_{m=0}^{n-1}\sum_{w\in W_m} r_w^{-1}\big|f(F_w x) - f(F_w y)\big|^p$$

admits a converging subsequence,

$$\mathcal{E}(f)=\lim_{l\to\infty}\frac{1}{n_l}\sum_{m=0}^{n_l-1}\sum_{w\in W_m} r_w^{-1}\big|f(F_wx)-f(F_wy)\big|^p,$$

where the limit is defined to be the self-similar p-energy of a function f on K (see Section 3 for the standard notations of F_w, r_w and W_m). In particular, when $p=2$, the same construction gives the Dirichlet form on K. Following Kumagai's method [19], we will see that even in the non-regular case ($r_i>1$ for some i), the p-energy can also be well understood as defined on $L^p(K,\mu)$ providing the measure μ is suitably chosen.

In fact, the technique of Kusuoka and Zhou [21] was firstly invented to construct self-similar Dirichlet forms on the Sierpinski carpets, typical symmetric fractals with infinitely ramification property. Recently, by introducing necessary arguments replacing the "Knight move" construction [2, 3, 4], two of the authors successfully show a purely analytic construction of Dirichlet forms on a wider class of planar fractals named unconstrained Sierpinski carpets [7]. Subsequently, Kigami [18] and Shimizu [30] introduced the arguments and the notion of p-modules to the setting of p-energies for $1<p<\infty$ on Sierpinski carpets and some other square-based fractals. However, a theory for p-energy forms on general p.c.f. self-similar sets remains blank and is urgent to be set up. By using the method of Γ-convergence of Besov-type seminorms, Gao, Yu and Zhang [9] constructed p-energy forms on a class of p.c.f. self-similar sets under additional assumptions on the critical Besov space. The purpose of this paper is to attempt to establish general practical criterions for the existence and non-existence of p-energy forms on general p.c.f. self-similar sets.

We organize the structure of the paper as follows. It is roughly divided into four parts.

In Section 2-4, we consider the problem of finding an eigenform of the renormalization map. In Section 2, following the pioneering work [14], we introduce several classes of non-standard p-energy forms on finite sets, and prove that one can compare two forms via their associated p-effective resistances. In Section 3, we introduce the basic concepts of p.c.f. self-similar sets and the renormalization maps, and develop some basic properties. In Section 4, by modifying the idea of Kusuoka-Zhou [21], we obtain an equivalent condition for the existence of an eigenform, denoted as assumption (**A**).

In Section 5, we turn to the construction of p-energies on p.c.f. self-similar sets under assumption (**A**). In particular, in Section 5.2, 5.3, along with Appendix A, we will focus on the non-regular case.

The short section, Section 6, will deal with affine nested fractals. By verifying assumption (**A**), we show that for any given symmetric renormalization factors, there always exists a symmetric p-energy form on the fractal.

Finally, in Section 7 and 8, for general p.c.f. self-similar sets, we prove criterion for assumption (**A**). We will follow the spirit of Sabot in [29] and utilize the technique of preserved equivalent relations in the proof. However, many of the arguments (adapted from the $p=2$ case) need essential modifications due to the wild properties of discrete p-energy forms. Readers may notice that Metz has improved Sabot's results in an earlier work [25], and recently Peirone presented a new proof [28] based on a fixed point theorem of anti-attracting maps.

Throughout this paper, on any set X, we write $l(X)$ for the space of real-valued functions on X; for $f\in l(X)$, we write $\mathrm{Osc}(f)=\sup\{|f(x)-f(y)|:x,y\in X\}$, the oscillation of f on X; if (X,d) is a metric space, we write $C(X)$ for the space of continuous functions on X. We always make the convention that $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$. For two variables a and b, we sometimes use $a\gtrsim b$ to mean $a\geq Cb$ for some constant $C>0$, and $a\lesssim b$ in a similar way. We write $a\asymp b$ if both $a\gtrsim b$ and $a\lesssim b$ hold. We also denote $a\vee b:=\max\{a,b\}$ and $a\wedge b:=\min\{a,b\}$.

2. Discrete p-energies

In this section, we define three classes of discrete p-energies on a finite set A:

$$\mathcal{M}_p(A)\supset\mathcal{Q}_p(A)\supset\mathcal{S}_p(A).$$

Here $\mathcal{S}_p(A)$ stands for standard energies of the form $E^{(p)}(f)=\sum_{x,y\in A}c_{x,y}|f(x)-f(y)|^p$, that are natural analogs to discrete Dirichlet forms ($p=2$ case). Unfortunately, when $p\neq 2$, the trace of standard forms is in general non-standard, so we define $\mathcal{Q}_p(A)$ for the closure of traces of standard forms, where "$\mathcal{Q}$" stands for quasi-standard. $\mathcal{M}_p(A)$ is the most general class introduced in [14], which can be avoided in the statement of theorems, but provides us freedom in the proofs (essential for Sections 7 and 8). We also write $\widetilde{\mathcal{M}}_p(A)$ for the class including degenerate forms, extending $\mathcal{M}_p(A)$.

We will list the basic properties of these p-energies that we need to use in Section 3-6, and we will return to talk more in Section 7.

First, we follow [14] to introduce a large class of discrete p-energies on a finite set A. The main result in this section is Proposition 2.6, which is a p-version of [6, Lemma 2.7]. In earlier celebrated work by Sabot [29], there was also a weaker version ([29, Lemma 1.19]).

Definition 2.1. *Let A be a finite set with $\#A\geq 2$ and $1<p<\infty$. A functional $E^{(p)}$ on $l(A)$ is called a p-energy on A if it satisfies*
(i). **Non-negativity:** $E^{(p)}(f)\geq 0$ *for all* $f\in l(A)$*;*
(ii). **Convexity:** $E^{(p)}(tf+(1-t)g)\leq tE^{(p)}(f)+(1-t)E^{(p)}(g)$ *for all* $f,g\in l(A)$ *and* $t\in(0,1)$*;*
(iii). **Homogeneity of degree** p**:** $E^{(p)}(tf)=|t|^pE^{(p)}(f)$ *for all* $f\in l(A)$ *and* $t\in\mathbb{R}$*;*
(iv). **invariant under addition of constants:** $E^{(p)}(f+t)=E^{(p)}(f)$ *for all* $f\in l(A)$ *and* $t\in\mathbb{R}$*;*
(v). **Markov property:** $E^{(p)}(\bar{f})\leq E^{(p)}(f)$ *for any* $f\in l(A)$*, where* $\bar{f}=(f\vee 0)\wedge 1$*;*
(vi). **Non-degeneracy:** $E^{(p)}(f)=0$ *if and only if* f *is constant on* A*.*

We will write $\mathcal{M}_p(A)$ for the collection of all p-energies $E^{(p)}$ satisfying (i)–(vi). We also write $\widetilde{\mathcal{M}}_p(A)$ for the collection of $E^{(p)}$ satisfying (i)–(v). Note that from (iii) and (iv), it is easy to see that $E^{(p)}(1)=0$.

The following proposition lists some simple properties of p-energies which can be checked directly and we leave the proof to readers.

Proposition 2.2. *Let A be a finite set and $B\subset A$.*

(a). If $E^{(p)}_1,E^{(p)}_2\in\widetilde{\mathcal{M}}_p(A)$, then $E^{(p)}_1+E^{(p)}_2\in\widetilde{\mathcal{M}}_p(A)$; if further $E^{(p)}_1$ or $E^{(p)}_2$ is in $\mathcal{M}_p(A)$, then $E^{(p)}_1+E^{(p)}_2\in\mathcal{M}_p(A)$.

(b). If $E^{(p)}\in\widetilde{\mathcal{M}}_p(B)$, then $E'^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, where $E'^{(p)}:l(A)\to[0,\infty)$ is defined by

$$E'^{(p)}(f)=E^{(p)}(f|_B),\quad \text{for } f\in l(A).$$

(c). If $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, then $E'^{(p)}\in\widetilde{\mathcal{M}}_p(B)$, where $E'^{(p)}:l(B)\to[0,\infty)$ is defined by

$$E'^{(p)}(f)=\inf\big\{E^{(p)}(f'):f'\in l(A),f'|_B=f\big\},\qquad \textit{for } f\in l(B).$$

We will always write this $E'^{(p)}$ as $[E^{(p)}]_B$ and call it the trace of $E^{(p)}$ to B. In addition, $[E^{(p)}]_B\in\mathcal{M}_p(B)$ providing that $E^{(p)}\in\mathcal{M}_p(A)$.

(d). For $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and $f\in l(A)$, if we denote $f_+=f\vee 0$ and $f_-=f_+-f$, then $E^{(p)}(f_+)\le E^{(p)}(f)$ and $E^{(p)}(f_-)\le E^{(p)}(f)$.

Definition 2.3. *Let $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$. For any $x\neq y\in A$, we define the p-effective resistance (p-resistance for short) between x,y to be*

$$R^{(p)}(x,y)=\frac{1}{\inf\big\{E^{(p)}(f):f(x)=0,f(y)=1\big\}}\in(0,\infty],$$

and write $R^{(p)}(x,x)=0$ for any $x\in A$ by convention.

We have the following simple properties of p-resistances.

Proposition 2.4. *(a). For $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, $R^{(p)}$ is symmetric, i.e. $R^{(p)}(x,y)=R^{(p)}(y,x)$ for any $x,y\in A$.*

(b). For $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, if for some $x\neq y\in A$, $R^{(p)}(x,y)<\infty$, then $R^{(p)}(x,y)$ is realized by some function $f\in l(A)$, i.e. there exists a function f satisfying $f(x)=0$, $f(y)=1$ and $E^{(p)}(f)=R^{(p)}(x,y)^{-1}$.

(c). For $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and $x,y,z\in A$ such that $R^{(p)}(x,y)$, $R^{(p)}(x,z)$, $R^{(p)}(y,z)<\infty$, it holds that

$$\big(R^{(p)}(x,y)\big)^{1/p}\le\big(R^{(p)}(x,z)\big)^{1/p}+\big(R^{(p)}(z,y)\big)^{1/p}.\tag{2.1}$$

In particular, if $E^{(p)}\in\mathcal{M}_p(A)$, $\{R^{(p)}(\cdot,\cdot)\}^{1/p}$ is a metric on A.

Proof. (a). By the homogeneity of $E^{(p)}$, for any $f\in l(A)$, we have $E^{(p)}(f)=E^{(p)}(-f)$, and also $E^{(p)}(-f)=E^{(p)}(1-f)$ by the invariance under addition of constants, hence $E^{(p)}(f)=E^{(p)}(1-f)$, and this implies that $R^{(p)}(x,y)=R^{(p)}(y,x)$ for any $x,y\in A$.

(b). Let $f_n\in l(A)$ be a sequence of functions satisfying $f_n(x)=0$, $f_n(y)=1$ and $0\le f_n\le 1$ such that

$$\lim_{n\to\infty}E^{(p)}(f_n)=R^{(p)}(x,y)^{-1}.$$

Since f_n are uniformly bounded, we may choose a subsequence, which still denote by f_n, such that for any $z\in A$, $\lim_{n\to\infty}f_n(z)$ exists, denoted by $f(z)$. Then the function f is as required by the continuity of $E^{(p)}$ (see Lemma A.1 for a proof of a stronger result).

(c). Indeed, let $f\in l(A)$ such that $f(x)=0$, $f(y)=1$ and $E^{(p)}(f)=\frac{1}{R^{(p)}(x,y)}$. Then by the homogeneity of $E^{(p)}$, we have $|f(x)-f(z)|\le\big(R^{(p)}(x,z)E^{(p)}(f)\big)^{\frac1p}=\Big(\frac{R^{(p)}(x,z)}{R^{(p)}(x,y)}\Big)^{\frac1p}$ and $|f(z)-f(y)|\le\big(R^{(p)}(z,y)E^{(p)}(f)\big)^{\frac1p}=\Big(\frac{R^{(p)}(z,y)}{R^{(p)}(x,y)}\Big)^{\frac1p}$. Then (2.1) follows from the triangle inequality $|f(x)-f(y)|\le|f(x)-f(z)|+|f(z)-f(y)|$. □

In $p=2$ the quadratic case, a discrete Dirichlet form on A is uniquely determined by the associated resistance metric [17]. But for $p\neq 2$, $\widetilde{\mathcal{M}}_p(A)$ is a large class, a p-energy is not determined by the p-resistance. However, we still have useful estimates from the p-resistance, which we will see soon in Proposition 2.6.

Lemma 2.5. *(a). For any $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and $f\in l(A)$, we have $E^{(p)}(|f|)\leq 2^pE^{(p)}(f)$.*

(b). For any $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and $f,g\in l(A)$, we have

$$\begin{cases}E^{(p)}(f\wedge g)\leq 2^{2p-1}\big(E^{(p)}(f)+E^{(p)}(g)\big),\\ E^{(p)}(f\vee g)\leq 2^{2p-1}\big(E^{(p)}(f)+E^{(p)}(g)\big).\end{cases}$$

(c). There exists a constant $C>0$ depending on $\#A$ and p, such that for any $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and $B\subsetneq A$, we have

$$C^{-1}\sum_{x\in B,y\in A\setminus B}R^{(p)}(x,y)^{-1}\leq E^{(p)}(1_B)\leq C\sum_{x\in B,y\in A\setminus B}R^{(p)}(x,y)^{-1}.$$

Proof. (a). It follows from Proposition 2.2 (d) that $E^{(p)}(f_+)\leq E^{(p)}(f)$ and $E^{(p)}(f_-)\leq E^{(p)}(f)$. So by the homogeneity and convexity of $E^{(p)}$, we have $E^{(p)}(|f|)=E^{(p)}(f_++f_-)\leq 2^{p-1}\big(E^{(p)}(f_+)+E^{(p)}(f_-)\big)\leq 2^pE^{(p)}(f)$.

(b). We only deal with the "$f\wedge g$" case, since the "$f\vee g$" case is similar. By using homogeneity, convexity of $E^{(p)}$, and by (a), it follows

$$\begin{aligned}E^{(p)}(f\wedge g)&=E^{(p)}\left(\frac{f+g-|f-g|}{2}\right)\leq 2^{-1}\big(E^{(p)}(f+g)+E^{(p)}(|f-g|)\big)\\&\leq 2^{p-1}\big(E^{(p)}(f+g)+E^{(p)}(f-g)\big)\\&\leq 2^{2p-1}\big(E^{(p)}(f)+E^{(p)}(g)\big).\end{aligned}$$

(c). For the lower bound, note that for any $x\in B$ and $y\in A\setminus B$, we have

$$R^{(p)}(x,y)^{-1}\leq E^{(p)}(1_B).$$

By summing up the above inequality over all $x\in B$ and $y\in A\setminus B$ and observe that $\#A$, $\#B$ are finite, we get the desired estimate.

For the upper bound, for each $x\in B$ and $y\in A\setminus B$, let $f_{x,y}\in l(A)$ satisfy $f_{x,y}(x)=1$, $f_{x,y}(y)=0$, $0\leq f_{x,y}\leq 1$, and $E^{(p)}(f_{x,y})=R^{(p)}(x,y)^{-1}$. Observe that $1_B=\max\limits_{x\in B}\min\limits_{y\in A\setminus B}f_{x,y}$. By applying (b) a few times to $E^{(p)}(1_B)=E^{(p)}(\max\limits_{x\in B}\min\limits_{y\in A\setminus B}f_{x,y})$, we obtain the inequality as required. □

Proposition 2.6. *There exists a constant $C>0$ depending only on $\#A$ and p, such that for any $E_1^{(p)},E_2^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ and non-constant function $f\in l(A)$, we have*

$$C^{-1}\inf_{x\neq y\in A}\frac{R_2^{(p)}(x,y)}{R_1^{(p)}(x,y)}\leq\frac{E_1^{(p)}(f)}{E_2^{(p)}(f)}\leq C\sup_{x\neq y\in A}\frac{R_2^{(p)}(x,y)}{R_1^{(p)}(x,y)},$$

where $R_i^{(p)}$ is the p-resistance associated with $E_i^{(p)}$ for $i=1,2$, and we do not count $\frac{\infty}{\infty}$ in the supremum or infimum.

Proof. Assume $f(A)=\{l_j\}_{j=1}^m$, where we order $l_1<l_2<\cdots<l_m$ with $2\leq m\leq\#A$. Then, we have

$$f=l_1+\sum_{j=2}^m\big(l_j-l_{j-1}\big)\cdot 1_{\{f\geq l_j\}}.$$

On one hand, by applying the right inequality in Lemma 2.5 (c), for $i=1,2$, we have

$$E_i^{(p)}(f)\leq C_1\sum_{j=2}^m E_i^{(p)}\big((l_j-l_{j-1})\cdot 1_{\{f\geq l_j\}}\big)\leq C_2\sum_{j=2}^m (l_j-l_{j-1})^p\sum_{x\in\{f\geq l_j\},y\in\{f<l_j\}}R_i^{(p)}(x,y)^{-1},$$

where $C_1,C_2>0$ are constants only depending on $\#A$ and p.

On the other hand, by the Markov property and the left inequality in Lemma 2.5 (c), for $i=1,2$, we obtain

$$\begin{aligned}E_i^{(p)}(f)&\geq \max_{2\leq j\leq m}E_i^{(p)}\big((l_j-l_{j-1})\cdot 1_{\{f\geq l_j\}}\big)\\&\geq \frac{1}{\#A-1}\sum_{j=2}^m E_i^{(p)}\big((l_j-l_{j-1})\cdot 1_{\{f\geq l_j\}}\big)\\&\geq C_3\sum_{j=2}^m (l_j-l_{j-1})^p\sum_{x\in\{f\geq l_j\},y\in\{f<l_j\}}R_i^{(p)}(x,y)^{-1},\end{aligned}$$

where $C_3>0$ is a constant only depending on $\#A$ and p.

Now the proposition follows immediately from the above two estimates. □

The following is a direct corollary of Proposition 2.6.

Corollary 2.7. *There is a constant $C>0$ depending on $\#A$ and p, such that for any $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, we can find $\bar{E}^{(p)}\in\widetilde{\mathcal{M}}_p(A)$ of the form*

$$\bar{E}^{(p)}(f)=\sum_{x,y\in A,\ x\neq y}c_{x,y}\big|f(x)-f(y)\big|^p, \tag{2.2}$$

with $c_{x,y}\geq 0$ such that for any $f\in l(A)$,

$$C^{-1}\bar{E}^{(p)}(f)\leq E^{(p)}(f)\leq C\bar{E}^{(p)}(f). \tag{2.3}$$

Proof. We simply choose $c_{x,y}=\frac{1}{R^{(p)}(x,y)}$ for any $x\neq y\in A$, and denote $\bar{R}^{(p)}$ the p-resistance associated with $\bar{E}^{(p)}$. Clearly, $\bar{E}^{(p)}\in\widetilde{\mathcal{M}}_p(A)$.

We will show that $C_1R^{(p)}(x,y)\leq\bar{R}^{(p)}(x,y)\leq R^{(p)}(x,y)$ for some constant $C_1>0$ depending only on $\#A$ and p, then the corollary follows from Proposition 2.6.

The upper bound $\bar{R}^{(p)}(x,y)\leq R^{(p)}(x,y)$ is immediate from the definition. It remains to show

$$C_1R^{(p)}(x,y)\leq\bar{R}^{(p)}(x,y). \tag{2.4}$$

For this purpose, we fix x,y and let $\rho=\big(R^{(p)}(x,y)\big)^{\frac{1}{p}}$. Let $A_x\subsetneq A$ be the set of all points z having the property:

there exists a sequence of distinct points $z=z_0,z_1,\cdots,z_m=x$ such that

$$R^{(p)}(z_i,z_{i+1})<\Big(\frac{\rho}{\#A}\Big)^p.$$

Clearly, $x\in A_x$, and $y\notin A_x$ by Proposition 2.4 (c). We then define a function f taking values 0 on A_x and 1 on A_x^c. Then by $f(x)=0,f(y)=1$, we have

$$\bar{R}^{(p)}(x,y)^{-1}\leq\bar{E}^{(p)}(f). \tag{2.5}$$

Also, for any $z\in A_x$ and $w\in A_x^c$, we must have $R^{(p)}(z,w)\geq \left(\frac{\rho}{\#A}\right)^p$, and hence

$$\bar{E}^{(p)}(f)=\sum_{z\in A_x,w\in A_x^c}\frac{1}{R^{(p)}(z,w)}\leq \#A_x\cdot\#A_x^c\cdot\left(\frac{\#A}{\rho}\right)^p\leq (\#A)^{p+2}R^{(p)}(x,y)^{-1}. \tag{2.6}$$

Thus (2.4) holds by combining (2.5), (2.6) and taking $C_1=(\#A)^{-p-2}$. □

In most part of the paper, we will restrict ourselves to more concrete classes $\mathcal{S}_p(A),\mathcal{Q}_p(A)$ of p-energies. All the stories in Sections 3-6 can be stated with only these two classes.

Definition 2.8. *We introduce a norm* $\|\cdot\|_{\widetilde{\mathcal{M}}_p(A)}$ *on* $\widetilde{\mathcal{M}}_p(A)$ *as follows:*

$$\|E^{(p)}\|_{\widetilde{\mathcal{M}}_p(A)}=\sup_{f\in l(A)\setminus Constants}\frac{E^{(p)}(f)}{\max_{x,y\in A}|f(x)-f(y)|^p},\quad \forall E^{(p)}\in\widetilde{\mathcal{M}}_p(A).$$

(a). Define

$$\mathcal{S}_p(A)=\big\{E^{(p)}\in\mathcal{M}_p(A): \textit{There exist constants } c_{x,y}\geq 0 \textit{ depending only on } x,y \textit{ so that } E^{(p)}(f)=\sum_{x,y\in A}c_{x,y}|f(x)-f(y)|^p\big\}.$$

(b). Define

$$\mathcal{Q}'_p(A)=\big\{E^{(p)}\in\mathcal{M}_p(A): \textit{There exists } B\supset A \textit{ and } E_B^{(p)}\in\mathcal{S}_p(B) \textit{ so that } E^{(p)}=[E_B^{(p)}]_A\big\}.$$

Take $\mathcal{Q}_p(A)$ *to be the closure of* $\mathcal{Q}'_p(A)$ *in* $\big(\mathcal{M}_p(A),\|\cdot\|_{\widetilde{\mathcal{M}}_p(A)}\big)$.

Remark 1. The class $\widetilde{\mathcal{M}}_p(A)$ is a cone. The spanned space of $\widetilde{\mathcal{M}}_p(A)$ under $\|\cdot\|_{\widetilde{\mathcal{M}}_p(A)}$ is a complete normed space. However, $\mathcal{S}_p(A)\subset\mathcal{Q}_p(A)\subset\mathcal{M}_p(A)$ are all not closed in $\widetilde{\mathcal{M}}_p(A)$ under this norm.

Remark 2. Any $E^{(p)}\in\mathcal{S}_p(A)$ is uniformly convex ([13]), which means for any $\delta>0$ and any $f,g\in l(A)$ such that $E^{(p)}(f)=E^{(p)}(g)=1$ and $E^{(p)}(f-g)\geq\delta$, we have

$$E^{(p)}\left(\frac{f+g}{2}\right)\leq 1-\varepsilon$$

for some $\varepsilon>0$ depending on p and δ. So it is direct to check that any $E^{(p)}\in\mathcal{Q}_p(A)$ is also uniformly convex, hence strictly convex.

Remark 3. It follows from Corollary 2.7 that, for any $E^{(p)}\in\mathcal{M}_p(A)$, we can find $\bar{E}^{(p)}\in\mathcal{S}_p(A)$ such that the two energies are comparable.

We will list some important properties of $\mathcal{Q}_p(A)$ in Appendix A, which will be used in Section 5.2.

3. P.C.F. SELF-SIMILAR SETS AND THE RENORMALIZATION MAPS

In this section, we introduce the renormalization map in p-energy setting on the post-critically finite (p.c.f. for short) self-similar sets introduced by Kigami [16, 17]. Please refer to [11, 12, 22, 23, 24, 26, 27] and the references therein for fruitful previous works for the

$p = 2$ case, especially, the celebrated work of Sabot [29] on the existence and uniqueness of an eigenvector of this map for nested fractals.

