BOUNDARY VALUE PROBLEMS FOR HARMONIC FUNCTIONS ON DOMAINS IN P.C.F. SELF-SIMILAR SETS

QINGSONG GU AND HUA QIU

ABSTRACT. We study the boundary value problems for harmonic functions on open connected subsets of post-critically finite (p.c.f.) self-similar sets, on which the Laplacian is defined through a self-similar local regular Dirichlet form. For a p.c.f. self-similar set K, we prove that for any open connected subset $\Omega \subset K$ whose "geometric" boundary is a graph-directed self-similar set, there exists a finite number of matrices called *flux transfer matrices* whose products generate the hitting probability from a point in Ω to the "resistance" boundary $\partial\Omega$. The harmonic functions on Ω can be expressed by integrating functions on $\partial\Omega$ against the probability measures. Furthermore, we obtain a two-sided estimate of the energy of a harmonic function in terms of its values on $\partial\Omega$. This generalizes the known results on Sierpinski gasket to p.c.f. self-similar sets.

1. Introduction

Let Ω be a smooth domain in $\mathbb{R}^n$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator. It is known that the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has a unique solution u for any continuous function f on the boundary. In particular, if Ω is the open unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$, u has an expression of Poisson integral

$$u(x) = \int_{|y|=1} f(y)P(x,y)d\sigma(y), \tag{1.2}$$

where $d\sigma$ is the normalized surface measure on the unit sphere, $P(x,y) = \frac{1-|x|^2}{|x-y|^n}$ is the Poisson kernel. From the probabilistic point of view, the measure $P(x,y)d\sigma(y)$ represents the hitting probability of the killed Brownian motion from $x \in B$ to the sphere.

On fractals, a local regular Dirichlet form plays the role of the Dirichlet integral $\int_\Omega |\nabla u|^2 dx$ in a domain Ω of $\mathbb{R}^n$, and it has an infinitesimal generator Δ called the Laplacian. The construction of Dirichlet forms on fractals is motivated by the study of Brownian motions on self-similar sets in a probabilistic approach, with pioneering works of Kusuoka [19], Goldstein [9] and Barlow-Perkins [5] on the Sierpinski gasket and of Lindstrøm [21] on nested fractals, and also of Barlow-Bass [4] on the Sierpinski carpet. There is also a large

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literature on the topic based on Kigami's analytic approach on the post-critically finite (p.c.f.) self-similar sets (see [1, 11, 14, 15, 16, 23, 25, 26, 27] and the references therein).

Specifically, let K be a self-similar set generated by an iterated function system $\{F_i\}_{i=1}^N$ on a metric measure space. Most of the previous studies are about the Dirichlet forms $(\mathcal{E}, \mathcal{F})$ satisfying the energy self-similar identity, which means that there exist N positive real numbers $\{r_i\}_{i=1}^N$ called *energy renormalizing factors* such that for any function $u \in \mathcal{F}$, it holds that $u \circ F_i \in \mathcal{F}$ for any $i = 1, \ldots, N$, and

$$\mathcal{E}[u] = \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}[u \circ F_i],$$

where $\mathcal{E}[u] := \mathcal{E}(u, u)$. If further $r_i \in (0, 1)$ for each $1 \leq i \leq N$, then the form is called regular. Such forms are known to exist on some classes of self-similar sets having certain symmetric properties, for example, nested fractals [21, 26], affine nested fractals [8], and Sierpinski carpets [4].

For a given p.c.f. self-similar set K equipped with a regular self-similar Dirichlet form, we are concerned with the boundary value problems for harmonic functions on a domain Ω in K (which means Ω is a nonempty open connected subset of K). We mainly focus on two problems originated from classical analysis: one is to find the exact description of the hitting probability from a point in Ω to the boundary; the other is to estimate the energy of a harmonic function generated by its boundary values. From the analytic point of view, we should regard Ω as a resistance space, see the work of Kigami and Takahashi [18] on a particular Ω, the Sierpinski gasket (SG) minus its bottom line. This leads us to invite the topology given by the resistance metric to replace the underlying topology inherited from K. So in our investigation, we need to discriminate between two different boundaries of Ω, and we call them "resistance" boundary and "geometric" boundary later.

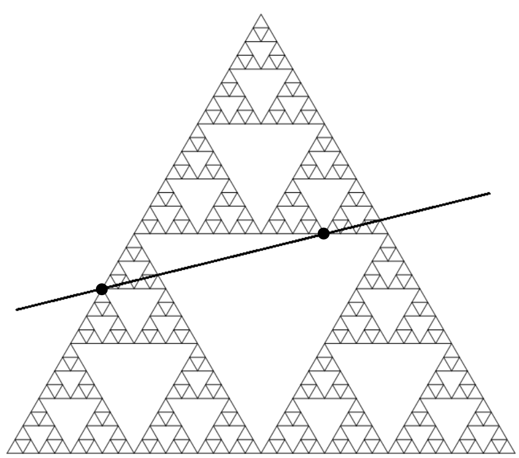

FIGURE 1. domains in the Sierpinski gasket

The study of such problems was initiated in [24, 10, 18] for typical domains in SG, see also [20, 6] for extensions in more generalized SGs. However, the techniques strongly depend on the specific structure of SG and the rigorous geometric structure of the domain. For a general p.c.f. self-similar set K, due to its self-similarity, it is natural to consider domains whose geometric boundaries are *graph-directed self-similar sets*, for example, domains in SG generated by "cutting" with an oblique line (see Figure 1 and Subsection 7.1). Another example is a family of domains in the Lindstrøm's snowflake whose boundaries are Koch curves (see Figure 2 with boundaries drawn in thick lines).

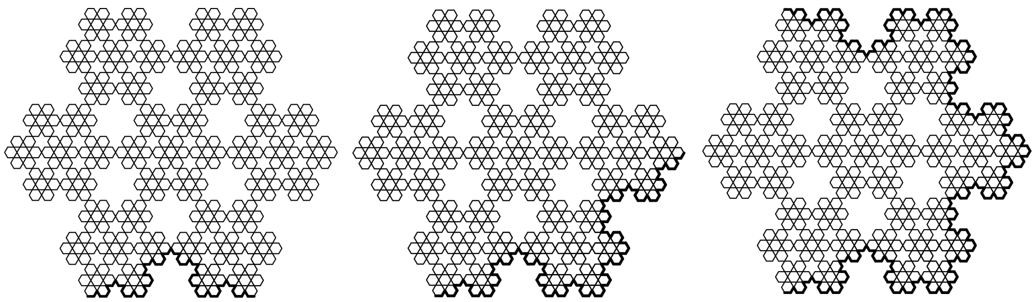

FIGURE 2. Domains in the Lindstrøm's snowflake

In this paper, we propose a condition called *boundary graph-directed condition* (BGD for short) for a finite collection of domains Ω_i, $1 \le i \le P$ with geometric boundaries D_i:

(BGD): *for* $1 \le i \le P$ *and* $1 \le k \le N$, *if* $\Omega_i \cap F_k(K) \neq \emptyset$ *and* $D_i \cap F_k(K) \neq \emptyset$, *then there exists* $1 \le j \le P$ *such that*

$$\Omega_i \cap F_k(K) = F_k(\Omega_j), \qquad D_i \cap F_k(K) = F_k(D_j). \tag{1.3}$$

Under this condition, we are able to solve the boundary value problems for general p.c.f. self-similar sets.

Firstly, to determine the hitting probability from a point in a domain Ω to its resistance boundary $\partial\Omega$, we introduce a finite number of matrices, called *flux transfer matrices*, and prove that the products of these matrices generate the hitting probability (see Theorem 5.4). We note that these matrices are determined not only by the Dirichlet form on the fractal but also by the graph-directed structure of the domain.

Secondly, using the hitting probability measures, we establish an equivalent characterization of energies of harmonic functions through their boundary values (see Theorem 6.2). We remark that a quite related problem is to consider the trace of functions with finite energy on a self-similar set to its subset. In [12], Hino and Kumagai proved a trace theorem for these functions on a self-similar set to its self-similar subsets, extending the result of Jonsson [13] for functions on SG to its bottom line.

We organize the paper as follows. In Section 2, we give some preliminaries for regular self-similar Dirichlet forms on p.c.f. self-similar sets and recall some basic properties of electric networks. In Section 3, we give several basic properties of the boundary graph-directed (BGD) condition to describe the geometric boundary of a domain in self-similar sets. In Section 4, for a domain satisfying BGD, we study its resistance boundary by virtue of resistance forms and characterize this boundary as a symbolic space. In Section 5, by introducing the flux transfer matrices, we prove Theorem 5.4 for the expression of hitting probabilities. In Section 6, we prove Theorem 6.2 for the energy estimate of harmonic functions in terms of their boundary values. Finally in Section 7, we present several examples.

Throughout the paper, we use the notation $f \lesssim (\gtrsim) g$ for two variables f and g if there exists a constant $C > 0$ such that $f \leq (\geq) Cg$, and also $f \asymp g$ if both $f \lesssim g$ and $f \gtrsim g$ hold. For a set A, we write $\ell(A)$ for the collection of functions on A.

2. **Preliminaries**

We first recall some notations about post-critically finite (p.c.f. for short) self-similar sets introduced by Kigami [15, 16]. Let $N \geq 2$ be an integer, $\{F_i\}_{i=1}^N$ be a finite collection of contractions on a complete metric space (X, d). The self-similar set associated with the iterated function system (IFS) $\{F_i\}_{i=1}^N$ is the unique nonempty compact set $K \subset X$ satisfying

$$K = \bigcup_{i=1}^N F_i(K).$$

We define the symbolic space as usual. Let $\Sigma = \{1, \ldots, N\}$ be the *alphabets*, Σ^n the set of *words* of length n (where $\Sigma^0 = \{\emptyset\}$ containing only the empty word), and Σ^∞ the set of *infinite words* $\omega = \omega_1\omega_2\cdots$. For $\omega = \omega_1\cdots\omega_n \in \Sigma^n$, we write $|\omega| = n$, $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$ and call $F_\omega(K)$ an *n-cell* ($F_\emptyset = \mathrm{Id}$). Let $\pi : \Sigma^\infty \to K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n\geq 1} F_{[\omega]_n}(K)$ with $[\omega]_n = \omega_1\cdots\omega_n$, a symbolic representation of $x \in K$ by ω.

Following [16], we define the critical set C and post-critical set $\mathcal{P}$ for K by

$$C = \pi^{-1}\left(\bigcup_{1\leq i<j\leq N} \Big(F_i(K) \cap F_j(K)\Big)\right), \qquad \mathcal{P} = \bigcup_{m\geq 1} \sigma^m(C),$$

where $\sigma : \Sigma^\infty \to \Sigma^\infty$ is the left shift operator, i.e. $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$. If $\mathcal{P}$ is finite, we call $\{F_i\}_{i=1}^N$ a *post-critically finite (p.c.f.)* IFS, and K a *p.c.f. self-similar set*. The boundary of K is defined by $V_0 = \pi(\mathcal{P})$. We also inductively denote

$$V_n = \bigcup_{i\in\Sigma} F_i(V_{n-1}), \qquad V_* = \bigcup_{n=0}^\infty V_n.$$

It is clear that $\{V_n\}_{n\geq 0}$ is an increasing sequence of sets and K is the closure of V_*. It is known that the metric space (K, d) has a fundamental neighborhood system $\{K_{n,x} : n \geq 0, x \in K\}$, where each $K_{n,x} = \bigcup_{\omega\in\Sigma^n : x\in F_\omega(K)} F_\omega(K)$, see [16, Proposition 1.3.6]. We always assume that (K, d) is connected.

Our basic assumption on a p.c.f. self-similar set K is the existence of a *regular harmonic structure*, which will generate a regular self-similar resistance form $(\mathcal{E}, \mathcal{F})$ with domain $\mathcal{F} = \{u \in C(K) : \mathcal{E}[u] := \mathcal{E}(u, u) < \infty\}$:

$$\mathcal{E}[u] = \sum_{i=1}^N \frac{1}{r_i}\mathcal{E}[u \circ F_i], \qquad u \in \mathcal{F}, \tag{2.1}$$

where $0 < r_i < 1, i = 1, \ldots, N$ are called *energy renormalizing factors*. By iterating (2.1), we see that for any $n \geq 1$,

$$\mathcal{E}[u] = \sum_{|\omega|=n} \frac{1}{r_\omega}\mathcal{E}[u \circ F_\omega], \qquad u \in \mathcal{F}, \tag{2.2}$$

where $r_\omega = r_{\omega_1} \cdots r_{\omega_n}$ for $\omega = \omega_1 \cdots \omega_n$. We call $\mathcal{E}_{F_\omega(K)}[u] := \frac{1}{r_\omega}\mathcal{E}[u \circ F_\omega]$ the *energy* of u on the cell $F_\omega(K)$.

We say a function $h \in \mathcal{F}$ *harmonic* in K providing that

$$\mathcal{E}[h] = \inf\{\mathcal{E}[u] : \ u \in \mathcal{F}, u|_{V_0} = h|_{V_0}\}.$$

Let A, B be two disjoint nonempty closed subsets of K, the *effective resistance* $R(A, B)$ between A and B is defined as

$$R(A, B)^{-1} := \inf\{\mathcal{E}[u] : \ u \in \mathcal{F}, u|_A = 0, u|_B = 1\}.$$

The infimum can be attained by a unique function which is harmonic in $K \setminus (A \cup B)$. We write $R(x, B) := R(\{x\}, B)$ and $R(x, y) := R(\{x\}, \{y\})$ when x, y are single points. When we only consider points, by setting $R(x, x) = 0$ for all $x \in K$, the resistance $R(\cdot, \cdot)$ is known to be a metric on K, which is called the *effective resistance metric*. It is known that (K, R) is equivalent to (K, d). In addition, $\mathrm{diam}_R(F_\omega(K)) \asymp r_\omega$ for any finite word ω, where $\mathrm{diam}_R(F_\omega(K))$ is the diameter of $F_\omega(K)$ under R.

For a Radon measure ν supported on K, the resistance form $(\mathcal{E}, \mathcal{F})$ turns out to be a Dirichlet form on $L^2(K, \nu)$, which determines a Laplacian Δ_ν. See [16, 17, 27] for details.

We then recall some basic facts in electric network theory. Let G be a finite set, and let $g : G \times G \to \mathbb{R}$ be a nonnegative function such that

$$g(p, q) = g(q, p), \ g(p, p) = 0, \quad p, q \in G.$$

For $p, q \in G$, we write $p \sim q$ if $g(p, q) > 0$, and say that (G, g) is *connected* if for any $p, q \in G$ there is a path $p = p_0 \sim p_1 \sim \cdots \sim p_n = q$. We always assume that (G, g) is connected, and call (G, g) an *electric network*.

For $u \in \ell(G)$, we define the *energy* of u on (G, g) to be

$$\mathcal{E}_G[u] := \frac{1}{2} \sum_{p,q \in G} g(p, q)(u(p) - u(q))^2.$$

By polarization, we can define $\mathcal{E}_G(u, v) = \frac{1}{4}\left(\mathcal{E}_G[u + v] - \mathcal{E}_G[u - v]\right)$ for $u, v \in \ell(G)$. Then $(\mathcal{E}_G, \ell(G))$ is a resistance form on G [16].

For $u \in \ell(G)$, we define the *Neumann derivative* of u (*flux* of ∇u, the flow associated with u, see [2]) at some vertex $p \in G$ as

$$(du)_p = \sum_{q \in G} g(p, q)(u(p) - u(q)). \tag{2.3}$$

Then clearly, for $u, v \in \ell(G)$,

$$\sum_{p \in G} v(p)(du)_p = \sum_{p \in G} u(p)(dv)_p, \tag{2.4}$$

and in particular,

$$\sum_{p \in G} (du)_p = 0. \tag{2.5}$$

For a resistance form $(\mathcal{E}, \mathcal{F})$ on K, it is known that the *trace* of $\mathcal{E}[\cdot]$ to a nonempty finite set $V \subset K$ is an electric network (V, g) determined by

$$\sum_{p,q \in V} g(p, q)(u(p) - u(q))^2 = \min\{\mathcal{E}[v] : \ v \in \mathcal{F}, v|_V = u\}, \quad u \in \ell(V),$$

while the unique function v minimizing the right hand side is harmonic in $K \setminus V$. In the following, with a little abuse of notation, sometimes we write $(dv)_p$ instead of $(du)_p$ for $p \in V$.

3. **Boundary graph-directed condition**

In this section, for a p.c.f. self-similar set K, we will introduce a condition for domains in K, named as boundary graph-directed condition, that will be concerned throughout the paper.