For simplicity, we will focus on the self-similar sets in $\mathbb{R}^d$. Let $\{F_i\}_{i=1}^N$ be a sequence of α_i-similitudes on $\mathbb{R}^d$, i.e. for $i = 1, \cdots, N$, $N \geq 2$, each $F_i : \mathbb{R}^d \to \mathbb{R}^d$ is of the form

$$F_i(x) = \alpha_i U_i x + a_i,$$

where $\alpha_i \in (0, 1)$, U_i is an orthogonal matrix, $a_i \in \mathbb{R}^d$. We call $\{F_i\}_{i=1}^N$ an *iterated function system* (i.f.s. for short). Let K be the associated *self-similar set*, i.e. K is the unique non-empty compact set in $\mathbb{R}^d$ satisfying

$$K = \bigcup_{i=1}^{N} F_i K.$$

We then define the associated symbolic space. Let $W = \{1, \cdots, N\}$ be the alphabet. Let $W_0 = \{\emptyset\}$, W_n be the set of *words* $w = w_1 \cdots w_n$ of length n with $w_i \in W$, $n \geq 1$ (we also denote $|w| = n$ for the *length* of w), and write $W_* = \bigcup_{n\geq 0} W_n$. For $w \in W_n$, we write $F_w = F_{w_1} \circ \cdots \circ F_{w_n}$, and call $F_w K$ an *n-cell* of K. Let Σ be the set of *infinite words* $\omega = \omega_1\omega_2 \cdots$ with $\omega_i \in W$, equipped with the usual product topology. There is a continuous surjection $\pi : \Sigma \to K$ defined by

$$\{\pi(\omega)\} = \bigcap_{n\geq 1} F_{[\omega]_n} K,$$

where for $\omega = \omega_1\omega_2 \cdots$ in Σ we write $[\omega]_n = \omega_1 \cdots \omega_n \in W_n$ for each $n \geq 1$.

Following [17], we define the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$ for K by

$$\mathcal{C} = \pi^{-1}\Big(\bigcup\nolimits_{i\neq j} \big(F_i K \cap F_j K\big)\Big), \quad \mathcal{P} = \bigcup\nolimits_{m\geq 1} \tau^m(\mathcal{C}),$$

where $\tau : \Sigma \to \Sigma$ is the left shift map defined as $\tau(\omega_1\omega_2 \cdots) = \omega_2\omega_3 \cdots$. If $\mathcal{P}$ is a finite set, we call $\{F_i\}_{i=1}^N$ a *post-critically finite* (p.c.f.) i.f.s., and K a *p.c.f. self-similar set*. In what follows, we always assume that K is a connected p.c.f. self-similar set. The *boundary* of K is defined to be $V_0 = \pi(\mathcal{P})$. We also define $V_n = \bigcup_{i=1}^N F_i V_{n-1}$ inductively, and write $V_* = \bigcup_{n\geq 0} V_n$. It is clear that V_* is a proper subset of K and the closure of V_* under the Euclidean metric is K.

When the fractal is symmetric, we will be particularly interested in symmetric p-energy forms. More precisely, we will consider a symmetric group $\mathscr{G}$ as follows:

($\mathscr{G}$-symmetry). Let $\mathscr{G}$ be a finite group of homeomorphisms $K \to K$. We say $(K, \{F_i\}_{i=1}^N)$ is *$\mathscr{G}$-symmetric* (or simply K is $\mathscr{G}$-symmetric) if for any $\sigma \in \mathscr{G}$ and $n \geq 0$, there is a permutation $\sigma^{(n)} : W_n \to W_n$ such that $\sigma(F_w K) = F_{\sigma^{(n)}(w)} K$ and $\sigma(F_w V_0) = F_{\sigma^{(n)}(w)} V_0$ for any $w \in W_n$.

To construct a p-energy form on K, it suffices to study the forms on $V_n, n \geq 0$. In the rest of this section, we introduce a renormalization map on the forms on $V_n, n \geq 0$, and study some basic properties. We will talk more in Section 7.

Remark. From now on, we study the limit of discrete p-energies on the expanding sequence $\{V_n\}_{n\geq 0}$. Note that all the constants that involved will depend only on $\#V_0$ and p. In the following of this paper, we will fix $p \in (1, \infty)$. For simplicity, we will write $\mathcal{M} = \mathcal{M}_p(V_0)$, $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_p(V_0)$, and omit the index p and superscript (p) when no confusion is caused.

We introduce a renormalization map $\mathcal{T}:\mathcal{M}(V_n)\to\mathcal{M}(V_n)$.

Definition 3.1. *Fix $\boldsymbol{r}=(r_1,r_2,\cdots,r_N)$ to be a positive vector. For $w=w_1\cdots w_n\in W_*$, we write $r_w=r_{w_1}\cdots r_{w_n}$. Let $n\geq 0$ and $E\in\widetilde{\mathcal{M}}(V_n)$.*

(a). We define $\Lambda E\in\widetilde{\mathcal{M}}(V_{n+1})$ as

$$\Lambda E(f)=\sum_{i=1}^N r_i^{-1}E(f\circ F_i),\qquad \textit{for } f\in l(V_{n+1}).$$

(b). We define $\mathcal{T}:\widetilde{\mathcal{M}}(V_n)\to\widetilde{\mathcal{M}}(V_n)$ as

$$\mathcal{T}E=[\Lambda E]_{V_n}.$$

Remark 1. There is some abuses of notations, as the definition of $\Lambda,\mathcal{T}$ actually depends on n. However, we admit the notations since they provide great convenience. In particular, we can use the notation $\Lambda^m:\widetilde{\mathcal{M}}(V_n)\to\widetilde{\mathcal{M}}(V_{n+m})$ for all $n\geq 0$. See the following Lemma 3.2 for example.

Remark 2. It follows from Proposition 2.2 that the operators Λ and $\mathcal{T}$ are well-defined.

More about ($\mathscr{G}$-symmetry). Assume K is $\mathscr{G}$-symmetric. We say $E\in\widetilde{\mathcal{M}}(V_n),n\geq 0$ is *$\mathscr{G}$-symmetric* if $E(f\circ\sigma)=E(f)$ for any $f\in l(V_n)$ and $\sigma\in\mathscr{G}$.

In addition, we assume $\boldsymbol{r}$ is $\mathscr{G}$-symmetric, i.e. $r_i=r_{\sigma^{(1)}(i)},\forall 1\leq i\leq N$. Then it is straightforward to check that if $E\in\widetilde{\mathcal{M}}(V_n),n\geq 0$ is $\mathscr{G}$-symmetric, then both ΛE and $[E]_{V_n}$ are $\mathscr{G}$-symmetric.

Lemma 3.2. *(a). Let $n\geq 0$ and $E\in\widetilde{\mathcal{M}}(V_{n+2})$, then $[[E]_{V_{n+1}}]_{V_n}=[E]_{V_n}$.*

(b). Let $n\geq 0$ and $E\in\widetilde{\mathcal{M}}(V_n)$, then $[\Lambda^m E]_{V_n}=\mathcal{T}^m E$ for any $m\geq 1$.

Proof. (a). It suffices to show $[[E]_{V_{n+1}}]_{V_n}(f)=[E]_{V_n}(f)$ for each $f\in l(V_n)$.

First, one can find $f_1\in l(V_{n+1})$ so that $f_1|_{V_n}=f$ and $[E]_{V_{n+1}}(f_1)=[[E]_{V_{n+1}}]_{V_n}(f)$; one can then find f_2 so that $f_2|_{V_{n+1}}=f_1$ and $E(f_2)=[E]_{V_{n+1}}(f_1)=[[E]_{V_{n+1}}]_{V_n}(f)$. So we have $[[E]_{V_{n+1}}]_{V_n}(f)\geq[E]_{V_n}(f)$ by definition.

Next, for any $g\in l(V_{n+2})$ satisfying $g|_{V_n}=f$, we have $E(g)\geq[E]_{V_{n+1}}(g|_{V_{n+1}})\geq[E]_{V_{n+1}}(f_1)=E(f_2)$, hence $[[E]_{V_{n+1}}]_{V_n}(f)\leq[E]_{V_n}(f)$.

(b). It is easy to see that $[\Lambda^m E]_{V_{n+m-1}}=\Lambda^{m-1}\mathcal{T}E$. In fact, let $f\in l(V_{n+m-1})$, one can define $f_1\in l(V_{n+m})$ such that $\mathcal{T}E(f\circ F_w)=\Lambda E(f_1\circ F_w)$ for each $w\in W_{m-1}$, then for any $g\in l(V_{n+m})$ such that $g|_{V_{n+m-1}}=f$ we have

$$\begin{aligned}\Lambda^m E(f_1)&=\sum_{w\in W_{m-1}}r_w^{-1}\Lambda E(f_1\circ F_w)\\&=\sum_{w\in W_{m-1}}r_w^{-1}\mathcal{T}E(f\circ F_w)\leq\sum_{w\in W_{m-1}}r_w^{-1}\Lambda E(g\circ F_w)=\Lambda^m E(g),\end{aligned}$$

hence $[\Lambda^m E]_{V_{n+m-1}}(f)=\Lambda^m E(f_1)=\sum_{w\in W_{m-1}}r_w^{-1}\mathcal{T}E(f\circ F_w)=\Lambda^{m-1}\mathcal{T}E(f)$.

By using (a), one can now repeat the argument to see (b). □

Finally, we end this section with some basic properties of $\mathcal{T}$, which are essentially due to Metz [23] for the $p=2$ case.

Definition 3.3. *(a). Let $E_1,E_2\in\widetilde{\mathcal{M}}$. We define*

$$\sup(E_1|E_2)=\sup\left\{\frac{E_1(f)}{E_2(f)}:f\in l(V_0),E_1(f)+E_2(f)>0\right\},$$
$$\inf(E_1|E_2)=\inf\left\{\frac{E_1(f)}{E_2(f)}:f\in l(V_0),E_1(f)+E_2(f)>0\right\}.$$

(b). Let $E\in\mathcal{M}$, we define

$$\theta(E)=\frac{\sup(\mathcal{T}E|E)}{\inf(\mathcal{T}E|E)}.$$

Following the routine argument on the $p=2$ setting, we can easily prove the following result.

Lemma 3.4. *(a). Let $E_1,E_2\in\mathcal{M}$, we have*

$$\begin{cases}\sup(\mathcal{T}E_1|\mathcal{T}E_2)\leq\sup(\Lambda E_1|\Lambda E_2)\leq\sup(E_1|E_2),\\ \inf(\mathcal{T}E_1|\mathcal{T}E_2)\geq\inf(\Lambda E_1|\Lambda E_2)\geq\inf(E_1|E_2).\end{cases}$$

(b). Let $E\in\mathcal{M}$, we have

$$\theta(\mathcal{T}E)\leq\theta(E).$$

(c). Let $E_1\in\widetilde{\mathcal{M}},E_2\in\mathcal{M}$, for $n\geq1$, we have

$$\inf(\mathcal{T}^nE_1|E_1)\leq\sup(\mathcal{T}^nE_2|E_2).$$

Proof. (a). For $f\in l(V_0)$, we denote by $f_i\in l(V_1)$ the minimal energy extension of f with respect to ΛE_i for $i=1,2$.

First, for any $g\in l(V_1)$, we have

$$\frac{\Lambda E_1(g)}{\Lambda E_2(g)}=\frac{\sum_{i=1}^N r_i^{-1}E_1(g\circ F_i)}{\sum_{i=1}^N r_i^{-1}E_2(g\circ F_i)}\leq\sup(E_1|E_2),$$

whence $\sup(\Lambda E_1|\Lambda E_2)\leq\sup(E_1|E_2)$.

Next, for any $f\in l(V_0)$, we have

$$\mathcal{T}E_1(f)=\Lambda E_1(f_1)\leq\Lambda E_1(f_2)\leq\sup(\Lambda E_1|\Lambda E_2)\Lambda E_2(f_2)=\sup(\Lambda E_1|\Lambda E_2)\mathcal{T}E_2(f),$$

so $\sup(\mathcal{T}E_1|\mathcal{T}E_2)\leq\sup(\Lambda E_1|\Lambda E_2)$.

Combining the above two arguments, we get the inequality for the "sup" part. Noticing that $\inf(E_1|E_2)=\sup(E_2|E_1)^{-1}$, we also get the inequality for the "inf" part.

(b) is an immediate consequence of (a):

$$\theta(\mathcal{T}E)=\frac{\sup(\mathcal{T}^2E|\mathcal{T}E)}{\inf(\mathcal{T}^2E|\mathcal{T}E)}\leq\frac{\sup(\mathcal{T}E|E)}{\inf(\mathcal{T}E|E)}=\theta(E).$$

(c). Choose $h\in l(V_0)$ so that $\frac{E_1(h)}{E_2(h)}=\sup(E_1|E_2)$. For $i=1,2$, we denote by $h_i\in l(V_n)$ a minimal energy extension of h with respect to Λ^nE_i. Then

$$\mathcal{T}^nE_1(h)=\Lambda^nE_1(h_1)\leq\Lambda^nE_1(h_2)\leq\sup(E_1|E_2)\Lambda^nE_2(h_2)=\frac{E_1(h)}{E_2(h)}\mathcal{T}^nE_2(h),$$

whence

$$\inf(\mathcal{T}^nE_1|E_1)\leq\frac{\mathcal{T}^nE_1(h)}{E_1(h)}\leq\frac{\mathcal{T}^nE_2(h)}{E_2(h)}\leq\sup(\mathcal{T}^nE_2|E_2).$$

□

4. The existence of an eigenform

In this section, we study the existence of an eigenvector (eigenform) of $\mathcal{T}$, i.e. to find $E\in\mathcal{M}$ and $\lambda>0$ such that
$$\mathcal{T}E=\lambda E.$$

Definition 4.1. *For $E\in\mathcal{M}$, we define*
$$\delta(E)=\frac{\min_{x\neq y}R(x,y)}{\max_{x\neq y}R(x,y)}.$$

In the rest of this paper, we refer to the following condition as (**A**):

(**A**). There exists $E\in\mathcal{M}$ such that $\inf_{n\geq 0}\delta(\mathcal{T}^nE)>0$.

This condition always holds for affine nested fractals (Section 6). For general p.c.f. fractals, we have a strengthened version of Sabot's criteria (Section 8).

Similar as the notations $\mathcal{M}$, $\widetilde{\mathcal{M}}$, from now on, we abbreviate $\mathcal{S}_p(V_0)$, $\mathcal{Q}'_p(V_0)$ and $\mathcal{Q}_p(V_0)$ to $\mathcal{S}$, $\mathcal{Q}'$ and $\mathcal{Q}$, respectively.

Theorem 4.2. *Let $1<p<\infty$ and K be a p.c.f. self-similar set. Then there exists $E\in\mathcal{Q}$ and $\lambda>0$ such that $\mathcal{T}E=\lambda E$ if and only if (**A**) holds. In particular, λ is unique.*

In addition, if K and $\boldsymbol{r}$ are $\mathscr{G}$-symmetric, then we can also require that E is $\mathscr{G}$-symmetric.

We will prove the existence of an eigenform by the spirit of Kusuoka-Zhou's idea [21], using an averaging technique. The original proof was designed to construct a self-similar Dirichlet form directly, and has never been modified for the renormalization map before.

We remark that in the $p=2$ case, the ω-limit technique was applied to show the existence of an eigenform by using the Brouwer's fixed point theorem (see [26, Theorem 4.1], [23, Proposition 4.4] and [25, Lemma 14]). This strategy can be directly applied here to show the existence of an eigenform $E\in\mathcal{M}$. However, it is hard to know whether the eigenform is in $\mathcal{Q}$ since it is not clear whether $\mathcal{Q}$ is a convex cone.

We first state an easy fact about the norm $\|\cdot\|_{\widetilde{\mathcal{M}}}$.

Lemma 4.3. *Let $E_n\in\widetilde{\mathcal{M}},n\geq 1$.*

(a). If $\sup_{n\geq 1}\|E_n\|_{\widetilde{\mathcal{M}}}<\infty$, then there is a subsequence $n_l,l\geq 1$ and $E\in\widetilde{\mathcal{M}}$ so that $\|E_{n_l}-E\|_{\widetilde{\mathcal{M}}}\to 0$ as $l\to\infty$.

(b). If $E\in\mathcal{M}$ and $\|E_n-E\|_{\widetilde{\mathcal{M}}}\to 0$ as $n\to\infty$, then $\|\mathcal{T}E_n-\mathcal{T}E\|_{\widetilde{\mathcal{M}}}\to 0$.

Proof. Let $M=\big\{f\in l(V_0):\mathrm{Osc}(f)=1,\sum_{x\in V_0}f(x)=0\big\}$, then M is a compact subset of $l(V_0)$ (with respect to the $\|\cdot\|_\infty$ norm).

(a). By the assumption, $\{E_n\}_n$ are uniformly bounded on M; and also for $f\neq g\in M$, we have
$$E_n(f)\leq tE_n\big(t^{-1}(f-g)\big)+(1-t)E_n\big((1-t)^{-1}g\big)=t^{1-p}E_n(f-g)+(1-t)^{1-p}E_n(g),$$
and similarly $E_n(g)\leq t^{1-p}E_n(f-g)+(1-t)^{1-p}E_n(f)$. Hence,
$$\big|E_n(f)-E_n(g)\big|\leq t^{1-p}E_n(f-g)+\big((1-t)^{1-p}-1\big)\cdot\max\big\{E_n(f),E_n(g)\big\}.$$

Let $t = \big(E_n(f-g)\big)^{1/p}$, noticing that $E_n(f-g) \leq \|f-g\|_\infty^p \|E_n\|_{\widetilde{\mathcal{M}}}$ and $E_n(f) \leq \|E_n\|_{\widetilde{\mathcal{M}}}, E_n(g) \leq \|E_n\|_{\widetilde{\mathcal{M}}}$, we can see

$$\big|E_n(f) - E_n(g)\big| \leq \|E_n\|_{\widetilde{\mathcal{M}}}^{1/p} \|f-g\|_\infty + \big((1 - \|E_n\|_{\widetilde{\mathcal{M}}}^{1/p}\|f-g\|_\infty)^{1-p} - 1\big) \cdot \|E_n\|_{\widetilde{\mathcal{M}}}.$$

Thus $\{E_n\}_n$ are equicontinuous on M.

Hence by Arzelà-Ascoli theorem, there is subsequence $\{n_l\}_{l\geq 1}$ so that E_{n_l} converges uniformly on M, and hence E_{n_l} converges to some $E \in \widetilde{\mathcal{M}}$ pointwisely on $l(V_0)$, noticing that any function in $l(V_0)$ can be decomposed into the sum of a constant function and a multiple of some function in M.

(b). Since M is compact and $E \in \mathcal{M}$, we have $S := \min_{f\in M} E(f) > 0$. Then

$$-\frac{\|E_n - E\|_{\widetilde{\mathcal{M}}}}{S} \leq \frac{E_n(f) - E(f)}{E(f)} \leq \frac{\|E_n - E\|_{\widetilde{\mathcal{M}}}}{S}, \quad \forall f \in M,$$

which implies

$$\frac{S - \|E_n - E\|_{\widetilde{\mathcal{M}}}}{S} \leq \frac{E_n(f)}{E(f)} \leq \frac{S + \|E_n - E\|_{\widetilde{\mathcal{M}}}}{S}, \quad \forall f \in M.$$

Hence, by Lemma 3.4 (a),

$$\frac{S - \|E_n - E\|_{\widetilde{\mathcal{M}}}}{S} \leq \frac{\mathcal{T}E_n(f)}{\mathcal{T}E(f)} \leq \frac{S + \|E_n - E\|_{\widetilde{\mathcal{M}}}}{S}, \quad \forall f \in M,$$

which is equivalent to

$$\frac{\big|\mathcal{T}E_n(f) - \mathcal{T}E(f)\big|}{\mathcal{T}E(f)} \leq \frac{\|E_n - E\|_{\widetilde{\mathcal{M}}}}{S}, \quad \forall f \in M.$$

Thus, $\|\mathcal{T}E_n - \mathcal{T}E\|_{\widetilde{\mathcal{M}}} \leq \frac{\|\mathcal{T}E\|_{\widetilde{\mathcal{M}}}}{S}\|E_n - E\|_{\widetilde{\mathcal{M}}}$. □

The following lemma is an alternative to [21, Theorem 7.16] by Kusuoka and Zhou.

Lemma 4.4. *Assume (**A**), and let $E \in \mathcal{M}$. Then*

(a). $\inf_{n\geq 0} \delta(\mathcal{T}^n E) > 0$.

(b). There exist $\lambda > 0$ and $C > 0$ such that

$$C^{-1}\lambda^n E \leq \mathcal{T}^n E \leq C\lambda^n E, \quad \forall n \geq 0.$$

Proof. (a) is an immediate consequence of Proposition 2.6 and Lemma 3.4 (a).

(b). Denote by R_n and R_0 the p-resistances w.r.t. $\mathcal{T}^n E$ and E respectively. Let $M_n = \sup_{x\neq y\in V_0} \frac{R(x,y)}{R_n(x,y)}$, and from (a), we denote $\delta = \inf_{n\geq 0}\delta(\mathcal{T}^n E) > 0$. Then

$$M_n = \sup_{x\neq y\in V_0} \frac{R(x,y)}{R_n(x,y)} \geq \inf_{x\neq y\in V_0} \frac{R(x,y)}{R_n(x,y)} \geq \delta^2 \sup_{x\neq y\in V_0} \frac{R(x,y)}{R_n(x,y)} = \delta^2 M_n. \tag{4.1}$$

Then by applying Proposition 2.6 to $\mathcal{T}^n E$ and E, we see that there exists $C_1 > 1$ such that

$$C_1^{-1} M_n E \leq \mathcal{T}^n E \leq C_1 M_n E, \qquad \forall n \geq 0.$$

By acting $\mathcal{T}^m$ to the above inequality and using it again, we obtain

$$C_1^{-2} M_n M_m E \leq C_1^{-1} M_n \mathcal{T}^m E \leq \mathcal{T}^{n+m} E \leq C_1 M_n \mathcal{T}^m E \leq C_1^2 M_n M_m E, \quad \forall m, n \geq 0,$$

whence by Proposition 2.6 again, there exists $C_2 > 1$ such that

$$C_2^{-1} M_n M_m \le M_{n+m} \le C_2 M_n M_m.$$

It follows by Fekete's lemma that

$$\lim_{n\to\infty} M_n^{\frac{1}{n}} = \inf_{n\ge 1} (C_2 M_n)^{\frac{1}{n}} = \sup_{n\ge 1} (C_2^{-1} M_n)^{\frac{1}{n}} =: \lambda > 0,$$

and thus

$$C_2^{-1}\lambda^n \le M_n \le C_2 \lambda^n, \qquad \forall n \ge 1.$$

This together with (4.1) implies that

$$C_3^{-1}\lambda^n R_n(x,y) \le R(x,y) \le C_3 \lambda^n R_n(x,y), \qquad \forall x \ne y \in V_0 \tag{4.2}$$

for some $C_3 > 1$. (b) then follows by (4.2) and Proposition 2.6. □

Proof of Theorem 4.2. By Lemma 4.4 (a), we can choose $E_0 \in \mathcal{S}$. In addition, if both K and $\boldsymbol{r}$ are $\mathscr{G}$-symmetric, we can require that E_0 is also $\mathscr{G}$-symmetric, so all the constructions in the proof will be $\mathscr{G}$-symmetric.

Let λ be the same constant in Lemma 4.4 (b) for E_0. We follow the idea of Kusuoka and Zhou [21] to consider the averaged energy

$$\mathcal{E}_n(f) = \frac{1}{n+1} \sum_{m=0}^{n} \lambda^{-m} \Lambda^m E_0(f|_{V_m}),$$

where $f \in l(V_n)$. In [21], a normalized limit energy of $\mathcal{E}_n$ on $L^2(K,\mu)$ is considered, where μ is the normalized Hausdorff measure restricted on K. Thanks to the p.c.f. structure, here we can avoid considering the limit in the sense of energies on $L^p(K,\mu)$. We only need to consider the limit of

$$E_n = [\mathcal{E}_n]_{V_0} \in \mathcal{Q}'.$$

Claim. There is $C > 0$ so that $C^{-1}E_0 \le E_n \le CE_0$.

First, by Lemma 3.2 (b), one can see that

$$E_n \ge \frac{1}{n+1} \sum_{m=0}^{n} \lambda^{-m} [\Lambda^m E_0]_{V_0} = \frac{1}{n+1} \sum_{m=0}^{n} \lambda^{-m} \mathcal{T}^m E_0.$$

So by Lemma 4.4 (b), $E_n \ge C_1^{-1} E_0$, where $C_1 > 0$ is the constant in Lemma 4.4 (b).

Next, for $f \in l(V_0)$, let $f_n \in l(V_n)$ be an extension of f which is a p-energy minimizer with respect to $\Lambda^n E_0$. Then, by Lemma 3.2 (b) and Lemma 4.4 (b), we have

$$\begin{aligned} E_n(f) \le \mathcal{E}_n(f_n) &= \frac{1}{n+1} \sum_{m=0}^{n} \lambda^{-m} \Lambda^m E_0(f_n|_{V_m}) \\ &\le C_1 \frac{1}{n+1} \lambda^{-n} \sum_{m=0}^{n} \Lambda^m \mathcal{T}^{n-m} E_0(f_n|_{V_m}) = C_1 \lambda^{-n} \mathcal{T}^n E_0(f) \le C_1^2 E_0(f). \end{aligned}$$

This finishes the proof of the Claim.