Recall that graph-directed self-similar sets are generalized from self-similar sets. Let $(\mathcal{A}, \Gamma)$ be a *directed graph* (allowing loops) with a finite set of *vertices* $\mathcal{A} = \{1, \ldots, P\}$ and a finite set of *directed edges* Γ. For any $\gamma \in \Gamma$, if γ is a directed edge from i to j for some $i, j \in \mathcal{A}$, we set $I(\gamma) = i$ and $T(\gamma) = j$ and call them *initial vertex* and *terminal vertex* of γ separately. For $i, j \in \mathcal{A}$, denote $\Gamma(i) = \{\gamma \in \Gamma : I(\gamma) = i\}$ and $\Gamma(i, j) = \{\gamma \in \Gamma : I(\gamma) = i, \ T(\gamma) = j\}$. We assume each $\Gamma(i)$ is nonempty and each edge γ is associated with a contraction Φ_γ on (X, d). Then there exists a unique vector of nonempty compact sets $\{D_i\}_{i=1}^P$ in (X, d), called *graph-directed self-similar sets* [22], satisfying

$$D_i = \bigcup_{j=1}^{P} \bigcup_{\gamma \in \Gamma(i,j)} \Phi_\gamma(D_j), \qquad 1 \le i \le P. \tag{3.1}$$

Let $m \geq 1$, a finite word $\boldsymbol{\gamma} = \gamma_1\gamma_2 \cdots \gamma_m$ with $\gamma_i \in \Gamma$ for $i = 1, \ldots, m$ is called *admissible* if $T(\gamma_i) = I(\gamma_{i+1})$ for any $i = 1, \ldots, m-1$; we set $|\boldsymbol{\gamma}| = m$, write $I(\boldsymbol{\gamma}) = I(\gamma_1)$, $T(\boldsymbol{\gamma}) = T(\gamma_m)$, and define $\Phi_{\boldsymbol{\gamma}} = \Phi_{\gamma_1} \circ \cdots \circ \Phi_{\gamma_m}$. We denote by Γ_m the set of all admissible words with length m, and write $\Gamma_0 = \{\emptyset\}$ containing only the empty word by convention. For $0 \le n \le m$, we denote $[\boldsymbol{\gamma}]_n = \gamma_1 \cdots \gamma_n$ the n-th step truncation of $\boldsymbol{\gamma}$. For $i \in \mathcal{A}$, we also denote $\Gamma_m(i) = \{\boldsymbol{\gamma} \in \Gamma_m : I(\boldsymbol{\gamma}) = i\}$. Denote by $\Gamma_* = \bigcup_{m=0}^{\infty} \Gamma_m$ the set of all finite admissible words.

We then apply the above definition to a particular situation, domains in p.c.f. self-similar sets. Let $(K, \{F_i\}_{i=1}^N)$ be a p.c.f. self-similar set. For $P \geq 1$, let $\{\Omega_1, \Omega_2, \ldots, \Omega_P\}$ be a vector of connected nonempty open subsets of K such that each Ω_i has a nonempty boundary with respect to the metric d, which is denoted as D_i. Later we call D_i the *geometric boundary* of Ω_i. We assume that $\{(\Omega_i, D_i)\}_{1 \le i \le P}$ satisfy the following BGD condition (see also (1.3)):

(BGD): *for* $1 \le i \le P$ *and* $1 \le k \le N$, *if* $\Omega_i \cap F_k(K) \neq \emptyset$ *and* $D_i \cap F_k(K) \neq \emptyset$, *then there exists* $1 \le j \le P$ *such that*

$$\Omega_i \cap F_k(K) = F_k(\Omega_j), \qquad D_i \cap F_k(K) = F_k(D_j).$$

According to the configuration of K and $\{\Omega_i\}_{i=1}^P$, we define the directed graph on $\mathcal{A} = \{1, \ldots, P\}$ as follows. For each pair (i, j) in the BGD condition, we set γ to be a directed edge from i to j with the contraction map $\Phi_\gamma = F_k$. Denote by Γ the set of all directed edges γ between vertices in $\mathcal{A}$. In this way, we have a directed graph $(\mathcal{A}, \Gamma)$ and a set of contractions $\{\Phi_\gamma\}_{\gamma \in \Gamma}$ such that for each γ, there is some $k \in \{1, \ldots, N\}$ satisfying $\Phi_\gamma = F_k$. Furthermore, $\{D_i\}_{i=1}^P$ satisfy the equations (3.1) with these $\{\Phi_\gamma\}_{\gamma \in \Gamma}$, and hence, $\{D_i\}_{i=1}^P$ is a vector of graph-directed self-similar sets.

Proposition 3.1. *Assume $\{\Omega_i\}_{i=1}^P$ satisfies the BGD condition.*

(i). If $\Omega_i \cap V_0 \neq \emptyset$, then $\Omega_j \cap V_0 \neq \emptyset$ provided that $\Gamma(i,j) \neq \emptyset$;

(ii). There exists $n_0 \geq 1$ such that $\Omega_{T(\boldsymbol{\gamma})} \cap V_0 \neq \emptyset$ for all $n \geq n_0$ and $\boldsymbol{\gamma} \in \Gamma_n$.

Proof. (i). Assume $\Omega_i \cap V_0 \neq \emptyset$ and $\gamma \in \Gamma(i,j)$. We consider two possible cases to achieve $\Omega_j \cap V_0 \neq \emptyset$.

Case 1. $\Phi_\gamma(\Omega_j) = \Omega_i$. By $\Omega_i \cap V_0 \neq \emptyset$, we can find some $p_k \in \Phi_\gamma(\Omega_j) \cap V_0 \subset \Phi_\gamma(V_0)$. This implies that $\Phi_\gamma(\Omega_j \cap V_0) \neq \emptyset$ and hence $\Omega_j \cap V_0 \neq \emptyset$.

Case 2. $\Phi_\gamma(\Omega_j) \subsetneq \Omega_i$. If $\Omega_j \cap V_0 = \emptyset$, we must have $\Omega_i \cap \Phi_\gamma(V_0) = \Omega_i \cap \Phi_\gamma(K) \cap \Phi_\gamma(V_0) = \Phi_\gamma(\Omega_j) \cap \Phi_\gamma(V_0) = \emptyset$, where we used the BGD condition in the second equality. Then $\Phi_\gamma(\Omega_j)$ and $\Omega_i \setminus \Phi_\gamma(\Omega_j)$ are two nonempty open subsets in Ω_i, which contradicts the connectedness of Ω_i.

(ii). We pick $n \geq 1$ sufficiently large such that $\Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})}) \subsetneq \Omega_i$ for all $1 \leq i \leq P$ and $\boldsymbol{\gamma} \in \Gamma_n(i)$. Then the proof is similar to that of Case 2 in (i). □

Proposition 3.2. *Assume $\{\Omega_i\}_{i=1}^P$ satisfies the BGD condition. Then each Ω_i is arcwise connected.*

Proof. By [16, Theorem 1.6.2], the connectedness of K implies that K and any cell $F_\omega(K)$ are arcwise connected. Hence each open set Ω_i is locally arcwise connected, and so each arcwise connected component of Ω_i is open. Since Ω_i is connected, Ω_i only has one arcwise connected component, so that Ω_i is arcwise connected. □

We will also use the notation of *infinite admissible words* $\boldsymbol{\gamma} = \gamma_1\gamma_2\cdots$ with $T(\gamma_i) = I(\gamma_{i+1})$ for all $i \geq 1$. We denote by Γ_∞ the collection of all infinite admissible words and $\Gamma_\infty(i) = \{\boldsymbol{\gamma} = \gamma_1\gamma_2\cdots \in \Gamma_\infty : \ I(\gamma_1) = i\}$ for $i = 1, \ldots, P$.

For $\boldsymbol{\gamma} = \gamma_1\gamma_2\cdots, \boldsymbol{\eta} = \eta_1\eta_2\cdots \in \Gamma_\infty$ with $\boldsymbol{\gamma} \neq \boldsymbol{\eta}$, let $\boldsymbol{\gamma} \wedge \boldsymbol{\eta}$ be the common prefix of $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$, then

$$|\boldsymbol{\gamma} \wedge \boldsymbol{\eta}| = \min\{i \geq 1 : \ \gamma_i \neq \eta_i\} - 1.$$

Define

$$\rho(\boldsymbol{\gamma}, \boldsymbol{\eta}) = \begin{cases} 2^{-|\boldsymbol{\gamma}\wedge\boldsymbol{\eta}|}, & \boldsymbol{\gamma} \neq \boldsymbol{\eta}, \\ 0, & \boldsymbol{\gamma} = \boldsymbol{\eta}. \end{cases}$$

Then by a routine argument, ρ is a metric on Γ_∞ and (Γ_∞, ρ) is a complete metric space.

For $i \in \{1, \ldots, P\}$, there is a natural surjective map

$$\iota_i : \Gamma_\infty(i) \to D_i$$

given by $\iota_i(\boldsymbol{\gamma}) = x$ with $\{x\} = \bigcap_{n\geq 1} \Phi_{[\boldsymbol{\gamma}]_n}(K)$, where $[\boldsymbol{\gamma}]_n = \gamma_1\cdots\gamma_n$ is the n-th step truncation of $\boldsymbol{\gamma}$. It is known that ι_i is continuous (see for example [16, Theorem 1.2.3]).

4. Resistance boundary and geometric boundary

In this section, we will discuss the relation of two "boundaries" of a domain $\Omega \subset K$, the geometric boundary and the resistance boundary. We will call them *d-boundary* and *R-boundary* for short.

Let Ω be a domain in K. For a function $u \in C(\Omega)$, by considering Ω as a countable disjoint union of cells, we define *the energy of u on* Ω to be the summation of energies of u on each of the cells and denote it as $\mathcal{E}_\Omega[u]$ (might equal to $+\infty$). It is known that $\mathcal{E}_\Omega[u]$ does not depend on the partition of disjoint cells in Ω. Denote $\mathcal{F}_\Omega = \{u \in C(\Omega) : \ \mathcal{E}_\Omega[u] <$

$\infty\}$. By polarization, we define $\mathcal{E}_\Omega(u,v) = \frac{1}{4}\left(\mathcal{E}_\Omega[u+v] - \mathcal{E}_\Omega[u-v]\right)$ for $u, v \in \mathcal{F}_\Omega$. It is direct to check that $(\mathcal{E}_\Omega, \mathcal{F}_\Omega)$ is a resistance form on Ω.

Define the *effective resistance metric* $R_\Omega(x,y)$ for two points x, y in Ω with respect to $\mathcal{E}_\Omega$: for $x, y \in \Omega$ and $x \neq y$,

$$R_\Omega(x,y)^{-1} := \inf\{\mathcal{E}_\Omega[u] : \ u \in \mathcal{F}_\Omega, u(x) = 0, u(y) = 1\},$$

and $R(x,x) = 0$ by convention. Then $R_\Omega(\cdot,\cdot)$ is a metric on Ω [16]. Let $\widetilde{\Omega}$ be the completion of Ω under R_Ω, and denote $\partial\Omega = \widetilde{\Omega} \setminus \Omega$, the *R-boundary* of Ω. Recall that there is another resistance metric $R(\cdot,\cdot)$ on Ω inherited from that on K.

Lemma 4.1. *Let $A \subset \Omega$ be a nonempty closed subset. Then there exists $C > 1$ depending on A such that*

$$R(x,y) \leq R_\Omega(x,y) \leq CR(x,y), \quad \forall x, y \in A. \tag{4.2}$$

In addition, (A, R_Ω) is homeomorphic to (A, R) and (A, d).

Proof. By definition,

$$\begin{aligned} R_\Omega(x,y)^{-1} &= \inf\{\mathcal{E}_\Omega[u] : \ u \in \mathcal{F}_\Omega, u(x) = 0, u(y) = 1\} \\ &\leq \inf\{\mathcal{E}[u] : \ u \in \mathcal{F}, u(x) = 0, u(y) = 1\} = R(x,y)^{-1}, \end{aligned}$$

we see that $R(x,y) \leq R_\Omega(x,y)$.

On the other hand, since A is closed in Ω, fix an $n \geq 1$ sufficiently large and a finite number of n-cells $\{F_{\omega^{(k)}}(K)\}_{k=1}^m$ such that

$$A \subset \bigcup_{k=1}^m F_{\omega^{(k)}}(K) \subset \Omega.$$

We can also require that $\bigcup_{k=1}^m F_{\omega^{(k)}}(K)$ is connected by the (arcwise) connectedness of Ω.

For any two points $x, y \in A$, we choose two n-cells (may be equal), say $F_\omega(K)$ and $F_{\omega'}(K)$, in $\{F_{\omega^{(k)}}(K)\}_{k=1}^m$ such that $x \in F_\omega(K)$ and $y \in F_{\omega'}(K)$. Let u be the unique function in $\mathcal{F}_\Omega$ such that $\mathcal{E}_\Omega[u] = R_\Omega(x,y)^{-1}$ and $u(x) = 0$, $u(y) = 1$. Define a function $v \in \mathcal{F}$ such that $v|_{F_{\omega^{(k)}}(K)} = u|_{F_{\omega^{(k)}}(K)}$ for each $1 \leq k \leq m$, $v = 0$ on $V_n \setminus \left(\bigcup_{k=1}^m F_{\omega^{(k)}}(V_0)\right)$ and $v \circ F_\tau$ is harmonic for each $\tau \in \Sigma^n \setminus \{\omega^{(1)}, \ldots, \omega^{(m)}\}$. Then $v(x) = 0$, $v(y) = 1$ and

$$R(x,y)^{-1} \leq \mathcal{E}[v] = \sum_{k=1}^m \mathcal{E}_{F_{\omega^{(k)}}(K)}[u] + \sum_{\tau \in \Sigma^n \setminus \{\omega^{(1)}, \ldots, \omega^{(m)}\}} \mathcal{E}_{F_\tau(K)}[v]. \tag{4.3}$$

Since v attains values 0 and 1 in the cells $F_\omega(K)$ and $F_{\omega'}(K)$ separately, and the union of cells $\bigcup_{k=1}^m F_{\omega^{(k)}}(K)$ is connected, we see that $\sum_{k=1}^m \mathcal{E}_{F_{\omega^{(k)}}(K)}[u] \geq C_1$ for some $C_1 > 0$ depending on n and $\{r_i\}_{i=1}^N$. Also note that $0 \leq v \leq 1$ in each of the cells $F_\tau(K)$ for $\tau \in \Sigma^n \setminus \{\omega^{(1)}, \ldots, \omega^{(m)}\}$, and v is harmonic in each $F_\tau(K)$, we obtain $\sum_{\tau \in \Sigma^n \setminus \{\omega^{(1)}, \ldots, \omega^{(m)}\}} \mathcal{E}_{F_\tau(K)}[v] \leq C_2$ for some $C_2 > 0$ depending on n and $\{r_i\}_{i=1}^N$. Hence the right hand side of (4.3) is bounded from above by

$$C \sum_{k=1}^m \mathcal{E}_{F_{\omega^{(k)}}(K)}[u] \leq C\mathcal{E}_\Omega[u] = CR_\Omega(x,y)^{-1}, \tag{4.4}$$

for some $C > 1$ depending on n and $\{r_i\}_{i=1}^N$.

Combining (4.3) and (4.4), we obtain the second inequality of (4.2).

From (4.2), we see that (A, R_Ω) is homeomorphic to (A, R) and so is also homeomorphic to (A, d). □

Let $\{\Omega_i\}_{i=1}^P$ be a finite collection of domains in K with d-boundaries $\{D_i\}_{i=1}^P$ satisfying the BGD condition.

We say a (finite or infinite) sequence of cells $\{F_{\omega^{(k)}}(K)\}_{k\geq 1}$ *a chain of cells* if $F_{\omega^{(k)}}(K)\cap F_{\omega^{(k+1)}}(K)\neq\emptyset$ for all $k\geq 1$. For a finite chain of cells $\{F_{\omega^{(k)}}(K)\}_{k=1}^m$ with $x\in F_{\omega^{(1)}}(K)$ and $y\in F_{\omega^{(m)}}(K)$, we say it *connects x and y*.

Lemma 4.2. *There exists $n_1\geq 1$ such that for each Ω_i with $\Omega_i\cap V_1\neq\emptyset$ and $x,y\in\Omega_i\cap V_1$, there exists a chain of n_1-cells $\{F_{\omega^{(k)}}(K)\}_{k=1}^m$ in Ω_i connecting x and y.*

Proof. By Proposition 3.2, each Ω_i is arcwise connected. Hence for any $x,y\in\Omega_i\cap V_1$, there exists a curve joining x and y in Ω_i (a continuous map $f:\ [0,1]\to\Omega_i$ such that $f(0)=x$, $f(1)=y$). By that Ω_i is pre-compact, an ε-neighborhood of the curve is contained in Ω_i, which gives a desired chain of n-cells for large n. Since the numbers of Ω_i and pairs $x,y\in\Omega_i\cap V_1$ are finite, we see that there exists a common n_1 as required. □

In the following, we write $r_{\max}=\max\{r_i:\ 1\leq i\leq N\}$ and $r_{\min}=\min\{r_i:\ 1\leq i\leq N\}$.

Proposition 4.3. *Each (Ω_i,R_{Ω_i}) is a bounded metric space.*

Proof. By Proposition 3.1(ii), we choose n_0 such that $\Omega_{T(\boldsymbol{\gamma})}\cap V_0\neq\emptyset$ for all $\boldsymbol{\gamma}\in\Gamma_n$ with $n\geq n_0$. Let

$$\mathcal{B}=\{T(\boldsymbol{\gamma}):\ \boldsymbol{\gamma}\in\Gamma_n,n\geq n_0\}.$$

We first prove that (Ω_i,R_{Ω_i}) is bounded for each $i\in\mathcal{B}$.

For any $x=\pi(\omega)\in\Omega_i$ with $\omega\in\Sigma^\infty$, let $m\geq 0$ be such that $F_{[\omega]_{m+1}}(K)\subset\Omega_i$ and $F_{[\omega]_m}(K)\not\subset\Omega_i$. By the BGD condition, $F_{[\omega]_m}(K)\cap\Omega_i=\Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})})$ for some $\boldsymbol{\gamma}\in\Gamma_m(i)$. So $x\in F_{[\omega]_{m+1}}(K)\subset\Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})})$.

Then by Lemma 4.2, we have the following two facts.

Fact 1. For $0\leq k\leq m$, and $y\in\Phi_{[\boldsymbol{\gamma}]_k}\left(\Omega_{T([\boldsymbol{\gamma}]_k)}\cap V_0\right)$, $z\in\Phi_{[\boldsymbol{\gamma}]_{k+1}}\left(\Omega_{T([\boldsymbol{\gamma}]_{k+1})}\cap V_0\right)$ ($F_{[\omega]_{m+1}}(V_0)$ if $k=m$), there exists a chain of (n_1+k)-cells in $\Phi_{[\boldsymbol{\gamma}]_k}\left(\Omega_{T([\boldsymbol{\gamma}]_k)}\right)$ connecting y and z.