It is then from the claim that $\sup_{n\geq 1}\|E_n\|_{\widetilde{\mathcal{M}}}<\infty$, so by Lemma 4.3 (a), we can find a subsequence $\{E_{n_l}\}_{l\geq 1}$ that converges to some $E'\in\mathcal{Q}$ under norm $\|\cdot\|_{\widetilde{\mathcal{M}}}$. Note that for each $n\geq 1$, we have

$$\lambda^{-1}\Lambda\mathcal{E}_n-\mathcal{E}_n=\frac{1}{n+1}(\lambda^{-n-1}\Lambda^{n+1}E_0-E_0).$$

So that

$$\lambda^{-1}\mathcal{T}E_n\geq E_n-\frac{1}{n+1}E_0.$$

By taking the limit in the subsequence $\{E_{n_l}\}_{l\geq 1}$ of each side, and using Lemma 4.3 (b), we see that $E'\leq\lambda^{-1}\mathcal{T}E'\leq\lambda^{-2}\mathcal{T}^2E'\leq\cdots$. We can use Lemma 4.4 (b) again to see the orbit $\lambda^{-n}\mathcal{T}^nE'$ is bounded from above, so we have $E=\lim\limits_{n\to\infty}\lambda^{-n}\mathcal{T}^nE'$ exists in $\mathcal{Q}$ (noticing that $\mathcal{Q}$ is closed in $\mathcal{M}$, although neither of which is complete), and clearly by Lemma 4.3 (b) again,

$$\mathcal{T}E=\lim_{n\to\infty}\lambda^{-n}\mathcal{T}^{n+1}E'=\lambda\lim_{n\to\infty}\lambda^{-n}\mathcal{T}^nE'=\lambda E.$$

This shows that $\mathcal{T}$ has a $\mathscr{G}$-symmetric eigenform.

Finally λ is unique since if there is another eigenform $\bar{E}$ such that $\mathcal{T}\bar{E}=\bar{\lambda}\bar{E}$ for some $\bar{\lambda}>0$, then by Lemma 3.4 (a), $\mathcal{T}^n\bar{E}\asymp\mathcal{T}^nE$ for any $n\geq 0$, which will force $\lambda=\bar{\lambda}$. □

5. p-energies on p.c.f. self-similar sets

In this section, we construct p-energies on p.c.f. self-similar sets, which covers the 2-energy forms (Dirichlet forms) as a special case. We apply the idea of [21] in this section.

Our story in this section is still based on Assumption (**A**). However, for convenience, we further require that $\lambda=1$ in Theorem 4.2 (by multiplying $\boldsymbol{r}$ with a constant). Let us introduce a condition ($\mathbf{A}'$) for convenience, which is essentially the same as (**A**) by Theorem 4.2.

($\mathbf{A}'$). (**A**) holds and $\boldsymbol{r}$ is properly chosen so that there is $E\in\mathcal{Q}$ such that $\mathcal{T}E=E$.

Another condition based on ($\mathbf{A}'$) is called regular condition, under which, we will see the whole story happens on $C(K)$.

(**R**). Assuming ($\mathbf{A}'$), we say $(E,\boldsymbol{r})$ is *regular* if $r_i<1$ for all $1\leq i\leq N$.

Remark. Assuming ($\mathbf{A}'$), by the later Lemma 5.4, we can always find some i such that $r_i<1$.

Theorem 5.1. *Assume ($\mathbf{A}'$). Let $E_0\in\mathcal{S}$ be defined as $E_0(f)=\frac{1}{2}\sum_{x\neq y}\big|f(x)-f(y)\big|^p,\forall f\in l(V_0)$. Define*

$$\mathcal{F}=\{f\in l(V_*):\sup_{n\geq 1}\Lambda^nE_0(f|_{V_n})<\infty\}.$$

Then there exists a subsequence $n_l,l\geq 1$ such that $\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(f|_{V_m})$ converges for every $f\in\mathcal{F}$. Define

$$\mathcal{E}(f)=\lim_{l\to\infty}\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(f|_{V_m}),\quad\forall f\in\mathcal{F}.$$

(a). Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by $f \sim g$ if and only if $f-g$ is a constant. Then $(\mathcal{F}/\sim, \mathcal{E}^{1/p})$ is a Banach space. In addition, $(\mathcal{E},\mathcal{F})$ is self-similar, i.e.

$$\mathcal{E}(f)=\sum_{i=1}^{N} r_i^{-1}\mathcal{E}(f\circ F_i), \quad \forall f\in\mathcal{F}.$$

*(b). If (**R**) holds, then there exists $C>0$ such that for any $f\in\mathcal{F}$, for any $w\in W_*$,*

$$Osc(f\circ F_w)\leq C r_w^{1/p}\mathcal{E}(f\circ F_w)^{1/p},$$

Consequently, $\mathcal{F}$ embeds into $C(K)$ naturally (by continuous extension). In addition, $\mathcal{F}$ is dense in $C(K)$.

However, if (**R**) fails, we need to consider the following condition (**B**) which involves a given probability Radon measure μ on K.

(**B**). $\sum_{m=0}^{\infty}\kappa_m^{1/p}<\infty$, where $\kappa_m=\kappa_m(\mu,\boldsymbol{r}):=\max_{w\in W_m} r_w\mu(F_wK)$.

Remark. (**R**) always implies (**B**). Even if (**R**) fails, there are a lot of μ satisfying (**B**), for example, a self-similar probability measure μ on K satisfying $\mu(F_iK)r_i<1$ for each $1\leq i\leq N$.

Theorem 5.2. *Assume ($\boldsymbol{A'}$) and (**B**). Let $\mathcal{F}_c=\{f\in\mathcal{F}: f \text{ is uniformly continuous}\}$, and consider the natural embedding (by continuous extension) $\Psi:\mathcal{F}_c\to C(K)\subset L^p(K,\mu)$. Then Ψ can be extended continuously to $\mathcal{F}\to L^p(K,\mu)$ in the sense that if $\{f_n\}\subset\mathcal{F}_c$, $f\in\mathcal{F}$, $f_n\to f$ pointwisely on V_* and $\mathcal{E}(f_n-f)\to 0$, then $\Psi(f_n)\to\Psi(f)$ under $L^p(K,\mu)$ norm. In addition, Ψ is self-similar:*

$$\Psi(f\circ F_w)=\Psi(f)\circ F_w, \quad \forall w\in W_*, f\in\mathcal{F}.$$

By identifying $\mathcal{F}$ with $\Psi(\mathcal{F})$, $(\mathcal{E},\mathcal{F})$ becomes a closed p-energy form on $L^p(K,\mu)$ in the sense that $\mathcal{E}$ has the Markov property and is uniformly convex, and $(\mathcal{F},\mathcal{E}_1^{1/p})$ is closed and separable, where $\mathcal{E}_1$ is defined by $\mathcal{E}_1(f)=\mathcal{E}(f)+\|f\|_{L^p(K,\mu)}^p$. In addition, $(\mathcal{E},\mathcal{F})$ is regular and strongly local in the following senses:

(regular). *Let $\mathscr{C}=\mathcal{F}\cap C(K)$. $\mathscr{C}$ is dense in $C(K)$, and $\mathscr{C}$ is dense in $\mathcal{F}$ with respect to the norm $\mathcal{E}_1^{1/p}$.*
(strongly local). *For $f,g\in\mathscr{C}$, if f is constant on a neighbourhood of $supp(g)$, then $\mathcal{E}(f+g)=\mathcal{E}(f)+\mathcal{E}(g)$.*

Remark. Theorem 5.2 follows immediate from Theorem 5.1 if (**R**) holds. Readers can skip Subsections 5.2, 5.3 if they are only interested in the regular case.

We will prove Theorem 5.1 in Subsection 5.1, and prove Theorem 5.2 in Subsection 5.3.

5.1. **The limit form.** The proof of Theorem 5.1 is similar to the $p=2$ case. We will take the advantage of the existence of a fixed point of $\mathcal{T}$ guaranteed by Theorem 4.2 under assumption (**A**). For convenience, we simply write $E(f)$ instead of $E(f|_{V_n})$ for $E\in\widetilde{\mathcal{M}}_p(V_n)$ and $f\in l(V_*)$.

Proposition 5.3. *Assume all the same conditions as in Theorem 5.1. Let $E\in\mathcal{Q}$ be a fixed point of $\mathcal{T}$, i.e. $\mathcal{T}E=E$. Then,* *(a). For any $f \in l(V_*)$, we have $E(f) \leq \Lambda E(f) \leq \Lambda^2 E(f) \leq \cdots$. So the following functional $\mathcal{E}_*$ with extended real values is well defined,*

$$\mathcal{E}_*(f) = \lim_{n\to\infty} \Lambda^n E(f), \quad \forall f \in l(V_*).$$

(b). $\mathcal{F} = \{f \in l(V_*) : \mathcal{E}_*(f) < \infty\}$.

(c). $(\mathcal{F}/\sim, \mathcal{E}_*^{1/p})$ is a separable Banach space. In addition, if $\mathcal{E}_*(f_n - f) \to 0$ as $n \to \infty$, then we can find constants $\{c_n\}$ so that $f_n - c_n$ converges to f pointwisely on V_*.

*(d). If in addition (**R**) holds, then there exists $C > 0$ such that for any $f \in \mathcal{F}$, for any $w \in W_*$,*

$$Osc(f \circ F_w) \leq C r_w^{1/p} \mathcal{E}_*(f \circ F_w)^{1/p}.$$

Consequently, $\mathcal{F}$ embeds densely into $C(K)$.

Before proving Proposition 5.3, we introduce the concept of piecewise p-harmonic functions. For $n \geq 0$, $f \in l(V_n)$, we define $H_* f \in l(V_*)$ by

$$H_* f|_{V_n} = f, \quad \mathcal{E}_*(H_* f) = \min\{\mathcal{E}_*(g) : g \in l(V_*), g|_{V_n} = f\},$$

and call $H_* f$ a *p-harmonic extension* of f to V_* with respect to $\mathcal{E}_*$. Note that $H_* f$ is unique by Lemma A.2. Strictly speaking, the operator H_* depends on n, but we insist not using a superscript n for brevity and it will cause no confusion. Define

$$\mathscr{H} = \big\{h = H_* f : f \in l(V_n), n \geq 0\big\},$$

call it the collection of *piecewise p-harmonic functions* with respect to $\mathcal{E}_*$.

Proof of Proposition 5.3. (a) is trivial, (b) follows from the fact that $E \asymp E_0$.

(c). First, $\mathcal{E}_*(f) > 0$ for any $f \in \mathcal{F} \setminus Constants$, and it is straightforward to check $\mathcal{E}_*^{1/p}$ is a norm on $\mathcal{F}/\sim$. In fact, it is not hard to see that for each $n \geq 0$, $(\Lambda^n E)^{1/p}$ defines a seminorm since $\Lambda^n E \in \mathcal{Q}(V_n)$. So the limit $\mathcal{E}_*^{1/p}$ is also a seminorm.

Next, let $f_n \in l(V_*), n \geq 1$ and $\mathcal{E}_*(f_n - f_m) \to 0$ as $n, m \to \infty$, we show that there is a sequence of constants $c_n \in \mathbb{R}$ and $f \in \mathcal{F}$ so that

$$\begin{cases} \mathcal{E}_*(f_n - f) \to 0, \\ f_n - c_n \to f \text{ pointwisely.} \end{cases}$$

In fact, we fix $x \in V_0$, and choose $c_n = f_n(x)$. For each $l \geq 0$, we have $\Lambda^l E(f_n - f_m) \to 0$ as $n, m \to \infty$, so $\mathrm{Osc}\big((f_n - f_m)|_{V_l}\big) \to 0$ as $n, m \to \infty$, hence $f_n - c_n$ converges pointwisely on V_l. Let $f \in l(V_*)$ be the pointwise limit of $f_n - c_n$. Then we show $f \in \mathcal{F}$ and $\mathcal{E}_*(f_n - f) \to 0$. Indeed, for each $l \geq 0$, $\Lambda^l E$ is continuous on $l(V_*)$ with respect to the topology of pointwise convergence, and thus the monotone limit $\mathcal{E}_*$ is lower-semicontinuous (with extended real values on $l(V_*)$). Then the claim follows since $\mathcal{E}_*(f_n - f) \leq \liminf_{m\to\infty} \mathcal{E}_*(f_n - f_m)$ by the lower-semicontinuity of $\mathcal{E}_*$.

Finally, since $\mathcal{E}_*$ is the limit of $\Lambda^n E \in \mathcal{Q}_p(V_n)$ (in some reasonable sense), by Remark 2 after Definition 2.8, one can easily see that $\mathcal{E}_*$ is uniformly convex. So for any $f \in \mathcal{F}$, noticing that $\mathcal{E}_*(H_* f|_{V_n}) \to \mathcal{E}_*(f)$ as $n \to \infty$ and $\mathcal{E}_*\big(\frac{1}{2}(H_* f|_{V_n}) + \frac{1}{2} f\big) \geq \mathcal{E}_*(H_* f|_{V_n})$, we can see that $\mathcal{E}_*(H_* f|_{V_n} - f) \to 0$ as $n \to \infty$, so $\mathscr{H}/\sim$ is dense in $(\mathcal{F}/\sim, \mathcal{E}_*^{1/p})$. We can choose a countable dense subset of $\mathscr{H}/\sim$, so $(\mathcal{F}/\sim, \mathcal{E}_*^{1/p})$ is separable.

(d) follows from a routine argument of resistance estimate. For $x,y\in V_n$, define $R_{n,*}(x,y)$ to be the p-resistance associated with $\Lambda^n E$. Then $R_{n,*}(x,y)=R_{n+1,*}(x,y)=\cdots$, so for any $x,y\in V_*$, we can define $R_*(x,y)=R_{n,*}(x,y)$ providing $x,y\in V_n$ for some n.

By the renormalization process, we know $R_*(F_wx,F_wy)\leq r_wR_*(x,y)$, $\forall w\in W_*,x,y\in V_*$. By Proposition 2.4 (c), $R_*^{1/p}$ is a metric on V_*, so by using a chaining argument and noticing that $r_i<1,\forall 1\leq i\leq N$, we have

$$R_*(x,y)\leq C^p,\quad \forall x,y\in V_*,$$

for some $C>0$ independent of x,y. Hence $\mathrm{Osc}(f)\leq C\mathcal{E}_*^{1/p}(f)$ for any $f\in\mathcal{F}$. In particular, $\mathrm{Osc}(f\circ F_w)\leq C\mathcal{E}_*^{1/p}(f\circ F_w)\leq Cr_w^{1/p}\mathcal{E}_*^{1/p}(f)$, for any $w\in W_*$. □

Proof of Theorem 5.1. By Proposition 5.3 (c), we choose a countable dense subset $\tilde{\mathcal{F}}$ of $\mathcal{F}$ such that $\tilde{\mathcal{F}}/\sim$ is dense in $(\mathcal{F}/\sim,\mathcal{E}_*^{1/p})$. Then, noticing that $\frac{1}{n+1}\sum_{m=0}^n\Lambda^mE_0(f)\leq C_1\mathcal{E}_*(f)$ for each $f\in\mathcal{F}$ for some constant $C_1>0$ independent of f and n, by a diagonal argument, we can pick a subsequence $n_l,l\geq 1$ such that

$$\mathcal{E}(g):=\lim_{l\to\infty}\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(g)$$

exists for each $g\in\tilde{\mathcal{F}}$. Next, for any $f\in\mathcal{F}$ and $\varepsilon>0$, by the choice of $\tilde{\mathcal{F}}$, we can find $g\in\tilde{\mathcal{F}}$ such that $\mathcal{E}_*(f-g)<\varepsilon$. Hence for some $C_2>0$,

$$\begin{aligned}&\Big|\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(f)-\frac{1}{n_{l'}+1}\sum_{m=0}^{n_{l'}}\Lambda^mE_0(f)\Big|\\ \leq& C_2\cdot\Big(\Big|\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(g)-\frac{1}{n_{l'}+1}\sum_{m=0}^{n_{l'}}\Lambda^mE_0(g)\Big|\\ &+\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(f-g)+\frac{1}{n_{l'}+1}\sum_{m=0}^{n_{l'}}\Lambda^mE_0(f-g)\Big)\\ \leq& C_2(1+2C_1)\varepsilon,\end{aligned}$$

for any l,l' large enough, where in the last inequality we use the fact that $\lim\limits_{l\to\infty}\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(g)$ exists, and the fact that $\frac{1}{n+1}\sum_{m=0}^n\Lambda^mE_0(f-g)\leq C_1\mathcal{E}_*(f-g)$ for any $n\geq 0$. It follows immediately that $\mathcal{E}(f):=\lim\limits_{l\to\infty}\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0(f)$ is well defined for any $f\in\mathcal{F}$. In addition,

$$\mathcal{E}(f)=\lim_{l\to\infty}\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^{m+1}E_0(f)=\lim_{l\to\infty}\Lambda\left(\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0\right)(f)=\sum_{i=1}^N r_i^{-1}\mathcal{E}(f\circ F_i).\tag{5.1}$$

Also, $\mathcal{E}^{1/p}$ is a seminorm since it is the limit of the seminorms $(\frac{1}{n_l+1}\sum_{m=0}^{n_l}\Lambda^mE_0)^{1/p}$.

Since $\mathcal{E}\asymp\mathcal{E}_*$, (a) follows immediately from Proposition 5.3 (c) and (5.1); (b) follows immediately from Proposition 5.3 (d). □

Remark. Now that once Theorem 5.1 is proved, we can simply let

$$E=[\mathcal{E}]_{V_0}$$

where $[\mathcal{E}]_{V_0}(f)=\min\{\mathcal{E}(g):g\in\mathcal{F},g|_{V_0}=f\},\forall f\in l(V_0)$. Then it is not hard to check $E\in\mathcal{Q}$ and $\mathcal{T}E=E$, since by (5.1), $\mathcal{T}E=[\Lambda[\mathcal{E}]_{V_0}]_{V_0}=[[\Lambda\mathcal{E}]_{V_1}]_{V_0}=[\Lambda\mathcal{E}]_{V_0}=[\mathcal{E}]_{V_0}=E$, where $\Lambda\mathcal{E}$ is defined as $\Lambda\mathcal{E}(f)=\sum_{i=1}^N r_i^{-1}\mathcal{E}(f\circ F_i)$ for any $f\in\mathcal{F}$. So the temporary form $\mathcal{E}_*$ can be just replaced with $\mathcal{E}$, and in addition, $\mathscr{H}$ is the space of piecewise p-harmonic functions associated with $(\mathcal{E},\mathcal{F})$. However, we will still keep the notation H_*.

Also, we remark that in the proof of Proposition 5.3 (c), by using the uniformly convex property, we have shown $\mathscr{H}/\sim$ is dense in $(\mathcal{F}/\sim,\mathcal{E})$ and moreover $\mathcal{E}\big(H_*(f|_{V_m})-f\big)\to 0$ as $m\to\infty$ for any $f\in\mathcal{F}$. Although we will not use this fact, it explains the construction that we define the embedding of f into $L^p(K,\mu)$ with the limit of $H_*(f|_{V_m})$ in $L^p(K,\mu)$.

Finally, similar to the $p=2$ case (see [17, Proposition 3.1.8]), we should get an estimate $r_w<1$, if $\dot{w}=www\cdots\in\mathcal{P}$. This will help us to see that (**R**) holds on affine nested fractals for certain "good" $\boldsymbol{r}$. The following argument is due to [14, Theorem 5.9].

Lemma 5.4. *Assume ($\boldsymbol{A}'$), if $\dot{w}=www\cdots\in\mathcal{P}$, then $r_w<1$.*

Proof. Let $x=\pi(\dot{w})\in V_0$, and choose m large enough so that $F_w^mK\cap V_0=\{x\}$. We let $h=H_*h'$ with $h'\in l(V_0)$ and $h'(y)=\delta_{x,y},\forall y\in V_0$. In other words, $h\in\mathcal{F}$ is a p-harmonic function with the boundary value $1_{\{x\}}$ on V_0.

Next, we define some functions with the same boundary values as h on V_0. First, we let $f=H_*f'$, where $f'\in l(V_{m|w|})$ is defined as $f'(y)=\delta_{x,y},\forall y\in V_{m|w|}$. So by self-similarity, we immediately see

$$\mathcal{E}(f)=r_w^{-m}\mathcal{E}(h). \tag{5.2}$$

Define $g'\in l(V_{m|w|})$ as

$$g'(y)=\begin{cases}1, \text{ if } y\in F_w^mV_0,\\ 0, \text{ if } y\in V_{m|w|}\setminus(F_w^mV_0).\end{cases}$$

Let $g=H_*g'$. Then, since $f|_{K\setminus F_w^mK}\equiv 0$ and $g|_{F_w^mK}\equiv 1$, by using the strongly local property and the homogeneity of $\mathcal{E}$, it is not hard to see

$$u(t):=\mathcal{E}((1-t)f+tg)=(1-t)^p\mathcal{E}(f)+t^p\mathcal{E}(g),\quad \forall\ 0<t<1.$$

Since $\mathcal{E}(f)>0$, we can see that $\frac{d}{dt}u(0)<0$, and hence

$$\mathcal{E}(h)\le\min_{t\in[0,1]}\mathcal{E}((1-t)f+tg)<\mathcal{E}(f)=r_w^{-m}\mathcal{E}(h),$$

where the first inequality is because $h|_{V_0}=\big((1-t)f+tg\big)|_{V_0}$ and h is p-harmonic. Clearly, $\mathcal{E}(h)=E(h')>0$, so $r_w<1$. □

5.2. **Embedding $\mathscr{H}$ into $C(K)$.** In this subsection, we show that H_*f satisfies some estimate of local oscillation, hence one can embed $\mathscr{H}$ into $C(K)$ by continuous extension. In particular, $\mathscr{H}$ is a dense subspace of $C(K)$, and we can see $\mathscr{H}/\sim$ is dense in $\mathcal{F}/\sim$ with respect to $\mathcal{E}^{1/p}$.

The proof is based on a similar idea of Lemma 5.4, but much more complicated. The essential difficulty is solved by Lemma A.3 in Appendix A.

Proposition 5.5. *Assume ($\boldsymbol{A}'$). There exists $m\ge 1$ and $\eta<1$ so that $Osc(H_*f\circ F_w)<\eta Osc(f)$ for any non-constant function $f\in l(V_0)$ and $w\in W_m$.*

Proof. We choose some fixed $m \geq 1$ so that $\#(F_w V_0) \cap V_0 \leq 1$ for each $w \in W_m$.

First, let us fix a non-constant function $f \in l(V_0)$, and complete the proof by taking advantage of compactness. For convenience, we write $h_f = (H_* f)|_{V_m}$, we then claim that

$$\max_{w \in W_m} \mathrm{Osc}(h_f \circ F_w) < \mathrm{Osc}(f). \tag{5.3}$$

We will show this by contradiction. Indeed, if it is not true, we must have $\max_{w \in W_m} \mathrm{Osc}(h_f \circ F_w) = \mathrm{Osc}(f)$. So we can find $\tau \in W_m$ such that $\mathrm{Osc}(h_f \circ F_\tau) = \mathrm{Osc}(f)$, and hence there exist $A, B \subset V_0$ such that

$$h_f \circ F_\tau|_A = \min_{x \in V_0} f(x), \quad h_f \circ F_\tau|_B = \max_{x \in V_0} f(x).$$

By assumption, we have $\#((F_\tau V_0) \cap V_0) \leq 1$, hence one of $F_\tau(A)$ and $F_\tau(B)$ must not intersect V_0. Without loss of generality, we may assume that $F_\tau(B) \cap V_0 = \emptyset$ (otherwise consider $-f$). Then, we define $h_{f,t} = h_f + t \cdot 1_{F_\tau B}$ for $t \in \mathbb{R}$, where $1_{F_\tau B} \in l(V_m)$ is the indicator function of $F_\tau B$. We denote by $\frac{d}{dt-}$ the left derivative, i.e. $\frac{d}{dt-} u|_{t=0} = \lim_{t \nearrow 0} \frac{u(t)-u(0)}{t}$. Then

$$\frac{d}{dt-} \Lambda^m E(h_{f,t})|_{t=0} = \sum_{w \in W_m} r_w^{-1} \frac{d}{dt-} E(h_{f,t} \circ F_w)|_{t=0}. \tag{5.4}$$

Since $h_f \circ F_\tau|_B = \max_{x \in V_0} h_f(x) = \max_{x \in V_m} h_f(x)$, by applying Lemma A.3, one can see that each term in the right hand side of (5.4) is non-negative and in particular

$$\frac{d}{dt-} E(h_{f,t} \circ F_\tau)|_{t=0} > 0,$$

thus we obtain $\frac{d}{dt-} \Lambda^m E(h_{f,t})|_{t=0} > 0$. Hence, there is $t < 0$ such that $\Lambda^m E(h_{f,t}) < \Lambda^m E(h_{f,0}) = \Lambda^m E(h_f) = E(f)$. This is a contradiction since h_f is a p-harmonic extension of f and $h_{f,t}|_{V_0} = f$. Hence (5.3) holds.

To complete the proof, we notice that $\max_{w \in W_m} \mathrm{Osc}(H_* f \circ F_w)$ is continuous on $l(V_0)$ by Lemma A.2. Hence, by (5.3), we have

$$\max_{f \in M} \max_{w \in W_m} \mathrm{Osc}(H_* f \circ F_w) < 1,$$

where $M = \{f \in l(V_0) : \mathrm{Osc}(f) = 1, \sum_{x \in V_0} f(x) = 0\}$ is a compact subset of $l(V_0)$. □

Corollary 5.6. *Assume (**A′**). We have the natural embedding $\mathscr{H} \subset C(K)$.*

5.3. **Embedding $\mathcal{F}$ into $L^p(K, \mu)$.** The idea is essentially due to Kumagai [19]. Readers can also find the proof in the book [17, Section 3.4].

By Corollary 5.6, for $f \in l(V_*)$, $m \geq 0$, we can naturally embed $H_*(f|_{V_m})$ into $L^p(K, \mu)$. Indeed, $\Psi(H_*(f|_{V_m})) \in C(K) \subset L^p(K, \mu)$, where Ψ is the same operator in the statement of Theorem 5.2. For short, we write

$$P_m f = \Psi(H_*(f|_{V_m})), \quad f \in l(V_*), m \geq 0.$$

Lemma 5.7. *Assume (**A′**) and (**B**). Then $P_m f$ converges in $L^p(K, \mu)$ as $m \to \infty$ for any $f \in \mathcal{F}$.*

Proof. Fix $m \geq 0$, and let $f_{m+1} = (P_m f)|_{V_{m+1}}$ and $g_{m+1} = f|_{V_{m+1}}$. Then

$$\sum_{w \in W_m} r_w^{-1} \Lambda E(f_{m+1} \circ F_w - g_{m+1} \circ F_w) = \Lambda^{m+1} E(f_{m+1} - g_{m+1}) \leq 2^p \mathcal{E}(f).$$

Hence, recall that $\kappa_m = \max_{w\in W_m} r_w\mu(F_wK)$, there is a constant $C>0$ independent of f such that

$$\begin{aligned}\|P_{m+1}f-P_mf\|^p_{L^p(K,\mu)} &\leq \sum_{w\in W_m}\mu(F_wK)\|P_{m+1}f-P_mf\|^p_{L^\infty(F_wK,\mu)}\\ &= \sum_{w\in W_m}\mu(F_wK)\|f_{m+1}\circ F_w-g_{m+1}\circ F_w\|^p_{l^\infty(V_1)}\\ &\leq C\sum_{w\in W_m}\mu(F_wK)\Lambda E(f_{m+1}\circ F_w-g_{m+1}\circ F_w)\\ &\leq 2^pC\kappa_m\mathcal{E}(f),\end{aligned} \tag{5.5}$$

where we use Lemma A.2 in the equality. Hence by (**B**),

$$\sum_{m=0}^\infty\|P_{m+1}f-P_mf\|_{L^p(K,\mu)}\leq 2C^{1/p}\left(\sum_{m=0}^\infty\kappa_m^{1/p}\right)\mathcal{E}^{1/p}(f)<\infty, \tag{5.6}$$

whence P_mf converges in $L^p(K,\mu)$ as $m\to\infty$. □

By Lemma 5.7, we can extend Ψ to $\mathcal{F}\to L^p(K,\mu)$ by defining $\Psi(f)=\lim_{m\to\infty}P_mf$.

Then for $f\in\mathcal{F}_c$, P_mf converges to f uniformly, hence $\Psi(f)$ is simply the continuous extension of f, and the notation is in consistency with that in the statement of Theorem 5.2.

Lemma 5.8. *Assume ($\boldsymbol{A'}$) and ($\boldsymbol{B}$). The extended embedding operator $\Psi:\ \mathcal{F}\to L^p(K,\mu)$ is continuous in the sense that if $f_n\to f$ pointwisely and $\mathcal{E}(f_n-f)\to 0$, then $\Psi(f_n)\to\Psi(f)$ in $L^p(K,\mu)$. Hence Ψ is linear. Furthermore, $\Psi:\mathcal{F}\to L^p(K,\mu)$ is injective.*

Proof. We first prove the continuity of Ψ. Assume $f_n\to f$ pointwisely and $\mathcal{E}(f_n-f)\to 0$ as $n\to\infty$. Then, by (5.5) and Lemma A.2, for any $m\geq 1$,

$$\begin{aligned}\|\Psi(f_n)-\Psi(f)\|_{L^p(K,\mu)} &\leq \|P_mf_n-P_mf\|_{L^p(K,\mu)}+\|P_mf-\Psi(f)\|_{L^p(K,\mu)}+\|P_mf_n-\Psi(f_n)\|_{L^p(K,\mu)}\\ &\leq \|f_n-f\|_{l^\infty(V_m)}+C_1\Big(\sum_{m'=m}^\infty\kappa_{m'}^{1/p}\Big)\big(\mathcal{E}^{1/p}(f)+\mathcal{E}^{1/p}(f_n)\big),\end{aligned}$$

for some $C_1>0$. Hence, $\Psi(f_n)\to\Psi(f)$ in $L^p(K,\mu)$.

Then we prove the injectivity of Ψ. The proof is due to [19], see also [17, Lemma 3.4.4]. The key observation is the following claim.

Claim. If $f\in\mathcal{F}$ and $\Psi(f)=0$, then $\|f|_{V_0}\|_{l^\infty(V_0)}\leq C_2\mathcal{E}^{1/p}(f)$ for some $C_2>0$ independent of f.

On one hand, by (5.5), we know that there is $C_3>0$ such that

$$\|P_0f\|_{L^p(K,\mu)}=\|\Psi(f)-P_0f\|_{L^p(K,\mu)}\leq C_3\mathcal{E}^{1/p}(f).$$

On the other hand, noticing that by Lemma A.2, $\Psi\circ H_*$ is continuous on $l(V_0)$. Hence, by letting $C_4=\max\{\|P_0f'\|^{-1}_{L^p(K,\mu)}: f'\in l(V_0),\|f'\|_{l^\infty(V_0)}=1\}$, we have

$$\|f|_{V_0}\|_{l^\infty(V_0)}\leq C_4\|P_0f\|_{L^p(K,\mu)}.$$

The claim follows immediately from the above two estimates.

To finish the proof, it suffices to apply Lemma 5.4. For each $x\in V_*$, there is $\tau,w\in W_*$ so that $x=\pi(\tau\dot{w})$. Hence, if $f\in\mathcal{F}$ and $\Psi(f)=0$, using the claim on $F_{\tau w^m}(V_0)$, for any $m\geq 1$, we have

$$|f(x)|\leq C_2\mathcal{E}^{1/p}(f\circ F_\tau\circ F_{w^m})\leq C_2(r_\tau r_w^m)^{1/p}\mathcal{E}^{1/p}(f).$$

Since $r_w<1$ by Lemma 5.4, we have $f(x)=0$, whence $f=0$ since the argument works for any $x\in V_*$. □

Proof of Theorem 5.2. By Lemma 5.8, $\Psi(\mathcal{F})$ is a subspace of $L^p(K,\mu)$. Since $P_m(f\circ F_w)=(P_{m+|w|}f)\circ F_w$ for any $w\in W_*$ and $m\geq 0$, we have $\Psi(f\circ F_w)=(\Psi f)\circ F_w$.

(*Closedness*). Let $f_n\in\mathcal{F},n\geq 1$ and assume $\mathcal{E}_1(\Psi(f_n)-\Psi(f_m))\to 0$ as $n,m\to\infty$, we need to show there is $f\in\mathcal{F}$ so that $\mathcal{E}_1(\Psi(f_n)-\Psi(f))\to 0$.

Since Ψ is linear by Lemma 5.8, we see that $\|\Psi(f_n-f_m)\|_{L^p(K,\mu)}\to 0$ as $n,m\to\infty$. In addition, by (5.5), we also see that $\|(\Psi-P_0)(f_n-f_m)\|_{L^p(K,\mu)}\to 0$. Hence, for some $C_1>0$, $\|f_n-f_m\|_{l^\infty(V_0)}\leq C_1\|P_0(f_n-f_m)\|_{L^p(K,\mu)}\to 0$ as $n,m\to\infty$.

Then, we apply Proposition 5.3 (c) to see that there is an $f\in\mathcal{F}$ so that $f_n\to f$ pointwisely, and $\mathcal{E}(f_n-f)\to 0$. Here we can choose $c_n=0$ in Proposition 5.3 since $f_n|_{V_0}$ converges pointwisely. Hence by Lemma 5.8, $\|\Psi(f_n)-\Psi(f)\|_{L^p(K,\mu)}\to 0$.

(*Regularity*). Since by Corollary 5.6, $\mathscr{H}\subset\mathscr{C}:=\mathcal{F}\cap C(K)$, so $\mathscr{C}$ is dense in $C(K)$. In addition, for each $f\in\mathcal{F}$, we have $\mathcal{E}_1(P_m(f)-\Psi(f))\to 0$ as $m\to\infty$, which implies that $\mathscr{C}$ is dense in $\mathcal{F}$ with respect to $\mathcal{E}_1^{1/p}$.

(*Markov property*). Write $\bar{f}=(f\wedge 1)\vee 0$ for short. It suffices to show that $\Psi(\bar{f})=\overline{\Psi(f)},\forall f\in\mathcal{F}$. To see this, for any $f\in\mathcal{F}$, we choose a sequence $f_n\in\mathcal{F}$ so that $\Psi(f_n)\in C(K)$ and $\mathcal{E}_1(\Psi(f_n)-\Psi(f))\to 0$. By continuity of $\Psi(f_n)$, we immediately have $\bar{f}_n=\overline{\Psi(f_n)}$. In addition, one can check

$$\|\overline{\Psi(f)}-\overline{\Psi(f_n)}\|_{L^p(K,\mu)}\leq\|\Psi(f)-\Psi(f_n)\|_{L^p(K,\mu)}\to 0,\tag{5.7}$$

$$\mathcal{E}(\bar{f}_n-\bar{f})\to 0,\text{ and }\bar{f}_n\to\bar{f}\text{ pointwisely.}\tag{5.8}$$

In particular, by (5.8) and Lemma 5.8, we know that $\overline{\Psi(f_n)}=\Psi(\bar{f}_n)\to\Psi(\bar{f})$ in $L^p(K,\mu)$. Hence, by (5.7), we have $\overline{\Psi(f)}=\Psi(\bar{f})$.

(*Strong locality*). It follows from a routine argument, noticing that if $f,g\in\mathscr{C}$ and f is constant on a neighbourhood of support of g, then $\Lambda^nE(f+g)=\Lambda^nE(f)+\Lambda^nE(g)$ for any n large enough. □

6. Affine nested fractals

In this section, we prove the existence of p-energy for any $p\in(1,\infty)$ on a class of highly symmetric p.c.f. fractals, namely the affine nested fractals. This class of fractals is firstly introduced in [8], which is a generalization of nested fractals introduced by Lindstrøm [22] in that the contraction ratios of the i.f.s. are allowed to be distinct.

The affine nested fractals are featured with the reflection symmetries interchanging essential fixed points. For convenience, we introduce the following notations for reflections in this section.

Definition 6.1. *Let $x,y\in\mathbb{R}^d$ and $x\neq y$. Define*

$$H_{x,y}=\{z\in\mathbb{R}^d:\ |x-z|=|y-z|\},$$

and denote the reflection with respect to $H_{x,y}$ by $\sigma_{x,y}$. In addition, for convenience, we write

$$H_x^y=\{z\in\mathbb{R}^d:\ |x-z|\leq|y-z|\}$$

for the closed half space containing x. Similarly, $H_y^x=\{z\in\mathbb{R}^d:\ |y-z|\leq|x-z|\}$.

In the following, we take the short definition of affine nested fractals from the book [17, Section 3.8], noticing that we assume $K\subset\mathbb{R}^d$ and F_i are similitudes for simplicity.

Definition 6.2. *Let K be a p.c.f. self-similar set associated with the i.f.s. $\{F_i\}_{i=1}^N$. We say $(K,\{F_i\}_{i=1}^N)$ is an affine nested fractal if $(K,\{F_i\}_{i=1}^N)$ has $\mathscr{G}$-symmetry, where $\mathscr{G}$ is the symmetry group generated by $\{\sigma_{x,y}:\ x\neq y\in V_0\}$.*

Our main result in this section is the following.

Theorem 6.3. *Let $(K,\{F_i\}_{i=1}^N)$ be an affine nested fractal. If the renormalization factor $\boldsymbol{r}$ is $\mathscr{G}$-symmetric, then condition ($\boldsymbol{A}$) holds.*

We refer to [17, Section 3.8] for basic properties of affine nested fractals, and an alternative proof of the existence of Dirichlet forms on nested fractals (essentially due to [22] and [8]). In particular, we need an easy geometric fact about affine nested fractals ([17, Lemma 3.8.6], see also [22, IV.9 Lemma] for nested fractals), which we state as Lemma 6.4 below.

Consider the distance set of points in V_0: $\{|x-y|:\ x,y\in V_0,x\neq y\}$, and we arrange them in order as $\ell_0<\ell_1<\cdots<\ell_m$, i.e. $\ell_0=\min\{|x-y|:\ x,y\in V_0,x\neq y\}$ is the minimal distance between two points in V_0, and ℓ_m is the maximal distance. For any two points $x,y\in V_0$, a sequence $x=x_0,x_1,\cdots,x_k=y$ in V_0 such that $|x_i-x_{i+1}|=\ell_0$ for $i=0,1,\cdots,k-1$ is called a *strict* 0*-walk* between x and y.

Lemma 6.4. *Let $(K,\{F_i\}_{i=1}^N,V_0)$ be an affine nested fractal. Then there exists a strict 0-walk between x and y for any $x,y\in V_0$, $x\neq y$.*

Proof of Theorem 6.3. Let $E\in\mathcal{S}$ be defined as

$$E(f)=\frac{1}{2}\sum_{x,y\in V_0:|x-y|=\ell_0}\big|f(x)-f(y)\big|^p,\quad\forall f\in l(V_0).$$

For $n\geq 0$, let R_n be the p-resistance associated with Λ^nE. Let $p_1,p_2\in V_0$ be such that $|p_1-p_2|=\ell_0$. We claim that

$$\frac{1}{2}R_n(p_1,p_2)\leq R_n(x,y)\leq N^pR_n(p_1,p_2),\qquad\forall x\neq y\in V_0,\forall n\geq 0.\tag{6.1}$$

As a consequence, $\inf_{n\geq 0}\delta(\mathcal{T}^nE)\geq\frac{1}{2}N^{-p}$, and hence condition (**A**) holds. The upper bound in (6.1) will be proved by Lemma 6.4, and the lower bound in (6.1) can be proved by using an analogous reflection principle.

"*Upper bound*". Indeed, for any $x,y\in V_0$, $x\neq y$, let $x=x_0,x_1,\cdots,x_k=y$ be a strict 0-walk in V_0, whose existence is guaranteed by Lemma 6.4. Note that $k\leq N$. Then by the triangle inequality of $R_n^{1/p}$ (Proposition 2.4(c)), we obtain

$$R_n(x,y)^{1/p} \leq \sum_{i=0}^{k-1} R_n(x_i,x_{i+1})^{1/p} = kR_n(p_1,p_2)^{1/p} \leq NR_n(p_1,p_2)^{1/p},$$

which gives the upper bound estimate in (6.1).

"*Lower bound*". If $|x-y|=\ell_0$, then $R_n(x,y)=R_n(p_1,p_2)$, so we have nothing to prove. Thus, we assume $|x-y|=\ell_k$ for some $1\leq k\leq m$. We let $z\in V_0$ so that $|x-z|=\ell_0$.

Let $n\geq 0$. We define $f\in l(V_n)$ such that

$$\begin{cases} f(x)=1,\ f(z)=0, \\ \Lambda^n E(f)=\min\{\Lambda^n E(g):\ g\in l(V_n),\ g(x)=1, g(z)=0\}. \end{cases}$$

Hence $\Lambda^n E(f)=\big(R_n(x,z)\big)^{-1}$. One can then define $h\in l(V_n)$ by

$$h(q)=\begin{cases} f(q), & \text{if } q\in H_z^y\cap V_n, \\ f\circ\sigma_{y,z}(q), & \text{if } q\in H_y^z\cap V_n. \end{cases}$$

In particular, since $|x-z|=\ell_0<\ell_k=|x-y|$, we have $x\in H_z^y$ hence $h(x)=1$. In addition, $h(y)=h(\sigma_{y,z}(z))=0$. So we also have $\big(R_n(x,y)\big)^{-1}\leq\Lambda^n E(h)$.

Finally, noting that we collect a term $|f(p)-f(q)|^p$ in $E(f)$ if and only if $|p-q|=\ell_0$, and hence the contributions of terms $|h(p)-h(q)|^p$ in $\Lambda^n E(h)$ for $p\in H_y^z\setminus H_{y,z}$ and $q\in H_z^y\setminus H_{y,z}$ is 0, so $\Lambda^n E(h)\leq 2\Lambda^n E(f)$. Combining all the information, we get

$$\big(R_n(x,y)\big)^{-1}\leq\Lambda^n E(h)\leq 2\Lambda^n E(f)=2\big(R_n(x,z)\big)^{-1}=2\big(R_n(p_1,p_2)\big)^{-1},$$

which gives the lower bound estimate in (6.1). □

7. Behavior of nearly degenerate p-energies

In Theorem 6.3, we obtain a uniform positive lower bound for $\delta(\Lambda^n E)$ on affine nested fractals by using the strong symmetry. On general p.c.f. self-similar sets, such an estimate is either not always true or hard to be established. Instead, we follow the idea of Sabot [29] to show that under suitable conditions, $\mathcal{T}$ will bounce near the boundary $\{E\in\widetilde{\mathcal{M}}:\delta(E)=0\}$, in other words when $\delta(E)$ is very small.

Before talking about Sabot's criteria, we return to study the general forms $E\in\widetilde{\mathcal{M}}_p(A)$ on a finite set A when $\delta(E)$ is very small. The key idea, due to [29], is to describe the distribution of the resistances using the equivalence relations.

7.1. **Equivalence relations.** Throughout this paper, we will use $\mathcal{J}$ to denote an equivalence relation on a finite set A, and write $A/\mathcal{J}$ for the set of equivalence classes. We will use the notation I for an equivalence class of $\mathcal{J}$, i.e. $I\in A/\mathcal{J}$, and notice that $I\subset A$. We say that $\mathcal{J}$ is *trivial* if it is one of the following two relations on A: $\mathcal{J}=0$ if $\#(A/\mathcal{J})=\#A$; $\mathcal{J}=1$ if $\#(A/\mathcal{J})=1$; otherwise $\mathcal{J}$ is called *non-trivial*. In addition, for two relations $\mathcal{J},\mathcal{J}'$, we say $\mathcal{J}\subset\mathcal{J}'$ if

$$x\mathcal{J}y\Longrightarrow x\mathcal{J}'y\qquad\text{for any } x,y\in A.$$

We introduce $\delta_{\mathcal{J}}(E^{(p)})$ to provide more information than $\delta(E^{(p)})$.

Definition 7.1. *Let $\mathcal{J}$ be a non-trivial equivalence relation on A, and let $E^{(p)} \in \mathcal{M}_p(A)$ with associated p-resistance $R^{(p)}$. We define*

$$\delta_{\mathcal{J}}(E^{(p)}) = \frac{\max_{x\mathcal{J}y} R^{(p)}(x,y)}{\min_{x\not\mathcal{J}y} R^{(p)}(x,y)}.$$

Of course, we are interested in the case that $\delta_{\mathcal{J}}(E^{(p)}) < 1$.

Lemma 7.2. *Let A be a finite set and $E^{(p)} \in \mathcal{M}_p(A)$. Let $\mathcal{J}, \mathcal{J}'$ be two non-trivial equivalence relations such that $\delta_{\mathcal{J}}(E^{(p)}) < 1$ and $\delta_{\mathcal{J}'}(E^{(p)}) < 1$, then either $\mathcal{J} \subset \mathcal{J}'$ or $\mathcal{J}' \subset \mathcal{J}$.*

Proof. Assume neither $\mathcal{J} \subset \mathcal{J}'$ nor $\mathcal{J}' \subset \mathcal{J}$ is true, then one can find x, y and x', y' such that

$$x\mathcal{J}y, \qquad x'\not\mathcal{J}y', \tag{7.1}$$

$$x\not\mathcal{J}'y, \qquad x'\mathcal{J}'y'. \tag{7.2}$$

Then, since $\delta_{\mathcal{J}} < 1$, $\frac{R^{(p)}(x,y)}{R^{(p)}(x',y')} < 1$ by (7.1); since $\delta_{\mathcal{J}'} < 1$, $\frac{R^{(p)}(x',y')}{R^{(p)}(x,y)} < 1$ by (7.2). Clearly, this is impossible. □

By a similar argument as in the proof of Corollary 2.7, we can prove the following lemma.

Lemma 7.3. *Let A be a finite set, $E^{(p)} \in \mathcal{M}_p(A)$ and $R^{(p)}$ be the effective resistance associated with $E^{(p)}$, and $x_1, x_2, y_1, y_2 \in A$. Let $0 < \delta < (\#A)^{-\#A(\#A-1)p/2}$. If $\frac{R^{(p)}(x_1,y_1)}{R^{(p)}(x_2,y_2)} < \delta$, then there is a non-trivial equivalence relation $\mathcal{J}$ such that $\delta_{\mathcal{J}}(E^{(p)}) < \delta^{\frac{2}{\#A(\#A-1)}} \cdot (\#A)^p$ and $x_1\mathcal{J}y_1, x_2\not\mathcal{J}y_2$.*

Remark. The lemma implies that if $\delta(E^{(p)}) < \delta$, then there is a non-trivial equivalence relation $\mathcal{J}$ such that $\delta_{\mathcal{J}}(E^{(p)}) < \delta^{\frac{2}{\#A(\#A-1)}} \cdot (\#A)^p$.

Proof. Define $\delta_0 := 2/(\#A(\#A-1))$. There are at most δ_0^{-1} different values of $R^{(p)}(x,y)$ between $R^{(p)}(x_1,y_1), R^{(p)}(x_2,y_2)$ which we list as $R^{(p)}(x_1,y_1) = R_1 < R_2 < \cdots < R_n = R^{(p)}(x_2,y_2)$ for some $n \le \delta_0^{-1}$. Since $\delta(E^{(p)}) = \frac{R_1}{R_n} < \delta$, we see that there is $1 \le k \le n-1$ such that $\frac{R_k}{R_{k+1}} < \delta^{\frac{1}{n-1}} < \delta^{\delta_0}$. Pick $R_k < r < R_{k+1}$ sufficiently close to R_k, we have the following three holds:

$$\text{either } R^{(p)}(x,y) < r, \text{ or } R^{(p)}(x,y) > \delta^{-\delta_0} r, \quad \forall x \ne y \in A, \tag{7.3}$$

$$R^{(p)}(x_1,y_1) = R_1 < r, \tag{7.4}$$

$$R^{(p)}(x_2,y_2) = R_n > \delta^{-\delta_0} r. \tag{7.5}$$

One can define an equivalence relation $\mathcal{J}$ on A so that $x\mathcal{J}y$ if and only if there exists a sequence of points $x = z_0, z_1, \cdots, z_m = y$ such that

$$R^{(p)}(z_i, z_{i+1}) < r, \quad \forall 0 \le i \le m-1.$$

On one hand, by (7.3), $x\not\mathcal{J}y$ implies $R^{(p)}(x,y) > \delta^{-\delta_0} r$. On the other hand, by the triangle inequality of $R^{(p)}(\cdot,\cdot)^{1/p}$, we have

$$R^{(p)}(x,y) < (\#A)^p r, \quad \forall x \ne y, x\mathcal{J}y. \tag{7.6}$$

Hence, $\delta_{\mathcal{J}}(E^{(p)}) < \delta^{\delta_0} \cdot (\#A)^p$.