Fact 2. For $k\geq m+1$, and $y\in F_{[\omega]_k}(V_0)$, $z\in F_{[\omega]_{k+1}}(V_0)$, there exists a chain of (n_1+k)-cells in $F_{[\omega]_k}(K)$ connecting y and z.

Note that the number of cells in each above chain is bounded from above by $M=N^{n_1}$. For convenience, we adjust the number of cells in each above chain to be M by adding some repeated cells in the chain. Hence from these two facts, there exists a chain of cells $\{F_{\omega^{(k)}}(K)\}_{k=1}^\infty$ in Ω_i and a sequence of points $\{x_k\}_{k=0}^\infty$ with $x_0\in F_{\omega^{(1)}}(V_0)\subset V_0$, $x_k\in F_{\omega^{(k)}}(V_0)\cap F_{\omega^{(k+1)}}(V_0)$ for $k\geq 1$ and $\lim_{k\to\infty}x_k=x$ (w.r.t. d), and $|\omega^{((l-1)M+1)}|=|\omega^{((l-1)M+2)}|=\cdots=|\omega^{(lM)}|=n_1+l-1$ for each $l\geq 1$. We have for any $u\in\mathcal{F}_{\Omega_i}$,

$$\begin{aligned}|u(x)-u(x_0)|&\leq\sum_{k=0}^\infty|u(x_k)-u(x_{k+1})|\\&\lesssim\sum_{k=1}^\infty r_{\omega^{(k)}}^{1/2}\mathcal{E}_{F_{\omega^{(k)}}(K)}[u]^{1/2}\leq\sum_{k=1}^\infty r_{\max}^{(n_1+k/M-1)/2}\mathcal{E}_{\Omega_i}[u]^{1/2}\lesssim\mathcal{E}_{\Omega_i}[u]^{1/2}.\end{aligned}$$

Hence we see that for $x,y\in\Omega_i$,

$$|u(x)-u(y)|\leq|u(x)-u(x_0)|+|u(y)-u(x_0)|\leq C\mathcal{E}_{\Omega_i}[u]^{1/2}$$

for some constant $C > 0$ independent of u, i, x and y. This gives that Ω_i is bounded under R_{Ω_i} for $i \in \mathcal{B}$.

Finally, for $i \in \mathcal{A}$, noticing that for each $\boldsymbol{\gamma} \in \Gamma_{n_0}(i)$, $T(\boldsymbol{\gamma}) \in \mathcal{B}$, and $\Omega_{T(\boldsymbol{\gamma})}$ is already bounded, by a similar chain argument as above, Ω_i is also bounded under R_{Ω_i}. □

Theorem 4.4. *For $i = 1, \ldots, P$, $(\partial\Omega_i, R_{\Omega_i})$ is homeomorphic to $(\Gamma_\infty(i), \rho)$.*

Proof. For $\boldsymbol{\gamma} \in \Gamma_*$, we write $\Omega_{\boldsymbol{\gamma}} := \Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})})$ for brevity. Let $x \in \partial\Omega_i$, and $\{x_n\}_{n\geq 1}$ be a sequence in Ω_i such that $\lim_{n\to\infty} x_n = x$ w.r.t. R_{Ω_i}.

Claim. For any $m \geq 1$, there exists a unique $\boldsymbol{\gamma} \in \Gamma_m(i)$ such that $x_n \in \Omega_{\boldsymbol{\gamma}}$ for all large enough n.

For $m \geq 1$, denote

$$U_{i,m} = \bigcup_{\boldsymbol{\gamma}\in\Gamma_m(i)} \Omega_{\boldsymbol{\gamma}}.$$

We prove this claim through two steps.

First we prove that, for any m, it always holds that $x_n \in U_{i,m}$ for all large enough n. Otherwise, there exists $m_0 \geq 1$ and a subsequence $\{x_{n_k}\}$ contained in $\Omega_i \setminus U_{i,m_0}$ which converges to x w.r.t. R_{Ω_i}. This gives that x is in the closure of $\Omega_i \setminus U_{i,m_0}$ under R_{Ω_i}, which is contained in Ω_i by using Lemma 4.1. A contradiction to $x \in \partial\Omega_i$.

Next, we turn to prove the claim. If it does not hold, then we can pick $m_1 \geq 1$ and $\boldsymbol{\gamma} \neq \boldsymbol{\eta} \in \Gamma_{m_1}(i)$ such that both $\Omega_{\boldsymbol{\gamma}}$ and $\Omega_{\boldsymbol{\eta}}$ contain infinitely many elements in the sequence $\{x_n\}$. By the above paragraph, we may require that $\Omega_{\boldsymbol{\gamma}} \cap \Omega_{\boldsymbol{\eta}} = \emptyset$ in addition. Considering a function u on Ω_i such that $u|_{\Omega_{\boldsymbol{\gamma}}} = 0$, $u|_{U_{i,m_1}\setminus\Omega_{\boldsymbol{\gamma}}} = 1$ and harmonic elsewhere in each m_1-cell in Ω_i, we have

$$R_{\Omega_i}(y, z) \geq \mathcal{E}_{\Omega_i}[u]^{-1} > 0 \tag{4.5}$$

for all $y \in \Omega_{\boldsymbol{\gamma}}$ and $z \in \Omega_{\boldsymbol{\eta}}$. This contradicts to that $\{x_n\}$ is a Cauchy sequence. So the claim holds.

By the claim, each sequence $\{x_n\}$ converging to $x \in \partial\Omega_i$ determines a unique infinite admissible word $\boldsymbol{\gamma} \in \Gamma_\infty(i)$. For two sequences $\{x_n\}$, $\{y_n\}$, if they determine two distinct words $\boldsymbol{\gamma}, \boldsymbol{\eta} \in \Gamma_\infty(i)$, then they must converge to distinct points in $\partial\Omega$, since from the above paragraph, $R_{\Omega_i}(x_n, y_n) \geq c_0 > 0$ for some $c_0 > 0$ and all large enough n. So for each $x \in \partial\Omega_i$, it determines a unique word $\boldsymbol{\gamma} \in \Gamma_\infty(i)$, we denote it as $\mathcal{T}(x)$.

For $\boldsymbol{\gamma} \in \Gamma_\infty(i)$, we pick a sequence $\{x_n\}$ in Ω_i such that $x_n \in \Phi_{[\boldsymbol{\gamma}]_n}(K)$ for all n. Note that

$$R_{\Omega_i}(x_n, x_{n+1}) \leq r_{[\boldsymbol{\gamma}]_n} R_{\Omega_{T([\boldsymbol{\gamma}]_n)}}\left(\Phi^{-1}_{[\boldsymbol{\gamma}]_n}(x_n), \Phi^{-1}_{[\boldsymbol{\gamma}]_n}(x_{n+1})\right) \leq C_1 r^n_{\max}$$

for some $C_1 > 0$ by Proposition 4.3, where $r_{[\boldsymbol{\gamma}]_n} = r_\omega$ with $\omega \in \Sigma^n$ uniquely determined by $F_\omega = \Phi_{[\boldsymbol{\gamma}]_n}$ under the BGD condition. We see that $\{x_n\}$ is a Cauchy sequence w.r.t. R_{Ω_i} and has a limit x in $\widetilde{\Omega}$. However, due to Lemma 4.1, $x \notin \Omega_i$. Hence $\mathcal{T}$ is a surjection.

Now we prove that $\mathcal{T}$ is a homeomorphism between $(\partial\Omega_i, R_{\Omega_i})$ and $(\Gamma_\infty(i), \rho)$. Pick $x \neq y \in \partial\Omega_i$, denote $\mathcal{T}(x) = \boldsymbol{\gamma}$ and $\mathcal{T}(y) = \boldsymbol{\eta}$ and $\boldsymbol{\kappa} = \boldsymbol{\gamma} \wedge \boldsymbol{\eta}$. Let $\{x_n\}, \{y_n\}$ be two sequences converging to x and y separately. Since $x_n, y_n \in \Omega_{\boldsymbol{\kappa}}$ for all large enough n, we have by Proposition 4.3,

$$R_{\Omega_i}(x_n, y_n) \leq C_1 r^{|\boldsymbol{\kappa}|}_{\max} = C_1 \rho(\boldsymbol{\gamma}, \boldsymbol{\eta})^{-\log r_{\max}/\log 2}. \tag{4.6}$$

On the other hand, since $\{x_n\}$, $\{y_n\}$ will enter into two disjoint offsprings of $\Omega_{\boldsymbol{\kappa}}$ for all large n, by using a same argument as (4.5), we have

$$R_{\Omega_i}(x_n, y_n) \geq C_2 r_{\min}^{|\boldsymbol{\kappa}|} = C_2 \rho(\boldsymbol{\gamma}, \boldsymbol{\eta})^{-\log r_{\min}/\log 2} \tag{4.7}$$

for some $C_2 > 0$. By combining (4.6) and (4.7) and letting $n \to \infty$, we have

$$C_2 \rho(\boldsymbol{\gamma}, \boldsymbol{\eta})^{-\log r_{\min}/\log 2} \leq R_{\Omega_i}(x, y) \leq C_1 \rho(\boldsymbol{\gamma}, \boldsymbol{\eta})^{-\log r_{\max}/\log 2}.$$

Combining this with the fact that $\mathcal{T}$ is a surjection, we have $(\partial\Omega_i, R_{\Omega_i})$ is homeomorphic to $(\Gamma_\infty(i), \rho)$. □

In a recent work [18, Theorem 4.5], Kigami and Takahashi obtained a similar result on a particular Ω, the SG minus its bottom line, by utilizing the binary tree structure of that domain.

Remark 1. *Recall that there is a continuous surjective map ι_i from $(\Gamma_\infty(i), \rho)$ to (D_i, d). This induces a continuous surjective map, still denoted as ι_i, from the R-boundary $(\partial\Omega_i, R_{\Omega_i})$ to the d-boundary (D_i, d). Hence if f is a continuous function on D_i, $f \circ \iota_i$ is a continuous function on $\partial\Omega_i$.*

Remark 2. *For $\boldsymbol{\gamma} \in \Gamma_*$, writing $I(\boldsymbol{\gamma}) = i$, $T(\boldsymbol{\gamma}) = j$, we define $\theta_\gamma : \Gamma_\infty(j) \to \Gamma_\infty(i)$ by $\theta_{\boldsymbol{\gamma}}(\boldsymbol{\eta})$ being the concatenation of $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ for each $\boldsymbol{\eta} \in \Gamma_\infty(j)$. By Theorem 4.4, with a slight abuse of notation, $\theta_{\boldsymbol{\gamma}}$ can be viewed as an injection from $\partial\Omega_j$ into $\partial\Omega_i$. It is direct to check that*

$$\Phi_{\boldsymbol{\gamma}} \circ \iota_j = \iota_i \circ \theta_{\boldsymbol{\gamma}}.$$

Also, $\{\partial\Omega_i\}_{i=1}^P$ satisfies the decomposition

$$\partial\Omega_i = \bigcup_{j=1}^{P} \bigcup_{\gamma \in \Gamma(i,j)} \theta_\gamma(\partial\Omega_j), \qquad 1 \leq i \leq P. \tag{4.8}$$

By a standard theory of resistance forms [16, Theorem 2.3.10], the resistance form $(\mathcal{E}_\Omega, \mathcal{F}_\Omega)$ on Ω (we omit the subscripts i for short) extends to be a resistance form on $\widetilde{\Omega}$, and each function in $\mathcal{F}_\Omega$ can be continuously extended to $\widetilde{\Omega}$. Furthermore, for a Radon measure ν on $\widetilde{\Omega}$, $(\mathcal{E}_\Omega, \mathcal{F}_\Omega)$ will generate a Dirichlet form on $L^2(\widetilde{\Omega}, \nu)$, which associates with a nonpositive self-adjoint operator Δ_ν called the Laplacian on $\widetilde{\Omega}$ (see [16, Theorem 2.4.2] or [17, Theorem 9.4]).

For a nonempty closed subset $A \subset \widetilde{\Omega}$, by [17, Lemma 8.2 and Theorem 8.4], for any $u_0 \in \mathcal{F}_\Omega|_A = \{v|_A : v \in \mathcal{F}_\Omega\}$, there exists a unique $u \in \mathcal{F}_\Omega$ such that $u|_A = u_0$ and

$$\mathcal{E}_\Omega[u] = \min\{\mathcal{E}_\Omega[v] : v \in \mathcal{F}_\Omega, v|_A = u_0\}.$$

The function u also satisfies

$$\Delta_\nu u = 0, \quad \text{ in } \widetilde{\Omega} \setminus A,$$

in the weak sense, i.e. $\mathcal{E}_\Omega(u, v) = 0$, for any $v \in \mathcal{F}_\Omega, v|_A = 0$. Say the function u is *harmonic* in $\widetilde{\Omega} \setminus A$, and call u a *harmonic extension* of u_0 from A to $\widetilde{\Omega}$. In particular, when $A = \partial\Omega$, we say u is a harmonic function in Ω with boundary values u_0.

In a standard way [17, Theorem 4.3], for a finite collection $G = \{A_1, \ldots, A_m\}$ of nonempty disjoint closed sets in $\widetilde{\Omega}$, we could trace $(\mathcal{E}_\Omega, \mathcal{F}_\Omega)$ to get a "shorted" resistance form $(\mathcal{E}_G, \ell(G))$ on G (viewing G as a finite point set). Precisely, by identifying $\ell(G)$ with

$$\{u|_{\bigcup_{i=1}^m A_i} : u \in \mathcal{F}_\Omega, u \text{ takes constant values on each } A_i\},$$

define for $f, g \in \ell(G)$, $\mathcal{E}_G(f,g) := \mathcal{E}_\Omega(H^G f, H^G g)$, where $H^G f$ (or $H^G g$) is the unique harmonic extension of f (or g) from $\bigcup_{i=1}^m A_i$ to $\widetilde{\Omega}$. Such a trace will induce an electric network on G. With a little abuse of notation, sometimes for $f \in \ell(G)$, we write $(dH^G f)_p$ instead of $(df)_p$, where p stands for some A_i .

In later sections, we always take G to be a collection of sets in the form of $\theta_\gamma(\partial\Omega_{T(\gamma)})$ with $\boldsymbol{\gamma} \in \Gamma_*$ together with some single points in Ω.

5. **Hitting probability**

Let $(K, \{F_i\}_{i=1}^N, V_0)$ be a p.c.f. self-similar set with $V_0 = \{p_1, \ldots, p_Q\}$ for some $Q \geq 2$. Let $(\mathcal{E}, \mathcal{F})$ be a regular self-similar resistance form on K satisfying (2.1).

Let $P \geq 1$ and $\{\Omega_i\}_{i=1}^P$ be a finite collection of domains in K with d-boundaries $\{D_i\}_{i=1}^P$ satisfying the BGD condition (1.3). Denote by $\{\partial\Omega_i\}_{i=1}^P$ the corresponding R-boundaries of $\{\Omega_i\}_{i=1}^P$ respectively. Denote $\mathcal{A} = \{1, \ldots, P\}$.

Flux transfer matrices. Let $(\mathcal{A}, \Gamma)$ be the directed graph induced from the BGD condition. For each $\gamma \in \Gamma(i,j)$, notice that by BGD, there is a contraction map Φ_γ such that $\Phi_\gamma(\Omega_j) \subset \Omega_i$. In the following, we associate each γ with a $Q \times Q$ real matrix M_γ, whose (k, ℓ)-entry represents:

the flux of the unit flow on $\widetilde{\Omega}_i$ from $\partial\Omega_i$ to p_k through $\Phi_\gamma(p_\ell)$ outwards from $\Phi_\gamma(\Omega_j)$.

For any $1 \leq k \leq Q$, if $p_k \notin \Omega_i$, we simply set the k-th row of M_γ to be zeros; otherwise, if $p_k \in \Omega_i$, let φ be the realization of $R_{\Omega_i}(\partial\Omega_i, p_k)$, i.e. φ is the unique function on $\widetilde{\Omega}_i$ such that $\varphi|_{\partial\Omega_i} = 0$, $\varphi(p_k) = 1$ and φ is harmonic in $\Omega_i \setminus \{p_k\}$ with $\mathcal{E}_{\Omega_i}[\varphi] = R_{\Omega_i}(\partial\Omega_i, p_k)^{-1}$. Let

$$v_k := R_{\Omega_i}(\partial\Omega_i, p_k)\varphi, \tag{5.9}$$

then v_k satisfies $(dv_k)_{p_k} = 1$. Since $\gamma \in \Gamma(i,j)$, we have $\Phi_\gamma(\Omega_j) \subset \Omega_i$. Consider the restriction of the function v_k on $\Phi_\gamma(\Omega_j)$, and denote it as $\tilde{v}_k$, then $\tilde{v}_k$ is harmonic in $\Phi_\gamma(\Omega_j) \setminus \Phi_\gamma(V_0)$. We define $M_\gamma(k,\ell) = (d\tilde{v}_k)_{\Phi_\gamma(p_\ell)}$ for each $p_\ell \in \Omega_j$ and $M_\gamma(k,\ell) = 0$ for $p_\ell \notin \Omega_j$.

We call $\{M_\gamma\}_{\gamma \in \Gamma(i)}$ the *flux transfer matrices* associated with domain Ω_i.