Furthermore, $x_1\mathcal{J}y_1$ by (7.4) and the definition of $\mathcal{J}$; $x_2\not\mathcal{J}y_2$ by (7.5) and

$$R^{(p)}(x,y)<\delta^{-\delta_0}r,\quad \forall x\neq y, x\mathcal{J}y,$$

where the latter inequality is due to (7.6) and the assumption $\delta<(\#A)^{-p\delta_0^{-1}}$. □

7.2. **Trace of p-energies.** Next, we consider the trace of p-energies (recall Proposition 2.2 (c)). In this subsection, we will fix an equivalence relation $\mathcal{J}$. The results are inspired by [29, Chapter 4]. Our proof is simplified by introducing a modified trace mapping.

Definition 7.4. *For $\delta>0$, $E^{(p)}\in\mathcal{M}_p(A)$, we say that $E^{(p)}\in\mathcal{M}_p(A,\mathcal{J},\delta)$ if the following two conditions hold:*
(a). $\delta_{\mathcal{J}}(E^{(p)})<\delta$.
(b). There exists $E_I^{(p)}\in\mathcal{M}_p(I)$ for each $I\in A/\mathcal{J}$, and a finite collection $\{E_i^{(p)}\}_{i=1}^m$ in $\widetilde{\mathcal{M}}_p(A)$ such that

$$E^{(p)}(f)=\sum_{I\in A/\mathcal{J}}E_I^{(p)}(f|_I)+\sum_{i=1}^m E_i^{(p)}(f),\qquad \text{for any } f\in l(A), \tag{7.7}$$

and for $i=1,\cdots,m$, $R_i^{(p)}(x,y)=\infty$ whenever $x\mathcal{J}y, x\neq y$, where $R_i^{(p)}$ is the p-resistance associated with $E_i^{(p)}$.

Clearly, $\{E^{(p)}\in\mathcal{S}_p(A):\delta_{\mathcal{J}}(E^{(p)})<\delta\}\subset\mathcal{M}_p(A,\mathcal{J},\delta)$.

We consider two kinds of traces of the p-energies in this paper: the usual trace $[E^{(p)}]_B$ (defined in Proposition 2.2 (c)) and a modified trace $<E^{(p)}>_B$ to be defined in the following Definition 7.5. Also see Lemma 7.6 (a) for an equivalent explanation of the modified trace.

Definition 7.5. *Let $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$. Let $B\subset A$, and let $\mathcal{J}|_B$ be the restriction of $\mathcal{J}$ on B. We write $I_{I'}$ for the unique equivalence class of $\mathcal{J}$ containing $I'\in B/\mathcal{J}|_B$.*

For each $I\in A/\mathcal{J}$ we choose $x_I\in I$. In addition, for each $I'\in B/\mathcal{J}|_B$, we require $x_{I_{I'}}\in I'$.

*(a). **Restriction on** $A/\mathcal{J}$: We define $[[E^{(p)}]]_{A/\mathcal{J}}\in\widetilde{\mathcal{M}}_p(A/\mathcal{J})$ by*

$$[[E^{(p)}]]_{A/\mathcal{J}}(u)=E^{(p)}\big(\sum_{I\in A/\mathcal{J}}u(I)1_I\big),\quad \forall u\in l(A/\mathcal{J}).$$

(b). With a little abuse of notation, for each $E_*^{(p)}\in\widetilde{\mathcal{M}}_p(A/\mathcal{J})$, we write

$$\begin{aligned}[E_*^{(p)}]_{B/\mathcal{J}|_B}(u')=\min\{&E_*^{(p)}(u):u\in l(A/\mathcal{J}),\ u(I)=u'(I'),\\ &\forall I'\in B/\mathcal{J}|_B, I\in A/\mathcal{J}\text{ satisfying } I'\subset I\}.\end{aligned}$$

*(c). **Modified trace:** Assume $E^{(p)}\in\mathcal{M}_p(A,\mathcal{J},\delta)$ with the decomposition (7.7). We can define*

$$<E^{(p)}>_B(f)=\sum_{I'\in B/\mathcal{J}|_B}[E_{I_{I'}}^{(p)}]_{I'}(f|_{I'})+[[[E^{(p)}]]_{A/\mathcal{J}}]_{B/\mathcal{J}|_B}(u_f),\quad \text{for any } f\in l(B),$$

where u_f is defined as $u_f(I')=f(x_{I_{I'}})$ for any $I'\in B/\mathcal{J}|_B$.

Remark 1. The modified trace $< E^{(p)} >_B$ actually depends on the decomposition (7.7) and the choice of $x_{I_{I'}}$, which are not reflected in the notation.

Remark 2. We will see in Proposition 7.7 that for δ small enough, $< E^{(p)} >_B \in \mathcal{M}_p(B, \mathcal{J}|_B, \delta)$ whenever $E^{(p)} \in \mathcal{M}_p(A, \mathcal{J}, \delta/2)$.

To help readers understand Definition 7.5, we provide some simple properties about the modified trace below.

Lemma 7.6. *We take the same setting as Definition 7.5. Assume* $E^{(p)} \in \mathcal{M}_p(A, \mathcal{J}, \delta)$ *with the decomposition (7.7).*

(a). Define $\tilde{E}^{(p)}$ *by*

$$\tilde{E}^{(p)}(f') = \sum_{I\in A/\mathcal{J}} E_I^{(p)}(f'|_I) + \sum_{i=1}^m E_i^{(p)}\Big(\sum_{I\in A/\mathcal{J}} f'(x_I)1_I\Big), \quad \forall f' \in l(A).$$

Then, $< E^{(p)} >_B = [\tilde{E}^{(p)}]_B$.

(b). For each $f \in l(B)$, *we can find* $\tilde{f} \in l(A)$ *such that*

$$\tilde{f}|_B = f, \text{ and } \tilde{E}^{(p)}(\tilde{f}) = < E^{(p)} >_B (f)$$

with two steps. First, we extend f *to* $\tilde{f}$ *on each* $I_{I'}, I' \in B/\mathcal{J}|_B$ *with respect to* $E^{(p)}_{I_{I'}}$ *separately; then, take* $\tilde{f}$ *to be a constant* c_I *on each remaining equivalence classes* I, *i.e.* $I \in A/\mathcal{J} \setminus \{I_{I'} : I' \in B/\mathcal{J}|_B\}$, *such that* $E^{(p)}\Big(\sum_{I\in A/\mathcal{J}} \tilde{f}(x_I)1_I\Big)$ *is minimized.*

Remark. $\sum_{i=1}^m E_i^{(p)}\big(\sum\limits_{I\in A/\mathcal{J}} f'(x_I)1_I\big) = E^{(p)}\big(\sum\limits_{I\in A/\mathcal{J}} f'(x_I)1_I\big)$ for any $f' \in l(A)$.

Proof. Let $\tilde{f}$ be the function defined in (b). Then it is straightforward to check that $\tilde{f}$ minimizes each term of $\tilde{E}^{(p)}$:

$$\begin{aligned} E_I^{(p)}(\tilde{f}|_I) &= \min\{E_I^{(p)}(f') : f' \in l(A), f'|_B = f\}, \quad \forall I \in A/\mathcal{J}, \\ E^{(p)}\Big(\sum_{I\in A/\mathcal{J}} \tilde{f}(x_I)1_I\Big) &= \min\{E^{(p)}\Big(\sum_{I\in A/\mathcal{J}} f'(x_I)1_I\Big) : f' \in l(A), f'|_B = f\}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} [\tilde{E}^{(p)}]_B(f) &= \min\{\tilde{E}^{(p)}(f') : f' \in l(A), f'|_B = f\} \\ &= \tilde{E}^{(p)}(\tilde{f}) = \sum_{I\in A/\mathcal{J}} E_I^{(p)}(\tilde{f}|_I) + E^{(p)}\Big(\sum_{I\in A/\mathcal{J}} \tilde{f}(x_I)1_I\Big) \\ &= \sum_{I'\in B/\mathcal{J}|_B} [E^{(p)}_{I_{I'}}]_{I'}(f|_{I'}) + [[[E^{(p)}]]_{A/\mathcal{J}}]_{B/\mathcal{J}|_B}(u_f) = < E^{(p)} >_B (f). \end{aligned}$$

Hence both (a) and (b) follow immediately. □

Proposition 7.7. *With the same setting as in Definition 7.5 (c), assume* $\delta < \min\{1, C^{-p}\}$ *for some constant* $C > 0$ *depending only on* $\#A$. *Then it holds that*

$$(1+\varepsilon)^{-1} \cdot < E^{(p)} >_B \le [E^{(p)}]_B \le (1+\varepsilon) \cdot < E^{(p)} >_B,$$

where $\varepsilon = \varepsilon(\delta) = \frac{1}{1-C\delta^{1/p}}\big(\frac{\delta^{-1/p}}{\delta^{-1/p}-1}\big)^{p-1} - 1 > 0$. *Clearly,* $\varepsilon(\delta) \to 0$ *as* $\delta \to 0$.

Proof. First, we provide some estimates.

Claim 1. Let $f_* \in l(A)$ and $g = \sum_{I\in A/\mathcal{J}} f_*(x_I)1_I$, we have

$$\sum_{i=1}^m E_i^{(p)}\big(f_* - g\big) < C\delta E^{(p)}(f_*),$$

for some constant $C > 0$ independent of f_*.

Proof of Claim 1. Observe that

$$\|f_* - g\|_{l^\infty(A)} \le \max_{I\in A/\mathcal{J}} \mathrm{Osc}(f_*|_I) \le \Big(\max_{x\mathcal{J}y} R^{(p)}(x,y)\cdot E^{(p)}(f_*)\Big)^{1/p}. \tag{7.8}$$

For each $E_i^{(p)}$, we can apply Corollary 2.7 to get an equivalent energy $\bar{E}_i^{(p)}$ of the form $\sum_{x\mathcal{J}y} c_{x,y}^{(i)}|u(x)-u(y)|^p$ (recall from Definition 7.4 (b) that we require $R_i^{(p)}(x,y)=\infty$ for any $x\mathcal{J}y, x\neq y$) which is comparable with $E_i^{(p)}$. Since $\sum_{i=1}^m \bar{E}_i^{(p)} \asymp \sum_{i=1}^m E_i^{(p)} \le E^{(p)}$,

$$R^{(p)}(x,y)^{-1} \gtrsim \sum_{i=1}^m c_{x,y}^{(i)}, \quad \forall x\not{\mathcal{J}}y.$$

We then arrive at

$$\begin{aligned}\sum_{i=1}^m E_i^{(p)}\left(f_*-g\right) &\lesssim \sum_{i=1}^m \bar{E}_i^{(p)}\left(f_*-g\right)\\ &= \sum_{i=1}^m \sum_{x\not{\mathcal{J}}y} c_{x,y}^{(i)}\Big(\big(f_*(x)-g(x)\big)-\big(f_*(y)-g(y)\big)\Big)^p\\ &\lesssim \sum_{x\not{\mathcal{J}}y} R^{(p)}(x,y)^{-1}(2\big\|f_*-g\big\|_{l^\infty(A)})^p \lesssim \frac{\big\|f_*-g\big\|_{l^\infty(A)}^p}{\min_{x\not{\mathcal{J}}y} R^{(p)}(x,y)}.\end{aligned} \tag{7.9}$$

Hence, by (7.8) and (7.9), we have for some constant $C>0$,

$$\sum_{i=1}^m E_i^{(p)}\left(f_*-g\right) \le C\frac{\max_{x\mathcal{J}y} R^{(p)}(x,y)}{\min_{x\not{\mathcal{J}}y} R^{(p)}(x,y)} E^{(p)}(f_*) < C\delta E^{(p)}(f_*),$$

where we have used $\delta_{\mathcal{J}}(E^{(p)}) < \delta$ in the last inequality. □

Claim 2. For any $f_* \in l(A)$, we have

$$\sum_{i=1}^m E_i^{(p)}(f_*) \le \left(\frac{\delta^{-1/p}}{\delta^{-1/p}-1}\right)^{p-1} \sum_{i=1}^m E_i^{(p)}\left(\sum_{I\in A/\mathcal{J}} f_*(x_I)1_I\right) + C\delta^{1/p}E^{(p)}(f_*), \tag{7.10}$$

$$\sum_{i=1}^m E_i^{(p)}\left(\sum_{I\in A/\mathcal{J}} f_*(x_I)1_I\right) \le \left(\frac{\delta^{-1/p}}{\delta^{-1/p}-1}\right)^{p-1} \sum_{i=1}^m E_i^{(p)}(f_*) + C\delta^{1/p}E^{(p)}(f_*). \tag{7.11}$$

Proof of Claim 2. As in Claim 1, we write $g = \sum_{I\in A/\mathcal{J}} f_*(x_I)1_I$. Also, we let $\lambda = \frac{1}{\delta^{-1/p}-1}$. Using the p-homogeneity and convexity of each $E_i^{(p)}$, we have

$$E_i^{(p)}(f_*) = E_i^{(p)}(g + f_* - g) = (1+\lambda)^p E_i^{(p)}\left(\frac{1}{1+\lambda}g + \frac{\lambda}{1+\lambda}\cdot\frac{f_* - g}{\lambda}\right)$$
$$\leq (1+\lambda)^{p-1}E_i^{(p)}(g) + \left(1+\frac{1}{\lambda}\right)^{p-1} E_i^{(p)}(f_* - g)$$
$$E_i^{(p)}(g) = E_i^{(p)}(f_* + g - f_*) = (1+\lambda)^p E_i^{(p)}\left(\frac{1}{1+\lambda}f_* + \frac{\lambda}{1+\lambda}\cdot\frac{g - f_*}{\lambda}\right)$$
$$\leq (1+\lambda)^{p-1}E_i^{(p)}(f_*) + \left(1+\frac{1}{\lambda}\right)^{p-1} E_i^{(p)}(g - f_*).$$

The claim follows immediately by summing up the above inequalities over $i = 1, \cdots, m$, and then by using Claim 1. □

Now, we return to the proof of Proposition 7.7. Let $f \in l(B)$. We consider the following two different extensions of f to A:
1). $f_1 \in l(A)$ is the minimal energy extension of f to A with respect to $E^{(p)}$.
2). $f_2 \in l(A)$ is the function $\tilde{f}$ constructed in Lemma 7.6 (b).

Proof of " $[E^{(p)}]_B \leq (1+\varepsilon) < E^{(p)} >_B$*".* We have

$$E^{(p)}(f_2) = \sum_{I\in A/\mathcal{J}} E_I^{(p)}(f_2|_I) + \sum_{i=1}^m E_i^{(p)}(f_2)$$
$$\leq \sum_{I\in A/\mathcal{J}} E_I^{(p)}(f_2|_I) + \left(\frac{\delta^{-1/p}}{\delta^{-1/p}-1}\right)^{p-1}\sum_{i=1}^m E_i^{(p)}\left(\sum_{I\in A/\mathcal{J}} f_2(x_I)1_I\right) + C\delta^{1/p}E^{(p)}(f_2)$$
$$\leq (\frac{\delta^{-1/p}}{\delta^{-1/p}-1})^{p-1}\cdot < E^{(p)} >_B (f) + C\delta^{1/p}E^{(p)}(f_2),$$

where the first line is due to the decomposition (7.7), the second line is due to (7.10), and the last line is due to Lemma 7.6. Hence, if $\delta < \min\{1, C^{-p}\}$,

$$[E^{(p)}]_B(f) \leq E^{(p)}(f_2) \leq (\frac{1}{1-C\delta^{1/p}})\cdot(\frac{\delta^{-1/p}}{\delta^{-1/p}-1})^{p-1}\cdot < E^{(p)} >_B (f).$$

Proof of " $(1+\varepsilon)^{-1} < E^{(p)} >_B \leq [E^{(p)}]_B$*".* We have

$$< E^{(p)} >_B (f) \leq \sum_{I\in A/\mathcal{J}} E_I^{(p)}(f_1|_I) + \sum_{i=1}^m E_i^{(p)}\left(\sum_{I\in A/\mathcal{J}} f_1(x_I)1_I\right)$$
$$\leq \sum_{I\in A/\mathcal{J}} E_I^{(p)}(f_1|_I) + \left(\frac{\delta^{-1/p}}{\delta^{-1/p}-1}\right)^{p-1}\sum_{i=1}^m E_i^{(p)}(f_1) + C\delta^{1/p}E^{(p)}(f_1)$$
$$\leq \left((\frac{\delta^{-1/p}}{\delta^{-1/p}-1})^{p-1} + C\delta^{1/p}\right)\cdot [E^{(p)}]_B(f),$$

where the first line is due to Lemma 7.6 (a), the second line is due to (7.11) and the last line is due to the decomposition (7.7). □

8. p-version criteria of Sabot

In this section, we will prove the criteria of Sabot (in p-energy version) concerning the behavior of $\mathcal{T}$. The main theorem, Theorem 8.4, consists of two parts: under certain conditions, one has (**A**) hold; under some other conditions, (**A**) does not hold. Unfortunately, like the $p = 2$ case, the above two can not cover all possible situations.

Throughout this section, we fix $p \in (1, \infty)$ and omit the subscript p or superscript (p) as we did before.

Definition 8.1. *Let $\mathcal{J}$ be an equivalence relation on V_0, and $\mathscr{G}$ be a group action on V_0.*

(a). We say that $\mathcal{J}$ is a $\mathscr{G}$-relation if

$$x\mathcal{J}y \Longrightarrow \sigma(x)\mathcal{J}\sigma(y),$$

for any $x, y \in V_0$ and $\sigma \in \mathscr{G}$.

(b). We define $\mathcal{J}^{(1)}$ on V_1 as the minimal equivalence relation such that

$$x\mathcal{J}y \Longrightarrow F_i x \mathcal{J}^{(1)} F_i y,$$

for any $x, y \in V_0$ and $1 \le i \le N$. With some abuse of the notation "$\mathcal{T}$", we define $\mathcal{T}\mathcal{J} = \mathcal{J}^{(1)}|_{V_0}$.

(c). We say $\mathcal{J}$ is preserved if $\mathcal{T}\mathcal{J} = \mathcal{J}$; otherwise, $\mathcal{J}$ is non-preserved.

Definition 8.2. *Let $\mathcal{J}$ be a non-trivial preserved equivalence relation on V_0.*

(a). We identify each $f \in l(V_0/\mathcal{J})$ with the function $\sum_{I \in V_0/\mathcal{J}} f(I) 1_I$ in $l(V_0)$. Do similarly for $f \in l(V_1/\mathcal{J}^{(1)})$.

(b). The operator $\Lambda : \mathcal{M}(V_0) \to \mathcal{M}(V_1)$ is automatically extended to $\Lambda_{V_0/\mathcal{J}} : \mathcal{M}(V_0/\mathcal{J}) \to \mathcal{M}(V_1/\mathcal{J}^{(1)})$: for each $E \in \mathcal{M}(V_0/\mathcal{J})$, $\Lambda_{V_0/\mathcal{J}} E \in \mathcal{M}(V_1/\mathcal{J}^{(1)})$ is defined as

$$\Lambda_{V_0/\mathcal{J}} E(f) = \sum_{i=1}^{N} r_i^{-1} E(f \circ F_i), \qquad \text{for any } f \in l(V_1/\mathcal{J}^{(1)}),$$

and we define $\mathcal{T}_{V_0/\mathcal{J}} : \mathcal{M}(V_0/\mathcal{J}) \to \mathcal{M}(V_0/\mathcal{J})$ as $\mathcal{T}_{V_0/\mathcal{J}} E = [\Lambda_{V_0/\mathcal{J}} E]_{V_0/\mathcal{J}}$.

We define

$$\underline{\rho}_{V_0/\mathcal{J}} = \sup_{E \in \mathcal{M}(V_0/\mathcal{J})} \inf(\mathcal{T}_{V_0/\mathcal{J}} E | E).$$

(c). We denote $\widetilde{\mathcal{M}}(V_0, \mathcal{J}, 0)$ the collection of $E \in \widetilde{\mathcal{M}}$ that takes the form $E = \sum_{I \in A/\mathcal{J}} E_I$ where $E_I \in \mathcal{M}(I)$. We define

$$\overline{\rho}_{\mathcal{J}} = \inf_{E \in \widetilde{\mathcal{M}}(V_0, \mathcal{J}, 0)} \sup(\mathcal{T}E | E), \quad \underline{\rho}_{\mathcal{J}} = \sup_{E \in \widetilde{\mathcal{M}}(V_0, \mathcal{J}, 0)} \inf(\mathcal{T}E | E).$$

Concerning a preserved equivalence relation $\mathcal{J}$, we can naturally extend the results in Lemma 3.4 to the operator $\mathcal{T}_{V_0/\mathcal{J}}$ acting on $\mathcal{M}(V_0/\mathcal{J})$. To be precise, we state the following lemma, whose proof is similar to that of Lemma 3.4.

Lemma 8.3. *(a). Let $E_1, E_2 \in \mathcal{M}(V_0/\mathcal{J})$, we have*

$$\begin{cases}\sup(\mathcal{T}_{V_0/\mathcal{J}}E_1|\mathcal{T}_{V_0/\mathcal{J}}E_2) \leq \sup(E_1|E_2),\\ \inf(\mathcal{T}_{V_0/\mathcal{J}}E_1|\mathcal{T}_{V_0/\mathcal{J}}E_2) \geq \inf(E_1|E_2).\end{cases}$$

(b). Let $E_1, E_2 \in \mathcal{M}(V_0/\mathcal{J})$, for $n \geq 1$, we have

$$\inf(\mathcal{T}^n_{V_0/\mathcal{J}}E_1|E_1) \leq \sup(\mathcal{T}^n_{V_0/\mathcal{J}}E_2|E_2).$$

Remark. The facts in Lemma 8.3 will be used in the proof of Proposition 8.12 (b).

In [29, Theorem 5.1], for the $p = 2$ case, the following condition (**H**) is assumed for the existence result.

(**H**). There do not exist two non-trivial preserved $\mathscr{G}$-relations $\mathcal{J}, \mathcal{J}'$ such that $\mathcal{J} \subsetneq \mathcal{J}'$.

For general $p \in (1, \infty)$, still assuming (**H**), our criteria for the existence and non-existence results state as follows.

Theorem 8.4. *Let K be a p.c.f. self-similar set with $\mathscr{G}$-symmetry, where $\mathscr{G}$ is a group of homeomorphism on K (see Section 3). Assume the renormalization vector $\boldsymbol{r}$ (see Definition 3.1) is $\mathscr{G}$-symmetric.*

*(a). **Non-existence:** If $\underline{\rho}_{\mathcal{J}} > \underline{\rho}_{V_0/\mathcal{J}'}$ for some non-trivial preserved relations $\mathcal{J}$ and $\mathcal{J}'$ on V_0, then $\mathcal{T}$ does not have an eigenform in $\mathcal{M}$.*

*(b). **Existence:** Assume (**H**) holds. If $\overline{\rho}_{\mathcal{J}} < \underline{\rho}_{V_0/\mathcal{J}}$ for any non-trivial preserved $\mathscr{G}$-relation $\mathcal{J}$, then condition (**A**) holds, hence $\mathcal{T}$ has a $\mathscr{G}$-symmetric eigenform in $\mathcal{Q}$.*

Remark. For the $p = 2$ case in [29], the uniqueness of the form is also provided, while for general p, the uniqueness of the p-energy form, even on the Sierpinski gasket, is unsettled since the difference of two forms is not necessarily convex.

It is direct to verify Theorem 8.4 (a); while for (b), we need some more preparations.

Proof of Theorem 8.4 (a). We will prove by contradiction. Assume $E \in \mathcal{M}$ is an eigenform of $\mathcal{T}$, i.e. $\mathcal{T}E = \lambda E$ for some $\lambda > 0$.

On one hand, we see that

$$\lambda E(f) = \mathcal{T}E(f) \leq \mathcal{T}_{V_0/\mathcal{J}'}([[E]]_{V_0/\mathcal{J}'})(f), \quad \forall f \in l(V_0/\mathcal{J}'),$$

so that $\lambda \leq \inf\big(\mathcal{T}_{V_0/\mathcal{J}'}([[E]]_{V_0/\mathcal{J}'})\big|[[E]]_{V_0/\mathcal{J}'}\big) \leq \underline{\rho}_{V_0/\mathcal{J}'}$.

On the other hand, one can find $E' = \sum_{I \in V_0/\mathcal{J}} E'_I$ where $E'_I \in \mathcal{M}(I)$ for each $I \subset \mathcal{J}$ and any $\varepsilon > 0$ such that $\mathcal{T}E' \geq (\underline{\rho}_{\mathcal{J}} - \varepsilon)E'$. By Proposition 2.6, there is $C > 0$ such that $E' \leq CE$. Now, fix $f \in l(V_0)$ with $E'(f) > 0$. For any $n \geq 1$, we have

$$(\underline{\rho}_{\mathcal{J}} - \varepsilon)^n E'(f) \leq \mathcal{T}^n E'(f) \leq C\mathcal{T}^n E(f) = C\lambda^n E(f),$$

which implies that $\lambda \geq \underline{\rho}_{\mathcal{J}}$ noticing that ε is arbitrary.