Proposition 5.1. *For $1 \leq i \leq P$ and $1 \leq k \leq Q$ such that $p_k \in \Omega_i \cap V_0$, we have*

$$\sum_{\ell=1}^{Q} M_\gamma(k,\ell) > 0, \quad \forall \gamma \in \Gamma(i),$$

and

$$\sum_{\gamma \in \Gamma(i)} \sum_{\ell=1}^{Q} M_\gamma(k,\ell) = 1. \tag{5.10}$$

Proof. For $\gamma \in \Gamma(i)$, by the strong maximum principle, $v_k > 0$ (defined in (5.9)) on $\Phi_\gamma(\Omega_{T(\gamma)} \cap V_0) \subset \Omega_i$. This gives that $\sum_{\ell=1}^{Q} M_\gamma(k,\ell) = -(dv_k)_{\theta_\gamma(\partial\Omega_{T(\gamma)})} > 0$. The equation (5.10) follows immediately by (2.5) and $(dv_k)_{p_k} = 1$. □

From now, for brevity of notation, for $\boldsymbol{\gamma} \in \Gamma_m$, $m \geq 1$, we write

$$\Omega_{\boldsymbol{\gamma}} := \Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})}), \quad \partial\Omega_{\boldsymbol{\gamma}} := \theta_{\boldsymbol{\gamma}}(\partial\Omega_{T(\boldsymbol{\gamma})}). \tag{5.11}$$

Note that by (4.8), we have

$$\partial\Omega_i = \bigcup_{\boldsymbol{\gamma} \in \Gamma_m(i)} \partial\Omega_{\boldsymbol{\gamma}}, \qquad \text{for all } m \geq 1, \tag{5.12}$$

where the union is disjoint.

Now for those Ω_i with $\Omega_i \cap V_0 \neq \emptyset$, we will use the matrices $\{M_\gamma\}_{\gamma\in\Gamma(i)}$ to construct a class of positive measures $\{\mu_{i,k} : \ p_k \in \Omega_i \cap V_0, 1 \leq k \leq Q\}$ on $\partial\Omega_i$.

Definition 5.2. *For $\boldsymbol{\gamma} = \gamma_1 \cdots \gamma_m \in \Gamma_m(i)$, write $M_{\boldsymbol{\gamma}} = M_{\gamma_1} \cdots M_{\gamma_m}$. We define*

$$\mu_{i,k}(\partial\Omega_{\boldsymbol{\gamma}}) = \mathbf{e}_k^t M_{\boldsymbol{\gamma}} \mathbf{1}, \tag{5.13}$$

where $\mathbf{e}_k = (0, \dots, 1, \dots, 0)^t$ is the Q-dimensional unit vector whose k-th coordinate is 1, and $\mathbf{1}$ is the Q-dimensional vector with all entries equal to 1.

Note that $\mu_{i,k}(\partial\Omega_{\boldsymbol{\gamma}})$ is the summation of the k-th row of $M_{\boldsymbol{\gamma}}$.

Proposition 5.3. *For $p_k \in \Omega_i \cap V_0$, $\mu_{i,k}$ extends to be a probability measure on $\partial\Omega_i$. Moreover, we have the identity*

$$\mu_{i,k} = \sum_{\gamma\in\Gamma(i), 1\leq\ell\leq Q} M_\gamma(k,\ell)\mu_{T(\gamma),\ell} \circ \theta_\gamma^{-1}.$$

Proof. By Proposition 3.1(i), for $\boldsymbol{\gamma} \in \Gamma_m(i)$, $m \geq 1$, it holds that $\Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})} \cap V_0) \neq \emptyset$. On the other hand, by (5.10),

$$\sum_{\eta\in\Gamma(i)} \mu_{i,k}(\partial\Omega_\eta) = \sum_{\eta\in\Gamma(i)} \mathbf{e}_k^t M_\eta \mathbf{1} = \sum_{\eta\in\Gamma(i)} \sum_{\ell=1}^{Q} M_\eta(k,\ell) = 1,$$

and similarly for any $\boldsymbol{\gamma} \in \Gamma_*$ with $I(\boldsymbol{\gamma}) = i$,

$$\sum_{\eta\in\Gamma(T(\boldsymbol{\gamma}))} \mu_{i,k}(\partial\Omega_{\boldsymbol{\gamma}\eta}) = \mathbf{e}_k^t M_{\boldsymbol{\gamma}} \sum_{\eta\in\Gamma(T(\boldsymbol{\gamma}))} M_\eta \mathbf{1} = \mathbf{e}_k^t M_{\boldsymbol{\gamma}} \mathbf{1} = \mu_{i,k}(\partial\Omega_{\boldsymbol{\gamma}}).$$

Hence $\mu_{i,k}$ can be extended to be a probability measure on $\partial\Omega_i$.

Moreover, for $\boldsymbol{\gamma} = \gamma_1 \cdots \gamma_m \in \Gamma_*$, we have

$$\mu_{i,k}(\partial\Omega_{\boldsymbol{\gamma}}) = \mathbf{e}_k^t M_{\gamma_1} M_{\gamma_2\cdots\gamma_m} \mathbf{1} = \sum_{\ell=1}^{Q} M_{\gamma_1}(k,\ell)\mathbf{e}_\ell^t M_{\gamma_2\cdots\gamma_m}\mathbf{1} = \sum_{\ell=1}^{Q} M_{\gamma_1}(k,\ell)\mu_{T(\gamma_1),\ell}(\theta_{\gamma_1}^{-1}(\partial\Omega_{\boldsymbol{\gamma}})). \tag{5.14}$$

Summing up (5.14) with $\gamma_1 \in \Gamma(i)$, we obtain

$$\mu_{i,k} = \sum_{\gamma\in\Gamma(i)} \sum_{\ell=1}^{Q} M_\gamma(k,\ell)\mu_{T(\gamma),\ell} \circ \theta_\gamma^{-1},$$

which finishes the proof. □

We then prove that the probability measures $\{\mu_{i,k} : \ p_k \in \Omega_i \cap V_0, \ 1 \leq k \leq Q\}$ are exactly the hitting probabilities associated with Ω_i, $1 \leq i \leq P$. This is the main result in this section.

Theorem 5.4. *For $p_k \in \Omega_i \cap V_0$, the probability measure $\mu_{i,k}$ in Definition 5.2 is the hitting probability of p_k to the R-boundary $\partial\Omega_i$. Consequently, for any $f \in C(\partial\Omega_i)$, the unique harmonic function u on Ω_i generated by f, i.e. $u|_{\partial\Omega_i} = f$, satisfies*

$$u(p_k) = \int_{\partial\Omega_i} f(x)d\mu_{i,k}(x). \tag{5.15}$$

Proof. We only prove the result when f is a simple function on $\partial\Omega_i$, since the general case will follow by approximating with simple functions. Let $m \geq 1$ be an integer, assume that f is of the form

$$f = \sum_{\gamma \in \Gamma_m(i)} f_\gamma 1_{\partial\Omega_\gamma}, \qquad f_\gamma \in \mathbb{R}. \tag{5.16}$$

Then f is continuous on $\partial\Omega_i$. Let u be the unique harmonic extension of f on Ω_i. Let $v = R_{\Omega_i}(\partial\Omega_i, p_k)\varphi$, where φ is the realization of $R_{\Omega_i}(\partial\Omega_i, p_k)$. Then $v(p_k) = R_{\Omega_i}(\partial\Omega_i, p_k)$, $v|_{\partial\Omega_i} = 0$, $(dv)_{p_k} = 1$ and v is harmonic in $\Omega_i \setminus \{p_k\}$. Notice that both u and v are harmonic in $\Omega_i \setminus \{p_k\}$ and take finitely many different values on the boundary $\partial\Omega_i$. The trace of the energy $\mathcal{E}_{\Omega_i}[\cdot]$ to $\{\partial\Omega_\gamma : \gamma \in \Gamma_m(i)\} \cup \{p_k\}$ is an electric network, and thus we can apply (2.4) with u and v_k to obtain

$$\sum_{\gamma \in \Gamma_m(i)} u(\partial\Omega_\gamma)(dv)_{\partial\Omega_\gamma} + u(p_k)(dv)_{p_k} = \sum_{\gamma \in \Gamma_m(i)} v(\partial\Omega_\gamma)(du)_{\partial\Omega_\gamma} + v(p_k)(du)_{p_k} = 0,$$

where in the last equality we use that $v = 0$ on $\partial\Omega_i$ and $(du)_{p_k} = 0$ by the harmonicity of u at p_k. Then by $(dv)_{p_k} = 1$ and (5.16), we obtain from above that

$$\sum_{\gamma \in \Gamma_m(i)} f_\gamma (dv)_{\partial\Omega_\gamma} + u(p_k) = 0.$$

By using the definition of $\mu_{i,k}$, we have $(dv)_{\partial\Omega_\gamma} = -\mu_{i,k}(\partial\Omega_\gamma)$, so we obtain

$$u(p_k) = \sum_{\gamma \in \Gamma_m(i)} f_\gamma \mu_{i,k}(\partial\Omega_\gamma),$$

proving that (5.15) holds for any simple function f. □

Remark 1. *Recall the first remark after Theorem 4.4, a function $f \in C(D_i)$ naturally induces a function $f \circ \iota_i \in C(\partial\Omega_i)$. In this way, the harmonic function generated by $f \circ \iota_i$ can be viewed as a harmonic extension of f from D_i to Ω_i.*

Remark 2. *In Theorem 5.4, if u is harmonic in $\Omega_i \setminus V_0$, similarly to the proof of* (5.15), *for $p_k \in \Omega_i \cap V_0$, we have*

$$u(p_k) = \int_{\partial\Omega_i} f(x) d\mu_{i,k}(x) + \sum_{x \in \Omega_i \cap V_0} v(x)(du)_x, \tag{5.17}$$

where $v = R_{\Omega_i}(\partial\Omega_i, p_k)\varphi$ with φ being the realization of $R_{\Omega_i}(\partial\Omega_i, p_k)$.