Combining the above two observations, we have $\underline{\rho}_{\mathcal{J}} \leq \lambda \leq \underline{\rho}_{V_0/\mathcal{J}'}$, which contradicts the assumption $\underline{\rho}_{\mathcal{J}} > \underline{\rho}_{V_0/\mathcal{J}'}$. Hence $\mathcal{T}$ does not have an eigenform in $\mathcal{M}$. □

8.1. **Non-symmetric p-energies on the Sierpinski gasket.** In this subsection, we consider an example, the Sierpinski gasket.

Let $q_1 = (0,0)$, $q_2 = (1,0)$ and $q_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ be the three vertices of a unit triangle in $\mathbb{R}^2$. For $i = 1,2,3$, let F_i be the contraction on $\mathbb{R}^2$ of the form $F_i(x) = \frac{1}{2}(x - q_i) + q_i$. The self-similar set w.r.t. the i.f.s. $\{F_1, F_2, F_3\}$ is the Sierpinski gasket (SG) in $\mathbb{R}^2$.

In [14], Herman, Peirone and Strichartz constructed a fully symmetric p-energy on SG. Here, following [29], we consider a non-symmetric case. In the following we fix a vector of renormalization factors $\boldsymbol{r} = (r_1, r_2, r_3)$. In other words, the associated renormalization map $\mathcal{T}$ is defined by

$$\mathcal{T}E(f) = \min\{\Lambda E(g):\ g \in l(V_1), g|_{V_0} = f\}, \quad E \in \mathcal{M}(V_0), f \in l(V_0),$$

where

$$\Lambda E(g) = r_1^{-1}E(g\circ F_1) + r_2^{-1}E(g\circ F_2) + r_3^{-1}E(g\circ F_3), \quad g \in l(V_1).$$

Clearly, there are three non-trivial preserved relations $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$, which can be described as

$$\begin{cases} q_1 \not\mathcal{J}_1 q_2,\ q_2\mathcal{J}_1 q_3; \\ q_2 \not\mathcal{J}_2 q_3,\ q_1\mathcal{J}_2 q_3; \\ q_3 \not\mathcal{J}_3 q_1,\ q_1\mathcal{J}_3 q_2. \end{cases}$$

Noticing that for $i = 1,2,3$, there is only one p-energy form (up to a constant multiplier) in $\mathcal{M}(V_0/\mathcal{J}_i)$ and in $\widetilde{\mathcal{M}}(V_0, \mathcal{J}_i, 0)$. For convenience, we use $\{i,j,k\}$ to denote a permutation of $\{1,2,3\}$. By direct computation, we can see

$$\overline{\rho}_{\mathcal{J}_i} = \underline{\rho}_{\mathcal{J}_i} = \left(r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}}\right)^{1-p}, \quad \underline{\rho}_{V_0/\mathcal{J}_i} = \left(\left(\frac{1}{r_j} + \frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}\right)^{1-p}.$$

For convenience of readers, in the following, we briefly explain the computation.

We first consider the unique p-energy (up to a constant multiplier) $E_i \in \widetilde{\mathcal{M}}(V_0, \mathcal{J}_i, 0)$ defined as $E_i(f) = |f(q_j) - f(q_k)|^p$ for $f \in l(V_0)$. Then

$$\begin{aligned}\mathcal{T}E_i(f) &= \min_{x,y,z\in\mathbb{R}} \left\{r_j^{-1}|f(q_j) - x|^p + r_k^{-1}|x - f(q_k)|^p + r_i^{-1}|y - z|^p\right\} \\ &= \left(r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}}\right)^{1-p} |f(q_j) - f(q_k)|^p.\end{aligned}$$

From this, we find that

$$\overline{\rho}_{\mathcal{J}_i} = \underline{\rho}_{\mathcal{J}_i} = \mathcal{T}E_i(f)/E_i(f) = \left(r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}}\right)^{1-p}.$$

Next, we consider the unique p-energy (up to a constant multiplier) $E_i' \in \mathcal{M}(V_0/\mathcal{J}_i)$ defined as $E_i'(f) = \big|f(\{q_i\}) - f(\{q_j, q_k\})\big|^p$ for $f \in l(V_0/\mathcal{J}_i)$, and hence

$$\begin{aligned}\mathcal{T}E_i'(f) &= \min_{x\in\mathbb{R}} \left\{r_i^{-1}\big|f(\{q_i\}) - x\big|^p + \left(r_j^{-1} + r_k^{-1}\right)\big|x - f(\{q_j, q_k\})\big|^p\right\} \\ &= \left(\left(\frac{1}{r_j} + \frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}\right)^{1-p} \big|f(\{q_i\}) - f(\{q_j, q_k\})\big|^p.\end{aligned}$$

Thus we have

$$\underline{\rho}_{V_0/\mathcal{J}} = \mathcal{T}E_i'(f)/E_i'(f) = \left(\left(\frac{1}{r_j}+\frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}\right)^{1-p}.$$

Corollary 8.5. *Assume $r_i \geq r_j \geq r_k$ for $\{i,j,k\} = \{1,2,3\}$.*

(a). If $r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}} < \left(\frac{1}{r_j}+\frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}$, then there does not exist a non-degenerate p-energy eigenform of $\mathcal{T}$.

(b). If $r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}} > \left(\frac{1}{r_j}+\frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}$, then there does exist a non-degenerate p-energy eigenform of $\mathcal{T}$.

Proof. One can easily check that $\overline{\rho}_{\mathcal{J}_i} = \underline{\rho}_{\mathcal{J}_i} \geq \overline{\rho}_{\mathcal{J}_j} = \underline{\rho}_{\mathcal{J}_j} \geq \overline{\rho}_{\mathcal{J}_k} = \underline{\rho}_{\mathcal{J}_k}$ and $\underline{\rho}_{V_0/\mathcal{J}_i} \leq \underline{\rho}_{V_0/\mathcal{J}_j} \leq \underline{\rho}_{V_0/\mathcal{J}_k}$. So to apply Theorem 8.4, it suffices to compare $\overline{\rho}_{\mathcal{J}_i}$ and $\underline{\rho}_{V_0/\mathcal{J}_i}$.

(a). If $\underline{\rho}_{\mathcal{J}_i} = \left(r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}}\right)^{1-p} > \left(\left(\frac{1}{r_j}+\frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}\right)^{1-p} = \underline{\rho}_{V_0/\mathcal{J}_i}$. Then by Theorem 8.4 (a), there does not exist a non-degenerate p-energy eigenform of $\mathcal{T}$.

(b). If $\overline{\rho}_{\mathcal{J}_i} = \left(r_j^{\frac{1}{p-1}} + r_k^{\frac{1}{p-1}}\right)^{1-p} < \left(\left(\frac{1}{r_j}+\frac{1}{r_k}\right)^{\frac{1}{1-p}} + r_i^{\frac{1}{p-1}}\right)^{1-p} = \underline{\rho}_{V_0/\mathcal{J}_i}$. Then it also holds that $\overline{\rho}_{\mathcal{J}_j} < \underline{\rho}_{V_0/\mathcal{J}_j}$ and $\overline{\rho}_{\mathcal{J}_k} < \underline{\rho}_{V_0/\mathcal{J}_k}$. Hence by Theorem 8.4 (b), there exists a non-degenerate p-energy eigenform of $\mathcal{T}$. □

8.2. **Non-preserved $\mathscr{G}$-relations.** In the rest of this section, we assume that K is $\mathscr{G}$-symmetric, for some finite group of homeomorphism $\mathscr{G}$, in the sense that introduced in Section 3 (page 9). Let $\boldsymbol{r}$ be $\mathscr{G}$-symmetric, so that for any $\mathscr{G}$-symmetric p-energy $E \in \mathcal{M}$, $\mathcal{T}^n E$ is also $\mathscr{G}$-symmetric, $n \geq 1$. Then if $\mathcal{J}$ is not a $\mathscr{G}$-relation, it is straightforward to see that $\delta_{\mathcal{J}}(\mathcal{T}^n E) \geq 1$. Hence, to verify (**A**), we only need to care about $\mathscr{G}$-relations. In this subsection, we further show that we do not need to worry about non-preserved $\mathscr{G}$-relations (Proposition 8.7 (b)).

Lemma 8.6. *Let $E, E' \in \mathcal{M}$ and R, R' be the p-resistances associated with E, E' respectively. Then*

$$\inf(E|E') \leq \min_{x\neq y} \frac{R'(x,y)}{R(x,y)} \leq \max_{x\neq y} \frac{R'(x,y)}{R(x,y)} \leq \sup(E|E').$$

Proof. We pick $x_0 \neq y_0$ so that $\frac{R'(x_0,y_0)}{R(x_0,y_0)} = \max_{x\neq y} \frac{R'(x,y)}{R(x,y)}$, and let f be a function such that $f(x_0) = 1$, $f(y_0) = 0$ and $E'(f) = R'(x_0,y_0)^{-1}$. Noticing that $E(f) \geq R(x_0,y_0)^{-1}$, hence

$$\sup(E|E') \geq \frac{E(f)}{E'(f)} \geq \frac{R'(x_0,y_0)}{R(x_0,y_0)} = \max_{x\neq y} \frac{R'(x,y)}{R(x,y)}.$$

Similarly, we have $\inf(E|E') \leq \min_{x\neq y} \frac{R'(x,y)}{R(x,y)}$. □

Recall that for each $E \in \mathcal{M}$, $\theta(E) = \frac{\sup(\mathcal{T}E|E)}{\inf(\mathcal{T}E|E)}$, see Definition 3.3 (b).

Proposition 8.7. *Let $\mathcal{J}$ be a non-preserved equivalence relation on V_0.*

(a). There is a constant $C > 0$ such that $\delta_{\mathcal{J}^{(1)}}(\Lambda E) \leq C\delta_{\mathcal{J}}(E)$ for any $E \in \mathcal{M}$.

(b). Assume that $\mathcal{T}\mathcal{J} \neq \mathcal{J}$. Then there is a constant $C' > 0$ such that $\theta(E) \geq C'\delta_{\mathcal{J}}(E)^{-1}$ for any $E \in \mathcal{M}$.

Proof. Let $E\in\mathcal{M}$ and R be the p-resistance associated with E. For convenience, we denote R_1 the p-resistance associated with ΛE.

First, fix $x,y\in V_1$, and assume $x\mathcal{J}^{(1)}y$. We can find a path $x=z_0,z_1,z_2,\cdots,z_n=y$ such that $z_i\mathcal{J}^{(1)}z_{i+1}$ for each $0\leq i<n$. Then $n\leq \#V_1$ and $R_1(z_i,z_{i+1})\leq(\max_{1\leq i\leq N}r_i)\cdot(\max_{x'\mathcal{J}y'}R(x',y'))$. Hence, there exists $C_1>0$ independent of E,x,y such that

$$R_1(x,y)\leq C_1\cdot\max_{x'\mathcal{J}y'}R(x',y'). \tag{8.1}$$

Next, fix $x,y\in V_1$, and assume $x\not\mathcal{J}^{(1)}y$. Then, as in Corollary 2.7, we define $\bar{E}$ as

$$\bar{E}(f)=\sum_{x',y'\in V_0}R(x',y')^{-1}|f(x')-f(y')|^p,\quad\forall f\in l(V_0).$$

By Corollary 2.7, we know $E\leq C_2\bar{E}$ for some $C_2>0$ independent of E. Let $I^{(1)}_x\in V_1/\mathcal{J}^{(1)}$ be such that $x\in I^{(1)}_x$, and let $1_{I^{(1)}_x}\in l(V_1)$ be the indicator function of $I^{(1)}_x$. Then,

$$R_1(x,y)^{-1}\leq\Lambda E(1_{I^{(1)}_x})\leq C_2\Lambda\bar{E}(1_{I^{(1)}_x})\leq C_3\big(\min_{x'\not\mathcal{J}y'}R(x',y')\big)^{-1}, \tag{8.2}$$

for some $C_3>0$ independent of E,x,y.

(a) follows immediately by combining (8.2) with (8.1).

(b). From now on, we let $x,y\in V_0$. Then, by the definition, $x\mathcal{T}\mathcal{J}y$ if and only if $x\mathcal{J}^{(1)}y$. We will consider two different cases.

Case 1: $\mathcal{T}\mathcal{J}\not\subset\mathcal{J}$. In this case, we can find $x,y\in V_0$ such that $x\mathcal{J}^{(1)}y$ and $x\not\mathcal{J}y$.

On one hand, by (8.1), $R_1(x,y)\leq C_1\cdot\max_{x'\mathcal{J}y'}R(x',y')\leq C_1\delta_{\mathcal{J}}(E)R(x,y)$. Hence, by Lemma 8.6, we have $\sup(\mathcal{T}E|E)\geq C_1^{-1}\delta_{\mathcal{J}}(E)^{-1}$.

On the other hand, by letting $E'\in\mathcal{M}$ be defined by $E'(f)=\frac{1}{2}\sum_{x\neq y}\big|f(x)-f(y)\big|^p$ for $f\in l(V_0)$ and $C_4=\sup(\mathcal{T}E'|E')>0$, using Lemma 3.4 (c), we also have $\inf(\mathcal{T}E|E)\leq C_4$.

Thus, $\theta(E)\geq C_1^{-1}C_4^{-1}\cdot\delta_{\mathcal{J}}(E)^{-1}$, where C_1,C_4 are independent of E.

Case 2: $\mathcal{J}\not\subset\mathcal{T}\mathcal{J}$. In this case, there exists a pair $x\neq y\in V_0$ such that $x\mathcal{J}y$ and $x\not\mathcal{J}^{(1)}y$.

Then, by (8.2), $R_1(x,y)\geq C_3^{-1}\min_{x'\not\mathcal{J}y'}R(x',y')\geq C_3^{-1}\delta_{\mathcal{J}}(E)^{-1}R(x,y)$. Hence, by Lemma 8.6, we know $\inf(\mathcal{T}E|E)\leq C_3\delta_{\mathcal{J}}(E)$.

On the other hand, by letting E' be the same as in Case 1, and $C_5=\inf(\mathcal{T}E'|E')>0$, using Lemma 3.4 (c), we also have $\sup(\mathcal{T}E|E)\geq C_5$.

Thus, $\theta(E)\geq C_5C_3^{-1}\cdot\delta_{\mathcal{J}}(E)^{-1}$, where C_3,C_5 are independent of E.

Combining above two cases, we finish the proof by choosing $C'=\min\{C_1^{-1}C_4^{-1},C_5C_3^{-1}\}$. □

Remark. Noting that by Lemma 3.4 (b), we have $\theta(\mathcal{T}^nE)\leq\theta(E),\forall\ n\geq 0$. Together with Proposition 8.7 (b), we see that if $\mathcal{J}$ is non-preserved, then $\inf_{n\geq 0}\delta_{\mathcal{J}}(\mathcal{T}^nE)\geq C\theta(E)^{-1}$ for some $C>0$. In particular, by the remark after Lemma 7.3 (note that the equivalence relation $\mathcal{J}$ constructed in Lemma 7.3 is $\mathscr{G}$-symmetric provided that K and E are $\mathscr{G}$-symmetric), we can see that Theorem 8.4 (b) holds if there does not exist any non-trivial preserved $\mathscr{G}$-relations.

Before ending this subsection, we point out an easy consequence of Lemma 8.6.

Lemma 8.8. *Let $\mathcal{J}$ be a non-trivial equivalence relation on V_0. Then for any $E, E' \in \mathcal{M}$ such that $C_1 E \leq E' \leq C_2 E$ for some constant $C_2 \geq C_1 > 0$, we have*

$$\frac{C_2}{C_1}\delta_{\mathcal{J}}(E) \geq \delta_{\mathcal{J}}(E') \geq \frac{C_1}{C_2}\delta_{\mathcal{J}}(E).$$

As a consequence, $\delta_{\mathcal{J}}(\mathcal{T}E) \geq \theta(E)^{-1}\delta_{\mathcal{J}}(E)$.

Proof. We denote R and R' to be the p-resistances associated with E and E' respectively. Then, by Lemma 8.6, we have $R'(x,y) \geq C_2^{-1}R(x,y)$ for any $x \neq y$, hence

$$\max_{x\mathcal{J}y} R'(x,y) \geq C_2^{-1} \max_{x\mathcal{J}y} R(x,y);$$

by Lemma 8.6, we have $R'(x,y) \leq C_1^{-1}R(x,y)$ for any $x \neq y$, hence

$$\min_{x\not\mathcal{J}y} R'(x,y) \leq C_1^{-1} \min_{x\not\mathcal{J}y} R(x,y).$$

Whence

$$\delta_{\mathcal{J}}(E') = \frac{\max_{x\mathcal{J}y} R'(x,y)}{\min_{x\not\mathcal{J}y} R'(x,y)} \geq \frac{C_2^{-1}\max_{x\mathcal{J}y} R(x,y)}{C_1^{-1}\min_{x\not\mathcal{J}y} R(x,y)} = \frac{C_1}{C_2}\delta_{\mathcal{J}}(E).$$

Similarly, we have $\frac{C_2}{C_1}\delta_{\mathcal{J}}(E) \geq \delta_{\mathcal{J}}(E')$. Finally, $\delta_{\mathcal{J}}(\mathcal{T}E) \geq \theta(E)^{-1}\delta_{\mathcal{J}}(E)$ is due to the facts $\inf(\mathcal{T}E|E)\cdot E \leq \mathcal{T}E \leq \sup(\mathcal{T}E|E)\cdot E$ and $\theta(E) = \frac{\sup(\mathcal{T}E|E)}{\inf(\mathcal{T}E|E)}$. □

8.3. **Preserved $\mathscr{G}$-relations.** In this subsection, we consider preserved non-trivial $\mathscr{G}$-relations. We will use Proposition 7.7 in this part.

Definition 8.9. *Let $\mathcal{J}$ be a non-trivial preserved $\mathscr{G}$-relation and $E \in \mathcal{M}(V_0,\mathcal{J},\delta)$ for some $\delta > 0$ which admits the decomposition (7.7) that is*

$$E(f) = \sum_{I\in V_0/\mathcal{J}} E_I(f|_I) + \sum_{i=1}^m E_i(f), \qquad \text{for any } f \in l(V_0),$$

where $E_I \in \mathcal{M}(I)$ and $E_i \in \mathcal{M}$ such that $R_i(x,y) = \infty$ for any $x\mathcal{J}y$, $x \neq y$. We define

$$\tilde{\mathcal{T}}_{\mathcal{J}}E(f) = \mathcal{T}\Big(\sum_{I\in V_0/\mathcal{J}} E_I\Big)(f) + \mathcal{T}_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})\Big(\sum_{I\in V_0/\mathcal{J}} f(x_I)1_I\Big), \qquad \text{for any } f \in l(V_0),$$

where we fix $x_I \in I$ for each $I \in V_0/\mathcal{J}$.

To help readers better understand the definition, we discuss some simple properties of $\tilde{\mathcal{T}}_{\mathcal{J}}E$ in the following lemma. In the following, for an equivalence relation $\mathcal{J}$ on V_0, we use I to denote an equivalence class in $V_0/\mathcal{J}$, and use I' to denote an equivalence class in $V_1/\mathcal{J}^{(1)}$. Moreover, I'_I is the unique equivalence relation in $V_1/\mathcal{J}^{(1)}$ such that $I'_I \supset I$ for each $I \in V_0/\mathcal{J}$.

Lemma 8.10. *Let $\mathcal{J}$ be a preserved $\mathscr{G}$-relation and $E \in \mathcal{M}(V_0,\mathcal{J},\delta)$ for some $\delta > 0$.*

(a). For each $I' \in \mathcal{J}^{(1)}$, we let

$$E'_{I'}(f|_{I'}) = \sum_{1\leq j\leq N, I\in V_0/\mathcal{J}: F_jI\subset I'} r_j^{-1}E_I(f\circ F_j|_I), \qquad \forall f \in l(V_1).$$

Then, $\Lambda(\sum_{I\in V_0/\mathcal{J}} E_I)=\sum_{I'\in V_1/\mathcal{J}^{(1)}} E'_{I'}$*. Moreover,* ΛE *has the decomposition of type (7.7):*

$$\Lambda E(f)=\sum_{I'\in V_1/\mathcal{J}^{(1)}} E'_{I'}(f|_{I'})+\sum_{i=1}^m\sum_{j=1}^N r_j^{-1}E_i(f\circ F_j),\quad \text{for any } f\in l(V_1).$$

(b). Taking the decomposition in (a), then $\tilde{\mathcal{T}}_{\mathcal{J}}E=<\Lambda E>_{V_0}$*, where* $<\Lambda E>_{V_0}$ *is the modified trace defined in Definition 7.5 (c). Moreover,* $\tilde{\mathcal{T}}_{\mathcal{J}}E$ *takes the natural decomposition*

$$\tilde{\mathcal{T}}_{\mathcal{J}}E(f)=\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I)+[[[\Lambda E]]_{V_1/\mathcal{J}^{(1)}}]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}} f(x_I)1_I),\ \forall f\in l(V_0).$$

In particular, $\mathcal{T}\big(\sum_{I\in V_0/\mathcal{J}}E_I\big)(f)=\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I)$*.*

(c). For $n\geq 1$*, define* $\tilde{\mathcal{T}}^n_{\mathcal{J}}$ *by*

$$\tilde{\mathcal{T}}^n_{\mathcal{J}}E(f)=\mathcal{T}^n\big(\sum_{I\in V_0/\mathcal{J}}E_I\big)(f)+\mathcal{T}^n_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})(\sum_{I\in V_0/\mathcal{J}} f(x_I)1_I),\quad \text{for any } f\in l(V_0).$$

Then $\tilde{\mathcal{T}}^n_{\mathcal{J}}$ *can be viewed as the n-times composition of* $\tilde{\mathcal{T}}_{\mathcal{J}}$*, with the natural decomposition of* $\tilde{\mathcal{T}}_{\mathcal{J}}E$*.*

(d). For any $\varepsilon>0$*, there is* $\delta>0$ *so that*

$$(1+\varepsilon)^{-1}\cdot\tilde{\mathcal{T}}_{\mathcal{J}}E\leq\mathcal{T}E\leq(1+\varepsilon)\cdot\tilde{\mathcal{T}}_{\mathcal{J}}E$$

for any preserved $\mathscr{G}$*-relation* $\mathcal{J}$ *and* $E\in\mathcal{M}(V_0,\mathcal{J},\delta)$*.*

Proof. (a). For convenience, we view E_I,E'_I as defined on V_0 or V_1. Since Λ is linear, we have the immediate decomposition

$$\Lambda E(f)=\Lambda(\sum_{I\in V_0/\mathcal{J}}E_I)(f)+\sum_{i=1}^m\Lambda E_i(f),\qquad \forall f\in l(V_1).$$

The first term on the right side is exactly

$$\Lambda(\sum_{I\in V_0/\mathcal{J}}E_I)(f)=\sum_{j=1}^N r_j^{-1}\sum_{I\in V_0/\mathcal{J}}E_I(f\circ F_j)=\sum_{I'\in V_1/\mathcal{J}^{(1)}}E'_{I'}(f),\qquad \forall f\in l(V_1),$$

and the second term is just $\sum_{i=1}^m\sum_{j=1}^N r_j^{-1}E_i(f\circ F_j)$.

(b). Recalling Definition 7.5 (c), for any $f\in l(V_0)$, we have

$$\begin{aligned}\tilde{\mathcal{T}}_{\mathcal{J}}E(f)&=\mathcal{T}\big(\sum_{I\in V_0/\mathcal{J}}E_I\big)(f)+\mathcal{T}_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I)\\&=[\Lambda(\sum_{I\in V_0/\mathcal{J}}E_I)]_{V_0}(f)+[\Lambda_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I).\end{aligned}$$

Then (b) will follow from the two identities below.

$$[\Lambda(\sum_{I\in V_0/\mathcal{J}}E_I)]_{V_0}(f)=\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I),\tag{8.3}$$

$$[\Lambda_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I)=[[[\Lambda E]]_{V_1/\mathcal{J}^{(1)}}]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I).\tag{8.4}$$

For the first equation (8.3), from (a), we first have that $\Lambda(\sum_{I\in V_0/\mathcal{J}} E_I)=\sum_{I'\in V_1/\mathcal{J}^{(1)}} E'_{I'}$. Then we can find an extension $f'\in l(V_1)$ of f such that $f'|_{V_1\setminus(\bigcup_{I\in V_0/\mathcal{J}} I'_I)}\equiv 0$ and

$$E'_{I'_I}(f'|_{I'_I})=[E'_{I'_I}]_I(f|_I),\qquad \forall I\in V_0/\mathcal{J},$$

which can be achieved by taking the minimal energy extension on each I'_I. Hence,

$$\begin{aligned}\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I)&=\sum_{I'\in V_1/\mathcal{J}^{(1)}}E'_{I'}(f')\geq[\sum_{I'\in V_1/\mathcal{J}^{(1)}}E'_{I'}]_{V_0}(f)\\&=[\Lambda(\sum_{I\in V_0/\mathcal{J}}E_I)]_{V_0}(f)\geq\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I),\end{aligned}$$

and (8.3) follows.

The second equation (8.4) follows from the fact that

$$\Lambda_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})(g)=\Lambda E(g)=[[\Lambda E]]_{V_1/\mathcal{J}^{(1)}}(g)$$

for any $g\in l(V_1)$ of the form $g=\sum_{I'\in V_1/\mathcal{J}^{(1)}} g(I')1_{I'}$.

(c). From (b), we know that

$$\mathcal{T}\big(\sum_{I\in V_0/\mathcal{J}}E_I\big)(f)=\sum_{I\in V_0/\mathcal{J}}[E'_{I'_I}]_I(f|_I),\quad \forall f\in l(V_0).$$

Hence, there is a natural decomposition of $\tilde{\mathcal{T}}_{\mathcal{J}}E$ of type (7.7). In addition, by the definition of $\tilde{\mathcal{T}}_{\mathcal{J}}E$, we can see

$$\tilde{\mathcal{T}}_{\mathcal{J}}E(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I)=[[\tilde{\mathcal{T}}_{\mathcal{J}}E]]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I)=\mathcal{T}_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I).$$

Inductively, we have

$$\tilde{\mathcal{T}}^{n+1}_{\mathcal{J}}E(f)=\tilde{\mathcal{T}}^{n}_{\mathcal{J}}(\tilde{\mathcal{T}}_{\mathcal{J}}E)(f)=\mathcal{T}^n\big(\mathcal{T}(\sum_{I\in V_0/\mathcal{J}}E_I)\big)(f)+\mathcal{T}^n_{V_0/\mathcal{J}}(\mathcal{T}_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}}))(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I).$$

(d). By definition, $\mathcal{T}E=[\Lambda E]_{V_0}$; by (b), $\tilde{\mathcal{T}}_{\mathcal{J}}E=<\Lambda E>_{V_0}$. The lemma follows immediately from Proposition 7.7, noticing that $\delta_{\mathcal{J}^{(1)}}(\Lambda E)<C\delta$ for some constant $C>0$ by Proposition 8.7 (a). □

Remark. Let $E\in\mathcal{M}(V_0,\mathcal{J},\delta)$ for some $\delta>0$. Then E admits the standard decomposition $E(f)=\sum_{I\in V_0/\mathcal{J}}E_I(f|_I)+\sum_{i=1}^m E_i(f),\forall f\in l(V_0)$. For convenience, we write $E_{\mathcal{J}}(f)=\sum_{I\in V_0/\mathcal{J}}E_I(f|_I),\forall f\in l(V_0)$. Then by Lemma 8.10(c), we immediately have the following two claims.

1) If $f\in l(V_0)$ satisfies $f(x_I)=0,\forall I\in V_0/\mathcal{J}$, then $\tilde{\mathcal{T}}^n_{\mathcal{J}}E(f)=\mathcal{T}^nE_{\mathcal{J}}(f),\forall n\geq 1$.

2) If $f\in l(V_0)$ satisfies $f(x)=\sum\limits_{I\in V_0/\mathcal{J}}f(x_I)1_I$, then $\tilde{\mathcal{T}}^n_{\mathcal{J}}E(f)=\mathcal{T}^n_{V_0/\mathcal{J}}([[E]]_{V_0/\mathcal{J}})(f),\forall n\geq 1$.

Before proceeding to the proof of the main result of this subsection, we point out that as an easy application of Corollary 2.7, Proposition 7.7 and Lemma 8.8, for any $E\in\mathcal{M}(V_0)$, we can replace it with an $\hat{E}\in\mathcal{M}\big(V_0,\mathcal{J},C^2\delta_{\mathcal{J}}(E)\big)$ with comparable energy and good decomposition.

Corollary 8.11. *There exist $C>1$ depending on $\#V_0$, p and small $\delta>0$, such that for any $E\in\mathcal{M}(V_0)$ and any non-trivial equivalence relation $\mathcal{J}$, we can find $\hat{E}\in\mathcal{M}\big(V_0,\mathcal{J},C^2\delta_{\mathcal{J}}(E)\big)$ of the form*

$$\hat{E}(f)=\sum_{I\in V_0/\mathcal{J}}\hat{E}_I(f|_I)+[[\hat{E}]]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I),\qquad \text{for any } f\in l(V_0),\tag{8.5}$$

such that

$$C^{-1}E\leq\hat{E}\leq CE.$$

Proof. First, we apply Corollary 2.7 to find $\bar{E}\in\mathcal{S}$ such that $C_1^{-1}E\leq\bar{E}\leq C_1E$ for some C_1 depending only on $\#V_0$, whence $\bar{E}\in\mathcal{M}_p(V_0,\mathcal{J},C_1^2\delta)$ by Lemma 8.8. Then, we let $\hat{E}=<\bar{E}>_{V_0}$. By Proposition 7.7, $C_2^{-1}\bar{E}\leq\hat{E}\leq C_2\bar{E}$ for some $C_2>0$ depending only on C_1,δ and $\#V_0$, whence $C_1^{-1}C_2^{-1}E\leq\hat{E}\leq C_1C_2E$, and by Lemma 8.8 $\hat{E}\in\mathcal{M}\big(V_0,\mathcal{J},C_1^2C_2^2\delta_{\mathcal{J}}(E)\big)$. □

Remark. Let $E^{(p)}\in\mathcal{M}_p(V_0)$, and $E',E''\in\mathcal{M}_p\big(V_0,\mathcal{J},\delta\big)$ for some $\delta>0$ of the form

$$E'(f)=\sum_{I\in V_0/\mathcal{J}}E'_I(f|_I)+[[E']]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I),$$
$$E''(f)=\sum_{I\in V_0/\mathcal{J}}E''_I(f|_I)+[[E'']]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I).$$

Assume $E'\leq E''$, then it is straightforward to check

$$E'_I\leq E''_I,\ \forall I\in V_0/\mathcal{J},\qquad [[E']]_{V_0/\mathcal{J}}\leq[[E'']]_{V_0/\mathcal{J}}.$$

Hence, $\tilde{\mathcal{T}}_{\mathcal{J}}E'\leq\tilde{\mathcal{T}}_{\mathcal{J}}E''$.

The main result of this subsection is the following Proposition 8.12. For the special $p=2$ case, the proposition can be easily proved by using Lemma 8.10 (d). For general p, additional difficulty comes from the fact that the trace of $\mathcal{T}E$ may not be a good form even if $E\in\mathcal{Q}$.

Proposition 8.12. *For any $\varepsilon>0$, there is $\delta>0$ and a constant $C\geq1$ so that for any preserved non-trivial $\mathscr{G}$-relation $\mathcal{J}$ and $E\in\mathcal{M}$, if $n\geq1$ and $\delta_{\mathcal{J}}(\mathcal{T}^mE)<\delta$, $\forall\ 0\leq m\leq n$, then the following (a) and (b) hold.*

(a). There exists $f\in l(V_0)$ such that $Osc(f)=1$, $f(x_I)=0,\forall\ I\in V_0/\mathcal{J}$ and

$$\mathcal{T}^nE(f)\leq C^2(1+\varepsilon)^n\bar{\rho}^n_{\mathcal{J}}E(f).$$

(b). There exists $f\in l(V_0)$ such that $Osc(f)=1$, $f(x)=\sum_{I\in V_0/\mathcal{J}}f(x_I)1_I$ and

$$\mathcal{T}^nE(f)\geq C^{-2}(1+\varepsilon)^{-n}\underline{\rho}^n_{V_0/\mathcal{J}}E(f).$$

Proof. Let $n\in\mathbb{N}$ and $\varepsilon>0$ be the same as in the statement of the proposition. E satisfies the condition of Proposition 8.12, with the constant δ chosen as follows.

Choice of δ: By Lemma 8.10 (d), we can choose $\delta_0>0$ such that whenever $\hat{E}\in\mathcal{M}(V_0,\mathcal{J},\delta_0)$, it holds

$$(1+\varepsilon/2)^{-1}\tilde{\mathcal{T}}_{\mathcal{J}}\hat{E}\leq\mathcal{T}\hat{E}\leq(1+\varepsilon/2)\tilde{\mathcal{T}}_{\mathcal{J}}\hat{E}.\tag{8.6}$$

We also require that δ_0 is small enough so that Corollary 8.11 holds, and we let C be the same constant in Corollary 8.11 (for the fixed δ_0). We choose a large enough $n_0\in\mathbb{N}$ so that

$$C^2\cdot(1+\varepsilon/2)^{n_0}\leq(1+\varepsilon)^{n_0}.$$

We choose $\delta = C^{-2} \cdot (1+\varepsilon/2)^{-2n_0} \cdot \delta_0$.

Claim 1. Let $\hat{E} \in \mathcal{M}(V_0, \mathcal{J}, C^2\delta)$ and assume that $\delta(\mathcal{T}^m \hat{E}) \leq C^2\delta, \forall 0 \leq m < n$. Then for any $0 \leq m \leq (n_0 \wedge n)$, we have

$$(1+\varepsilon/2)^{-m} \tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E} \leq \mathcal{T}^m \hat{E} \leq (1+\varepsilon/2)^m \tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E}. \tag{8.7}$$

Proof of Claim 1. We will use induction. The case $m = 0$ is trivial. If (8.7) holds for some $0 \leq m \leq (n_0 \wedge n) - 1$, i.e.

$$(1+\varepsilon/2)^{-m} \tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E} \leq \mathcal{T}^m \hat{E} \leq (1+\varepsilon/2)^m \tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E}, \tag{8.8}$$

then consequently,

$$\delta_{\mathcal{J}}(\tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E}) \leq (1+\varepsilon/2)^{2m} \delta_{\mathcal{J}}(\mathcal{T}^m \hat{E}) \leq \delta_0. \tag{8.9}$$

Hence by (8.9) and applying (8.6) to $\tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E}$, we have

$$(1+\varepsilon/2)^{-1} \tilde{\mathcal{T}}_{\mathcal{J}}^{m+1} \hat{E} \leq \mathcal{T} \tilde{\mathcal{T}}_{\mathcal{J}}^m \hat{E} \leq (1+\varepsilon/2) \tilde{\mathcal{T}}_{\mathcal{J}}^{m+1} \hat{E}. \tag{8.10}$$

Now by acting $\mathcal{T}$ to (8.8) and comparing with (8.10), we obtain

$$(1+\varepsilon/2)^{-m-1} \tilde{\mathcal{T}}_{\mathcal{J}}^{m+1} \hat{E} \leq \mathcal{T}^{m+1} \hat{E} \leq (1+\varepsilon/2)^{m+1} \tilde{\mathcal{T}}_{\mathcal{J}}^{m+1} \hat{E}.$$

The induction is completed and hence Claim 1 holds. □

Now, we return to the setting of the question. Write $n = kn_0 + s$ for some $k \geq 0$ and $0 < s \leq n_0$, and we want to iterate Claim 1. The idea is to apply Claim 1 to $\mathcal{T}^{n_0 l} \hat{E}$ for each $0 \leq l \leq k$: applying Corollary 8.11, we can find $\hat{E}_l \in \mathcal{M}\big(V_0, \mathcal{J}, C^2 \delta_{\mathcal{J}}(\mathcal{T}^{n_0 l} E)\big)$ independent of $\mathcal{J}$, so that

$$C^{-1} \mathcal{T}^{n_0 l} E \leq \hat{E}_l \leq C \mathcal{T}^{n_0 l} E, \qquad \forall 0 \leq l \leq k. \tag{8.11}$$

Noticing that $\delta_{\mathcal{J}}(\mathcal{T}^m E) < \delta, \forall 0 \leq m < n$ as assumed in the proposition, and by acting $\mathcal{T}^m$ to (8.11), we always have

$$\delta_{\mathcal{J}}(\mathcal{T}^m \hat{E}_l) \leq C^2 \delta_{\mathcal{J}}(\mathcal{T}^{m+n_0 l} E) < C^2 \delta, \quad \forall 0 \leq m < n - n_0 l. \tag{8.12}$$

Then we apply Claim 1 to prove the following Claim 2.

Claim 2. $C^{-1}(1+\varepsilon)^{-n} \tilde{\mathcal{T}}_{\mathcal{J}}^n \hat{E}_0 \leq \mathcal{T}^n E \leq C(1+\varepsilon)^n \tilde{\mathcal{T}}_{\mathcal{J}}^n \hat{E}_0$.

Proof of Claim 2. By acting $\mathcal{T}^{n_0}$ to (8.11) with l replaced by $l-1$, we have

$$C^{-1} \mathcal{T}^{n_0} \mathcal{T}^{n_0(l-1)} E \leq \mathcal{T}^{n_0} \hat{E}_{l-1} \leq C \mathcal{T}^{n_0} \mathcal{T}^{n_0(l-1)} E, \qquad \forall 1 \leq l \leq k,$$

together with (8.11), we obtain

$$C^{-2} \mathcal{T}^{n_0} \hat{E}_{l-1} \leq \hat{E}_l \leq C^2 \mathcal{T}^{n_0} \hat{E}_{l-1}, \qquad \forall 1 \leq l \leq k.$$

Then by applying Claim 1 to $\hat{E}_{l-1}$ (noticing (8.12)), we see

$$\eta^{-1} \tilde{\mathcal{T}}_{\mathcal{J}}^{n_0} \hat{E}_{l-1} \leq C^{-2} \mathcal{T}^{n_0} \hat{E}_{l-1} \leq \hat{E}_l \leq C^2 \mathcal{T}^{n_0} \hat{E}_{l-1} \leq \eta \tilde{\mathcal{T}}_{\mathcal{J}}^{n_0} \hat{E}_{l-1}, \qquad \forall 1 \leq l \leq k, \tag{8.13}$$

where, for short, we write $\eta = C^2(1+\varepsilon/2)^{n_0} \leq (1+\varepsilon)^{n_0}$. In this step, cautious readers may worry about the definition of $\tilde{\mathcal{T}}_{\mathcal{J}}$, which depends on the decomposition of $\tilde{\mathcal{T}}_{\mathcal{J}}^{n_0} \hat{E}_{l-1}$ and $\hat{E}_l$. Fortunately, since both $\hat{E}_l$ (define in Corollary 8.11) and $\tilde{\mathcal{T}}_{\mathcal{J}}^{n_0} \hat{E}_{l-1}$ take the decomposition of the form (8.5), a comparison of each component is taken care of with the remark after Corollary 8.11.

Next, one can iterate (8.13) to see

$$\eta^{-k}\tilde{\mathcal{T}}_{\mathcal{J}}^{kn_0}\hat{E}_0 \le \cdots \le \eta^{-2}\tilde{\mathcal{T}}_{\mathcal{J}}^{2n_0}\hat{E}_{k-2} \le \eta^{-1}\tilde{\mathcal{T}}_{\mathcal{J}}^{n_0}\hat{E}_{k-1} \le \hat{E}_k \le \eta\tilde{\mathcal{T}}_{\mathcal{J}}^{n_0}\hat{E}_{k-1} \le \eta^2\tilde{\mathcal{T}}_{\mathcal{J}}^{2n_0}\hat{E}_{k-2} \le \cdots \le \eta^k\tilde{\mathcal{T}}_{\mathcal{J}}^{kn_0}\hat{E}_0,$$

whence, by applying Claim 1 again, using (8.11), we have

$$\begin{aligned}C^{-1}\eta^{-k}(1+\varepsilon/2)^{-s}\tilde{\mathcal{T}}_{\mathcal{J}}^{kn_0+s}\hat{E}_0 &\le C^{-1}(1+\varepsilon/2)^{-s}\tilde{\mathcal{T}}_{\mathcal{J}}^{s}\hat{E}_k \le C^{-1}\mathcal{T}^s\hat{E}_k\\ &\le \mathcal{T}^s\mathcal{T}^{n_0k}E\\ &\le C\mathcal{T}^s\hat{E}_k \le C(1+\varepsilon/2)^s\tilde{\mathcal{T}}_{\mathcal{J}}^{s}\hat{E}_k \le C\eta^k(1+\varepsilon/2)^s\tilde{\mathcal{T}}_{\mathcal{J}}^{kn_0+s}\hat{E}_0.\end{aligned}$$

The Claim follows immediately since $\eta \le (1+\varepsilon)^{n_0}$ and $n = kn_0 + s$. □

The rest of the proof is routine by using Claim 2 and the remark after Lemma 8.10. Recall that $\hat{E}_0$ has the decomposition (8.5):

$$\hat{E}_0(f) = \sum_{I\in V_0/\mathcal{J}}\hat{E}_{0,I}(f|_I) + [[E_0]]_{V_0/\mathcal{J}}(\sum_{I\in V_0/\mathcal{J}} f(x_I)1_I), \qquad \text{for each } f\in l(V_0).$$

(a). As in the remark after Lemma 8.10, we use the notation $\hat{E}_{\mathcal{J}} = \sum_{I\in V_0/\mathcal{J}}\hat{E}_{0,I}$ for short. We first show that $\inf(\mathcal{T}^n\hat{E}_{\mathcal{J}}|\hat{E}_{\mathcal{J}}) \le \overline{\rho}_{\mathcal{J}}^n$.

Indeed, if it is not true, there is $\rho > \overline{\rho}_{\mathcal{J}}$ such that

$$\inf(\mathcal{T}^n\hat{E}_{\mathcal{J}}|\hat{E}_{\mathcal{J}}) > \rho^n, \tag{8.14}$$

and by $\rho > \overline{\rho}_{\mathcal{J}}$, there exists $E' \in \widetilde{\mathcal{M}}(V_0,\mathcal{J},0)$ such that

$$\sup(\mathcal{T}E'|E') < \rho. \tag{8.15}$$

Now by Lemma 3.4 (a) and using (8.15), we see that

$$\sup(\mathcal{T}^nE'|E') = \sup_f \prod_{i=0}^{n-1}\frac{\mathcal{T}^{i+1}E'(f)}{\mathcal{T}^iE'(f)} \le (\sup(\mathcal{T}E'|E'))^n < \rho^n. \tag{8.16}$$

Combining (8.14) and (8.16), we obtain $\inf(\mathcal{T}^n\hat{E}_{\mathcal{J}}|\hat{E}_{\mathcal{J}}) > \sup(\mathcal{T}^nE'|E')$, a contradiction with Lemma 3.4 (c). So we have $\inf(\mathcal{T}^n\hat{E}_{\mathcal{J}}|\hat{E}_{\mathcal{J}}) \le \overline{\rho}_{\mathcal{J}}^n$.

Hence there exists f satisfying $\mathrm{Osc}(f) = 1$, $f(x_I) = 0, \forall I \in V_0/\mathcal{J}$ such that $\mathcal{T}^n\hat{E}_{\mathcal{J}}(f) \le \overline{\rho}_{\mathcal{J}}^n\hat{E}_{\mathcal{J}}(f)$. Then by the second inequality in Claim 2 and the remark after Lemma 8.10, we have

$$\begin{aligned}\mathcal{T}^nE(f) &\le C(1+\varepsilon)^n\tilde{\mathcal{T}}_{\mathcal{J}}^n\hat{E}_0(f)\\ &= C(1+\varepsilon)^n\mathcal{T}^n\hat{E}_{\mathcal{J}}(f) \le C(1+\varepsilon)^n\overline{\rho}_{\mathcal{J}}^n\hat{E}_{\mathcal{J}}(f) \le C^2(1+\varepsilon)^n\overline{\rho}_{\mathcal{J}}^nE(f).\end{aligned}$$

(b). With the help of Lemma 8.3 (see also Lemma 3.4), we can show that

$$\sup(\mathcal{T}_{V_0/\mathcal{J}}^n([[\hat{E}_0]]_{V_0/\mathcal{J}})|[[\hat{E}_0]]_{V_0/\mathcal{J}}) \ge \underline{\rho}_{V_0/\mathcal{J}}^n$$

by a similar argument to (a). If it is not true, we can find $\rho < \underline{\rho}_{V_0/\mathcal{J}}$ such that

$$\sup(\mathcal{T}_{V_0/\mathcal{J}}^n([[\hat{E}_0]]_{V_0/\mathcal{J}})|[[\hat{E}_0]]_{V_0/\mathcal{J}}) < \rho^n, \tag{8.17}$$

and $E' \in \mathcal{M}(V_0/\mathcal{J})$ such that

$$\inf(\mathcal{T}_{V_0/\mathcal{J}}E'|E') > \rho. \tag{8.18}$$

Using Lemma 8.3 (a) and (8.18), we arrive at

$$\inf(\mathcal{T}^n_{V_0/\mathcal{J}}E'|E') \geq (\inf(\mathcal{T}_{V_0/\mathcal{J}}E'|E'))^n > \rho^n.$$

This together with (8.17) gives $\inf(\mathcal{T}^n_{V_0/\mathcal{J}}E'|E') > \sup(\mathcal{T}^n_{V_0/\mathcal{J}}([[\hat{E}_0]]_{V_0/\mathcal{J}})|[[\hat{E}_0]]_{V_0/\mathcal{J}})$, contradicts Lemma 8.3 (b). So we have $\sup(\mathcal{T}^n_{V_0/\mathcal{J}}([[\hat{E}_0]]_{V_0/\mathcal{J}})|[[\hat{E}_0]]_{V_0/\mathcal{J}}) \geq \underline{\rho}^n_{V_0/\mathcal{J}}$.

Hence there exists $f \in l(V_0)$ of the form $f(x) = \sum_{I\in V_0/\mathcal{J}} f(x_I)1_I$ with $\mathrm{Osc}(f) = 1$, such that $\mathcal{T}^n_{V_0/\mathcal{J}}([[\hat{E}_0]]_{V_0/\mathcal{J}})(f) \geq \underline{\rho}^n_{V_0/\mathcal{J}}\hat{E}_0(f)$. Then by the first inequality in Claim 2 and the remark after Lemma 8.10, we have

$$\begin{aligned}\mathcal{T}^n E(f) &\geq C^{-1}(1+\varepsilon)^{-n}\tilde{\mathcal{T}}^n_{\mathcal{J}}\hat{E}_0(f)\\ &= C^{-1}(1+\varepsilon)^{-n}\mathcal{T}^n_{V_0/\mathcal{J}}([[\hat{E}_0]]_{V_0/\mathcal{J}})(f)\\ &\geq C^{-1}(1+\varepsilon)^{-n}\underline{\rho}^n_{V_0/\mathcal{J}}\hat{E}_0(f) \geq C^{-2}(1+\varepsilon)^{-n}\underline{\rho}^n_{V_0/\mathcal{J}}E(f).\end{aligned}$$

□

8.4. **Proof of Theorem 8.4 (b).** We now use Proposition 8.12 and Proposition 8.7 (b) to finish the proof of Theorem 8.4.

Proof of Theorem 8.4 (b). Let $E_0 \in \mathcal{M}$ be $\mathscr{G}$-symmetric. Our goal is to prove that

$$\inf_{n\geq 0}\delta(\mathcal{T}^n E_0) > 0. \tag{8.19}$$

Then, by Theorem 4.2, $\mathcal{T}$ has a $\mathscr{G}$-symmetric eigenform in $\mathcal{Q}$.

By Lemma 7.3, to prove (8.19), it suffices to show

$$\inf_{n\geq 0}\delta_{\mathcal{J}}(\mathcal{T}^n E_0) > 0. \tag{8.20}$$

for any equivalence relation $\mathcal{J}$ on V_0. We only need to take care of all preserved $\mathscr{G}$-relations: if $\mathcal{J}$ is not preserved, then by Proposition 8.7 (b) and by Lemma 3.4 (b), we have $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) \geq C_1\theta(\mathcal{T}^n E_0)^{-1} \geq C_1\theta(E_0)^{-1}, \forall n \geq 0$ for some constant $C_1 > 0$; if $\mathcal{J}$ is not a $\mathscr{G}$-relation, there exist $\sigma \in \mathscr{G}$ and $x, y \in V_0$, such that $x\mathcal{J}y$ and $\sigma(x)\not\mathcal{J}\sigma(y)$, whence $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) \geq 1, \forall n \geq 0$, noticing that $\mathcal{T}^n E_0$ is always $\mathscr{G}$-symmetric.

In the following, we fix a non-trivial preserved $\mathscr{G}$-relation $\mathcal{J}$ on V_0, and verify (8.20). We split the proof into the following two claims.

In Claim 1, we use the assumption (**H**).

Claim 1. Let $f \in l(V_0)$ be such that $Osc(f) = 1$, $f(x_I) = 0, \forall\ I \in V_0/\mathcal{J}$; let $g \in l(V_0)$ be such that $Osc(g) = 1$, $g(x) = \sum_{I\in V_0/\mathcal{J}} g(x_I)1_I$ (where $x_I \in I$ is the same as in Definition 8.9). Then for any $n \geq 0$ such that $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) < 1$,

$$C_2^{-1}\delta_{\mathcal{J}}(\mathcal{T}^n E_0) \leq \frac{\mathcal{T}^n E_0(g)}{\mathcal{T}^n E_0(f)} \leq C_2\delta_{\mathcal{J}}(\mathcal{T}^n E_0),$$

where the constant $C_2 > 0$ depends only on $\#V_0$ and $\theta(E_0)$.

Proof of Claim 1. Assume that $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) < 1$ holds and we write R for the p-resistance associated with $\mathcal{T}^n E_0$ for short. The claim then follows immediately from the definition of

$\delta_{\mathcal{J}}$ if we can show

$$\begin{cases} \frac{C_3^{-1}}{\min_{x\not\mathcal{J}y} R(x,y)} \leq \mathcal{T}^n E_0(g) \leq \frac{C_3}{\min_{x\not\mathcal{J}y} R(x,y)}, \\ \frac{C_3^{-1}}{\max_{x\mathcal{J}y} R(x,y)} \leq \mathcal{T}^n E_0(f) \leq \frac{C_3}{\max_{x\mathcal{J}y} R(x,y)}, \end{cases} \tag{8.21}$$

for some constant $C_3 > 0$ depending only on $\#V_0$ and $\theta(E_0)$.

First, one can use Corollary 2.7 to show

$$\begin{cases} \frac{C_4^{-1}}{\max_{x\not\mathcal{J}y} R(x,y)} \leq \mathcal{T}^n E_0(g) \leq \frac{C_4}{\min_{x\not\mathcal{J}y} R(x,y)}, \\ \frac{C_4^{-1}}{\max_{x\mathcal{J}y} R(x,y)} \leq \mathcal{T}^n E_0(f) \leq \frac{C_4}{\min_{x\mathcal{J}y} R(x,y)}, \end{cases} \tag{8.22}$$

for some $C_4 > 0$ depending only on $\#V_0$. The first formula of (8.22) can be proved as follows:

$$\begin{aligned} \frac{C_5^{-1}}{\max_{x\not\mathcal{J}y} R(x,y)} \leq C_5^{-1} \sum_{x\not\mathcal{J}y} \frac{\big|g(x)-g(y)\big|^p}{R(x,y)} &\leq \mathcal{T}^n E_0(g) \\ \leq C_5 \sum_{x\not\mathcal{J}y} \frac{\big|g(x)-g(y)\big|^p}{R(x,y)} &\leq \frac{C_5 \#V_0(\#V_0-1)}{\min_{x\not\mathcal{J}y} R(x,y)}, \end{aligned}$$

where we use the fact that $\mathrm{Osc}(g) = 1$ in the first and the fourth inequalities, we use the fact $\mathrm{Osc}(g|_I) = 0, \forall I \in V_0/\mathcal{J}$ and use Corollary 2.7 in the second and the third inequalities, with $C_5 > 0$ being the constant in Corollary 2.7. Similarly, the second formula of (8.22) can be proved as follows:

$$\begin{aligned} \frac{2^{-p} C_5^{-1}}{\max_{x\mathcal{J}y} R(x,y)} \leq C_5^{-1} \sum_{x\mathcal{J}y} \frac{\big|f(x)-f(y)\big|^p}{R(x,y)} &\leq \mathcal{T}^n E_0(f) \\ \leq C_5 \sum_{x\neq y} \frac{\big|f(x)-f(y)\big|^p}{R(x,y)} &\leq \frac{C_5 \#V_0(\#V_0-1)}{\min_{x\mathcal{J}y} R(x,y)}, \end{aligned}$$

where we use the fact that there exist $x\mathcal{J}y$ such that $|f(x)-f(y)| \geq 1/2$ in the first inequality, the second and the third inequalities are due to Corollary 2.7, and in the forth inequality we use the fact that $R(x',y') \geq \min_{x\mathcal{J}y} R(x,y), \forall x' \neq y'$ (since $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) < 1$).

It remains to show that there exists $C_6 > 0$ depending only on $\#V_0$ and $\theta(E_0)$ such that

$$\frac{\min_{x\not\mathcal{J}y} R(x,y)}{\max_{x\not\mathcal{J}y} R(x,y)} \geq C_6, \qquad \frac{\min_{x\mathcal{J}y} R(x,y)}{\max_{x\mathcal{J}y} R(x,y)} \geq C_6. \tag{8.23}$$

In fact, by Lemma 7.3, if $\frac{\min_{x\not\mathcal{J}y} R(x,y)}{\max_{x\not\mathcal{J}y} R(x,y)}$ is small enough (precise meaning depends on $\#V_0$ as discussed in Lemma 7.3), then one can find $\mathcal{J}' \neq \mathcal{J}$ such that

$$\delta_{\mathcal{J}'}(\mathcal{T}^n E_0) < \Big(\frac{\min_{x\not\mathcal{J}y} R(x,y)}{\max_{x\not\mathcal{J}y} R(x,y)}\Big)^{\frac{2}{\#V_0(\#V_0-1)}} (\#V_0)^p. \tag{8.24}$$

In addition, by Lemma 7.2, we know that $\mathcal{J}' \supset \mathcal{J}$, whence by (**H**), we know $\mathcal{J}'$ is not preserved or not $\mathscr{G}$-symmetric. Then, as discussed earlier in the proof of the theorem (using Lemma 3.4(b) and Proposition 8.7 (b)), we have $\delta_{\mathcal{J}'}(\mathcal{T}^n E_0) \geq \min\{C_1 \theta(E_0)^{-1}, 1\}$. Combining the estimate with (8.24), the lower bound of $\frac{\min_{x\not\mathcal{J}y} R(x,y)}{\max_{x\not\mathcal{J}y} R(x,y)}$ in (8.23) is proved. The lower bound of $\frac{\min_{x\mathcal{J}y} R(x,y)}{\max_{x\mathcal{J}y} R(x,y)}$ follows from a same argument.

To conclude, (8.21) follows from (8.22) and (8.23), and the claim follows from (8.21). □

In the following Claim 2, we use Proposition 8.12 and the assumption $\overline{\rho}_{\mathcal{J}} < \underline{\rho}_{V_0/\mathcal{J}}$. We choose small $\varepsilon > 0$ so that

$$(1+\varepsilon)^{-2}\frac{\underline{\rho}_{V_0/\mathcal{J}}}{\overline{\rho}_{\mathcal{J}}} > 1,$$

and let δ be the same as in Proposition 8.12.

Claim 2. If $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) < \delta$ for some n, then there exists $n < n' < \infty$ such that
(1). $\delta_{\mathcal{J}}(\mathcal{T}^{n'} E_0) \geq \delta$.
(2). $\delta_{\mathcal{J}}(\mathcal{T}^m E_0) \geq C_7\delta_{\mathcal{J}}(\mathcal{T}^n E_0), \forall n \leq m < n'$, for some $0 < C_7 \leq 1$ independent of n, m, n'.

Proof of Claim 2. Assume that $\delta_{\mathcal{J}}(\mathcal{T}^n E_0) < \delta$ holds, and we let

$$n' = \min\{m \geq n : \delta_{\mathcal{J}}(\mathcal{T}^m E_0) \geq \delta\},$$

where we use the setting $\min\emptyset = \infty$. Then, by Proposition 8.12, for any finite $n \leq m < n'$, we can find $f_m \in l(V_0)$ such that $\mathrm{Osc}(f_m) = 1$, $f_m(x_I) = 0, \forall\, I \in V_0/\mathcal{J}$, and

$$\mathcal{T}^m E_0(f_m) \leq C_8(1+\varepsilon)^{m-n}\overline{\rho}_{\mathcal{J}}^{m-n}\mathcal{T}^n E_0(f_m);$$

we can find $g_m \in l(V_0)$ such that $\mathrm{Osc}(g_m) = 1$, $g_m(x) = \sum_{I\in V_0/\mathcal{J}} g_m(x_I)1_I$, and

$$\mathcal{T}^m E_0(g_m) \geq C_8^{-1}(1+\varepsilon)^{-(m-n)}\underline{\rho}_{V_0/\mathcal{J}}^{m-n}\mathcal{T}^n E_0(g_m),$$

for some constant $C_8 > 0$. Hence, by applying Claim 1, we see

$$\begin{aligned}\delta_{\mathcal{J}}(\mathcal{T}^m E_0) &\geq C_2^{-1}\frac{\mathcal{T}^m E_0(g_m)}{\mathcal{T}^m E_0(f_m)}\\ &\geq C_2^{-1}C_8^{-2}\left((1+\varepsilon)^{-2}\frac{\underline{\rho}_{V_0/\mathcal{J}}}{\overline{\rho}_{\mathcal{J}}}\right)^{m-n}\frac{\mathcal{T}^n E_0(g_m)}{\mathcal{T}^n E_0(f_m)}\\ &\geq C_2^{-2}C_8^{-2}\left((1+\varepsilon)^{-2}\frac{\underline{\rho}_{V_0/\mathcal{J}}}{\overline{\rho}_{\mathcal{J}}}\right)^{m-n}\delta_{\mathcal{J}}(\mathcal{T}^n E_0).\end{aligned}$$

Then (2) follows immediately since $(1+\varepsilon)^{-2}\frac{\underline{\rho}_{V_0/\mathcal{J}}}{\overline{\rho}_{\mathcal{J}}} > 1$ by the choice of ε; it also follows that $n' < \infty$, since otherwise $\lim\limits_{m\to\infty}\delta_{\mathcal{J}}(\mathcal{T}^m E_0) = \infty$, which is impossible. □

Finally, we are ready to prove (8.20) by showing that

$$\delta_{\mathcal{J}}(\mathcal{T}^k E_0) \geq C_7\theta(E_0)^{-1}\min\{\delta, \delta_{\mathcal{J}}(E_0)\}, \qquad \forall k \geq 0. \tag{8.25}$$

In fact, we can verify this for each $k \geq 1$: if $\delta_{\mathcal{J}}(\mathcal{T}^k E_0) \geq \delta$, there is nothing to prove; if $\delta_{\mathcal{J}}(\mathcal{T}^k E_0) < \delta$, one let $\tilde{k} = \min\{m :\ 1 \leq m \leq k,\ \delta_{\mathcal{J}}(\mathcal{T}^s E_0) < \delta,\ \forall m \leq s \leq k\}$, then either $\tilde{k} = 1$ or $\delta_{\mathcal{J}}(\mathcal{T}^{\tilde{k}-1}E_0) \geq \delta$, and hence

$$\delta_{\mathcal{J}}(\mathcal{T}^{\tilde{k}-1}E_0) \geq \min\{\delta, \delta_{\mathcal{J}}(E_0)\}.$$

Then by using Lemma 8.8 and Lemma 3.4(b), we obtain

$$\delta_{\mathcal{J}}(\mathcal{T}^{\tilde{k}}E_0) \geq \theta(\mathcal{T}^{\tilde{k}-1}E_0)^{-1}\delta_{\mathcal{J}}(\mathcal{T}^{\tilde{k}-1}E_0) \geq \theta(E_0)^{-1}\min\{\delta, \delta_{\mathcal{J}}(E_0)\}.$$

Using Claim 2 with $n = \tilde{k}$ and noticing that $k < n'$, we see (8.25) holds. □

Appendix A. Properties of $\mathcal{Q}_p(A)$

In this appendix, we consider some useful properties concerning $E^{(p)}\in\mathcal{Q}_p(A)$, where A is a finite set. We begin with a lemma concerning the norm $\|\cdot\|_{\widetilde{\mathcal{M}}_p(A)}$.

Lemma A.1. *Let $E_n^{(p)}\in\widetilde{\mathcal{M}}_p(A),n\geq 1$ and $E^{(p)}\in\widetilde{\mathcal{M}}_p(A)$, and assume $\|E_n^{(p)}-E^{(p)}\|_{\widetilde{\mathcal{M}}_p(A)}\to 0$ as $n\to\infty$. Then, for any $f_n\in l(A),n\geq 1$ and $f\in l(A)$ such that $f_n\to f$ pointwisely, we have*

$$\lim_{n\to\infty}E_n^{(p)}(f_n)=E^{(p)}(f).$$

Proof. First, from the definition of $\|\cdot\|_{\widetilde{M}_p(A)}$, we have

$$|E_n^{(p)}(f)-E^{(p)}(f)|\leq \mathrm{Osc}(f,A)^p\|E_n^{(p)}-E^{(p)}\|_{\widetilde{M}_p(A)},\tag{A.1}$$

where $\mathrm{Osc}(f,A)=\max\{|f(x)-f(y)|:\ x,y\in A\}$. Then by the condition that $\lim_{n\to\infty}\|E_n^{(p)}-E^{(p)}\|_{\widetilde{\mathcal{M}}_p(A)}=0$, we have

$$\lim_{n\to\infty}\big|E_n^{(p)}(f)-E^{(p)}(f)\big|=0.\tag{A.2}$$

We then estimate $\big|E_n^{(p)}(f_n)-E_n^{(p)}(f)\big|$. Note that by (A.1), we see that $\big|E_n^{(p)}(f)-E^{(p)}(f)\big|$ is uniformly bounded for all $n\geq 1$ and for all f with $\mathrm{Osc}(f,A)=1$, and from this, there is $C_0>0$ such that

$$E_n^{(p)}(f)\leq C_0,\qquad \forall n\geq 1,\forall f\text{ with }\mathrm{Osc}(f,A)=1.\tag{A.3}$$

Since $f_n\to f$ as $n\to\infty$, we can write both $f_n=(1-\varepsilon_n)f+\varepsilon_n g_n$ and $f=(1-\varepsilon_n)f_n+\varepsilon_n g_n'$, where $\varepsilon_n\in(0,1)$ satisfying $\lim_{n\to\infty}\varepsilon_n=0$ and g_n,g_n' are uniformly bounded in n. Now using the convexity of $E_n^{(p)}$, we see that

$$\begin{aligned}E_n^{(p)}(f_n)&\leq(1-\varepsilon_n)E_n^{(p)}(f)+\varepsilon_nE_n^{(p)}(g_n),\\ E_n^{(p)}(f)&\leq(1-\varepsilon_n)E_n^{(p)}(f_n)+\varepsilon_nE_n^{(p)}(g_n'),\end{aligned}$$

which gives that

$$\varepsilon_nE_n^{(p)}(f_n)-\varepsilon_nE_n^{(p)}(g_n')\leq E_n^{(p)}(f_n)-E_n^{(p)}(f)\leq-\varepsilon_nE_n^{(p)}(f)+\varepsilon_nE_n^{(p)}(g_n).\tag{A.4}$$

Since f_n,g_n,g_n' are all uniformly bounded, by (A.3), we see that there is $C_0'>0$ such that $E_n^{(p)}(f)$, $E_n^{(p)}(f_n)$, $E_n^{(p)}(g_n)$, $E_n^{(p)}(g_n')$ are all bounded by C_0'. Thus using (A.4), we obtain

$$\big|E_n^{(p)}(f_n)-E_n^{(p)}(f)\big|\leq 2\varepsilon_nC_0',\tag{A.5}$$

which goes to 0 as $n\to\infty$.

Above all, (A.2) together with (A.5) implies that $\lim\limits_{n\to\infty}E_n^{(p)}(f_n)=E^{(p)}(f)$ as desired. □

We will need to compare the p-harmonic (minimal energy) extensions of different functions in Section 5. Noticing that by Remark 2 after Definition 2.8, $E^{(p)}\in\mathcal{Q}_p(B)$ is always strictly convex, so the p-harmonic extension of a function is always unique.

Lemma A.2. *Let $E^{(p)}\in\mathcal{Q}_p(B)$ and assume $A\subset B$. Let $f,g\in l(B)$ and assume $E^{(p)}(f)=[E^{(p)}]_A(f|_A)$, $E^{(p)}(g)=[E^{(p)}]_A(g|_A)$. Then*

$$f(x)\leq g(x),\forall x\in A\Longrightarrow f(x)\leq g(x),\forall x\in B.$$

Proof. First, we assume $E^{(p)}\in\mathcal{S}_p(B)$, so there are $c_{x,y}\geq 0$ depending on x,y such that $E^{(p)}(u)=\sum_{x,y\in B}c_{x,y}|u(x)-u(y)|^p,\forall u\in l(B)$. In this situation, the conditions $E^{(p)}(f)=[E^{(p)}]_A(f|_A)$, $E^{(p)}(g)=[E^{(p)}]_A(g|_A)$ imply that

$$\begin{cases}\sum_{y\neq x}c_{x,y}\cdot p\big|f(x)-f(y)\big|^{p-1}\cdot\operatorname{sgn}\big(f(x)-f(y)\big)=0,\quad\forall x\in B\setminus A,\\ \sum_{y\neq x}c_{x,y}\cdot p\big|g(x)-g(y)\big|^{p-1}\cdot\operatorname{sgn}\big(g(x)-g(y)\big)=0,\quad\forall x\in B\setminus A.\end{cases}\tag{A.6}$$

Now we show $\max_{x\in B}\big(f(x)-g(x)\big)=\max_{x\in A}\big(f(x)-g(x)\big)$, then the lemma (for this special case) follows. In fact, assume the equality does not hold, then we can find $x_0\in B\setminus A$, so that $f(x_0)-g(x_0)=\max_{x\in B}\big(f(x)-g(x)\big)>\max_{x\in A}\big(f(x)-g(x)\big)$. Then, we have $f(x_0)-f(x)\geq g(x_0)-g(x)$ for any $x\in B$, so that

$$\big|f(x_0)-f(x)\big|^{p-1}\cdot\operatorname{sgn}\big(f(x_0)-f(x)\big)\geq\big|g(x_0)-g(x)\big|^{p-1}\cdot\operatorname{sgn}\big(g(x_0)-g(x)\big).$$

Then, by (A.6) (taking $x=x_0$ there), we know that the above inequality must be equality, so we get $f(x_0)-f(x)=g(x_0)-g(x)$, or equivalently $f(x)-g(x)=f(x_0)-g(x_0)$, for any $x\in B$ such that $c_{x,x_0}>0$. We can repeat the above argument to see there is some $x\in A$ such that $f(x)-g(x)=f(x_0)-g(x_0)$, since B is finite. A contradiction.

Next, assume $E^{(p)}\in\mathcal{Q}'_p(B)$, so we can find $V\supset B$ and $E^{(p)}_V\in\mathcal{S}_p(V)$ such that $[E^{(p)}_V]_B=E^{(p)}$. Noticing that $E^{(p)}_V$ is strictly convex, we can extend f,g uniquely to $f_V,g_V\in l(V)$ so that $f_V|_B=f,g_V|_B=g$ and

$$\begin{cases}E^{(p)}_V(f_V)=[E^{(p)}_V]_B(f)=E^{(p)}(f)=[E^{(p)}]_A(f|_A),\\ E^{(p)}_V(g_V)=[E^{(p)}_V]_B(g)=E^{(p)}(g)=[E^{(p)}]_A(g|_A).\end{cases}$$

By the previous paragraph, we then get $f(x)\leq g(x),\forall x\in A$ implies $f(x)\leq g(x),\forall x\in V$ where $V\supset B$.

Finally, let $E^{(p)}\in\mathcal{Q}_p(B)$, so we can find $E^{(p)}_n\in\mathcal{Q}'_p(B)$ such that $\|E^{(p)}_n-E^{(p)}\|_{\widetilde{\mathcal{M}}_p(B)}\to 0$ as $n\to\infty$. Define $f_n,g_n\in l(B)$ as the unique functions such that $f_n|_A=f|_A$, $E^{(p)}_n(f_n)=[E^{(p)}_n]_A(f|_A)$ and $g_n|_A=g|_A$, $E^{(p)}_n(g_n)=[E^{(p)}_n]_A(g|_A)$, noticing that $E^{(p)}_n$ are strictly convex. In addition, $f_n\leq g_n$ by the previous paragraph. Then, by passing to a subsequence, we have $f_{n_l}\to f'$ and $g_{n_l}\to g'$ for some $f',g'\in l(B)$. We claim that $f'=f$ and $g'=g$ so the proof is completed. In fact, we have $f'|_A=f|_A$ and by Lemma A.1,

$$E^{(p)}(f')=\lim_{l\to\infty}E^{(p)}_{n_l}(f_{n_l})\leq\lim_{l\to\infty}E^{(p)}_{n_l}(f)=E^{(p)}(f),$$

and the same argument works for g. □

The following Lemma A.3 will play a key role in the proof of $\mathscr{H}\subset C(K)$ in Section 5.

Lemma A.3. *Let $E^{(p)}\in\mathcal{Q}_p(A)$ and $f\in l(A)$. Define*

$$\partial_{X,-}E^{(p)}(f)=\lim_{t\nearrow 0}\frac{E^{(p)}(f+t\cdot 1_X)-E^{(p)}(f)}{t},$$

where $X\subset A$ and $1_X\in l(A)$ is the indicator function of X.

Let $M_f=\big\{x\in A:f(x)=\max_{y\in A}f(y)\big\}$. Then $\partial_{X,-}E^{(p)}(f)\geq 0$ if $X\subset M_f$. In addition, if $M_f\neq A$, then $\partial_{M_f,-}E^{(p)}(f)>0$.

Proof. Define $u_X(t) = E^{(p)}(f + t \cdot 1_X)$, then u_X is a convex function with non-negative real values, so u_X has left derivatives $\frac{d}{dt-}u_X(t)$ everywhere. Hence $\partial_{X,-}E^{(p)}(f)$ is well defined.

We are only interested in the case that $X \neq A$, since otherwise the lemma is trivial. In this case, since $E^{(p)} \in \mathcal{Q}_p(A)$ is strictly convex, u_X is also strictly convex. Hence there is a unique s such that

$$u_X(s) = \min\{u_X(t) : t \in \mathbb{R}\}.$$

Clearly, if $X \subset M_f$, then $s \leq 0$ by the Markov property. In addition, by the strict convexity, we know that $\frac{d}{dt-}u_X(t) > 0$ if $t > s$. In particular, this holds when $X = M_f \neq A$. To deal with the case $t = s$, we introduce the following construction.

Construction. Let $A_* = (A/X) \cup \{o\}$, where $A/X := (A \setminus X) \cup \{X\}$ (we view X as a single point in A_*) and o is a new point outside A. For each $g_* \in l(A_*)$, we can define $g \in l(A)$ by

$$g(x) = \begin{cases} g_*(x), & \text{if } x \in A \setminus X, \\ g_*(X), & \text{if } x \in X. \end{cases}$$

Then, we define

$$E_*^{(p)}(g_*) = E^{(p)}(g) + \big|g_*(X) - g_*(o)\big|^p, \quad \forall g_* \in l(A_*).$$

It is easy to see that $E_*^{(p)} \in \mathcal{Q}_p(A_*)$.

Returning to our basic setting, one can see that $f|_X \equiv c$, where $c = \max_{y \in A} f(y)$. So, for each $\delta > 0$, one can define $f_{*,\delta} \in l(A_*)$ by

$$f_{*,\delta}|_{A\setminus X} = f|_{A \setminus X}, \quad f_{*,\delta}(X) = c, \quad \text{and} \quad f_{*,\delta}(o) = c - \delta.$$

Define $u_{X,*,\delta}(t) = E_*^{(p)}(f_{*,\delta} + t \cdot 1_X)$. Let s_δ be the unique value such that $u_{X,*,\delta}(s_\delta) = \min\{u_{X,*,\delta}(t) : t \in \mathbb{R}\}$. Now we claim that $s_\delta < 0$. Indeed, let $g \in l(A_*)$ be such that $g|_{A \setminus X} = c$, $g(o) = c - \delta$, and g be p-harmonic at X w.r.t. $E_*^{(p)}$. It is clear that $g(X) < c$ (by a same argument as the proof of Lemma 5.4). Now we compare $f_{*,\delta} + s_\delta \cdot 1_X$ with g. By Lemma A.2, we see that

$$c + s_\delta = f_{*,\delta}(X) + s_\delta \cdot 1_X(X) \leq g(X) < c.$$

Finally, by the same argument as before, we see

$$0 < \frac{d}{dt-}u_{X,*,\delta}(0) = \partial_{X,-}E_*^{(p)}(f_{*,\delta}) = \partial_{X,-}E^{(p)}(f) + p\delta^{p-1}.$$

Since δ can be arbitrarily small, we have $\partial_{X,-}E^{(p)}(f) \geq 0$ as desired. □

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