The following property says that for fixed i, the measures $\mu_{i,k}$ are actually equivalent for different $p_k \in \Omega_i$. For convenience, we will also write the measure $\mu_{i,k}$ as $\mu_{i,p}$ if we denote p_k by p.

Proposition 5.5. *For each $i \in \mathcal{A}$, assume $p, p' \in \Omega_i \cap V_0$ and let $\mu_{i,p}, \mu_{i,p'}$ be the associated probability measures. Then there exists a constant $C > 0$ such that for any measurable set $E \subset \partial\Omega_i$,*

$$C^{-1}\mu_{i,p}(E) \leq \mu_{i,p'}(E) \leq C\mu_{i,p}(E). \tag{5.18}$$

Proof. With out loss of generality, we may assume $E = \partial\Omega_\gamma$ for some $\gamma \in \Gamma_m(i)$, $m \geq 1$. Let u_γ be the harmonic function in Ω_i with boundary values

$$u_\gamma = \begin{cases} 1 & \text{in } \partial\Omega_\gamma, \\ 0 & \text{in } \partial\Omega_i \setminus \partial\Omega_\gamma. \end{cases}$$

Then $u_\gamma > 0$ in Ω_i by the strong maximum principle. Note that by Theorem 5.4, $\mu_{i,p}(E) = u_\gamma(p)$ and $\mu_{i,p'}(E) = u_\gamma(p')$. Then by Lemma 4.2, we pick $n_1 \geq 1$ such that p and p' can be connected by a chain of n_1-cells in Ω_i and we denote the union of these cells by A. Now consider $A \cap (\Omega_i \setminus A)$, which is a nonempty finite set in V_{n_1}, denoted as $\{q_1, \ldots, q_\ell\}$. Since u_γ is positive and harmonic in Ω_i, by viewing $\{q_1, \ldots, q_\ell\}$ as the boundary of A, we see that there is a positive probability vector $(w_1, \ldots, w_\ell)$ such that

$$u_\gamma(p) = \sum_{s=1}^{\ell} w_s u_\gamma(q_s), \tag{5.19}$$

where $\sum_{s=1}^{\ell} w_s = 1$ and $w_s > 0$ only depending on the resistance form and A. Similarly, there is a positive probability vector $(w'_1, \ldots, w'_\ell)$ such that

$$u_\gamma(p') = \sum_{s=1}^{\ell} w'_s u_\gamma(q_s). \tag{5.20}$$

Now since $q_s \in \Omega_i$, we have $u_\gamma(q_s) > 0$. Combining (5.19) and (5.20), we see at

$$\min_{1\leq s\leq \ell} \frac{w_s}{w'_s} \leq \frac{u_\gamma(p)}{u_\gamma(p')} \leq \max_{1\leq s\leq \ell} \frac{w_s}{w'_s},$$

which implies (5.18). □

6. Energy estimates

In this section, we characterize harmonic functions in Ω with finite energy in terms of their boundary values.

Let $\{(\Omega_i, D_i)\}_{i=1}^{P}$ be domains in a p.c.f. self-similar set $(K, \{F_i\}_{i=1}^N, V_0)$ satisfying the BGD condition (1.3) and $(\mathcal{E}, \mathcal{F})$ be a self-similar resistance form with energy renormalizing factors $\{r_i\}_{i=1}^N$, $0 < r_i < 1$.

Before proceeding, we give a property for energies of harmonic functions.

Lemma 6.1. *Let u be a harmonic function on K. We have*

$$\mathcal{E}[u] \asymp \sum_{p,q\in V_0} |u(p) - u(q)|^2 \asymp \sum_{p\in V_0} \left|(du)_p\right|^2,$$

where the positive constants in the two "$\asymp$"'s are independent of u.

Proof. The lemma follows from the fact that both of the last two terms are norms on $\ell(V_0)$ modulo constants. □

In the rest of this section, for a domain Ω_i, $i \in \mathcal{A}$, we omit the subscript i for brevity, denote it as Ω, and write D for its d-boundary, $\partial\Omega$ for its R-boundary. For a harmonic function u on Ω with boundary value f on $\partial\Omega$, our purpose is to estimate $\mathcal{E}_\Omega[u]$ from above and below in terms of f.

For two words $\boldsymbol{\eta}, \boldsymbol{\xi} \in \Gamma_m$ with $m \geq 1$, we write $\boldsymbol{\eta} \sim \boldsymbol{\xi}$ if $[\boldsymbol{\eta}]_{m-1} = [\boldsymbol{\xi}]_{m-1} = \boldsymbol{\gamma}$ for some $\boldsymbol{\gamma} \in \Gamma_{m-1}$ (we also write $\boldsymbol{\eta}^- = \boldsymbol{\xi}^- = \boldsymbol{\gamma}$). Note that the two m-cells $\Phi_{\boldsymbol{\eta}}(K)$ and $\Phi_{\boldsymbol{\xi}}(K)$ are contained in the same $(m-1)$-cell $\Phi_{\boldsymbol{\gamma}}(K)$. In the following, we denote $V^{(\boldsymbol{\gamma})} := V_0 \cap \Omega_{T(\boldsymbol{\gamma})}$ and $r_{\boldsymbol{\gamma}} = r_\omega$ with the unique $\omega \in \Sigma^{m-1}$ satisfying $F_\omega = \Phi_{\boldsymbol{\gamma}}$.

For $f \in C(\partial\Omega)$ and $p \in V^{(\boldsymbol{\gamma})}$, we denote

$$f_{\boldsymbol{\gamma},p} = \int_{\partial\Omega_{T(\boldsymbol{\gamma})}} f \circ \theta_{\boldsymbol{\gamma}} d\mu_{T(\boldsymbol{\gamma}),p}.$$

Our main result in this section is the following.

Theorem 6.2. *Let $(K,\{F_i\}_{i=1}^N,V_0)$ be a p.c.f. self-similar set equipped with a self-similar resistance form $(\mathcal{E},\mathcal{F})$ with energy renormalizing factors $\{r_i\}_{i=1}^N$. Let Ω be a domain in K with a graph-directed geometric boundary D and resistance boundary $\partial\Omega$. Assume $\Omega\cap V_0\neq\emptyset$. For $f\in C(\partial\Omega)$, let u be the harmonic extension of f in Ω. Then*

$$\mathcal{E}_\Omega[u]\asymp\sum_{m=0}^{\infty}\sum_{\gamma\in\Gamma_m}\frac{1}{r_\gamma}\sum_{\xi,\eta:\ \xi^-=\eta^-=\gamma}\ \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}}\left(f_{\xi,p}-f_{\eta,q}\right)^2, \tag{6.21}$$

where the constant in "$\asymp$" does not depend on u or f.

Proof. We first show "$\lesssim$" in (6.21). For a given $f\in C(\partial\Omega)$, we will construct a continuous function h in $\widetilde{\Omega}$ such that $h=f$ on $\partial\Omega$. For $m\geq 1$, let $\mathcal{W}_m=\{\omega\in\Sigma^m, F_\omega(K)\subset\Omega, F_{\omega^-}(K)\not\subset\Omega\}$ ($\omega^-=[\omega]_{m-1}$) and $\mathcal{W}_*=\bigcup_{m=1}^\infty\mathcal{W}_m$. Define $\Omega^m=\bigcup\{F_\omega(K):\ \omega\in\mathcal{W}_m\}$, i.e. the union of all m-cells that are contained in Ω_γ for some $\gamma\in\Gamma_{m-1}$. Then $\Omega=\bigcup_{m=1}^\infty\Omega^m$. Define the point set $T_m=\{\Phi_\gamma(V^{(\gamma)}):\ \gamma\in\Gamma_m\}$, and for each point $x=\Phi_\gamma(p)\in T_m$ with $p\in V^{(\gamma)}$, define $h(x)=f_{\gamma,p}$. By the BGD condition, we see that $\Lambda\Omega^m:=\Omega^m\cap(\bigcup\{F_\omega(K):\ \omega\in\mathcal{W}_*\setminus\mathcal{W}_m\})\subset T_m\cup T_{m-1}$. For each $m\geq 1$, we do harmonic extension of h on each Ω^m with the above defined boundary values on $\bigcup_{m=1}^\infty\Lambda\Omega^m$. Then h is a continuous function on Ω and converges to f near the resistance boundary $\partial\Omega$.

We then estimate $\mathcal{E}_{\Omega^m}[h]$ for $m\geq 1$. For $m=1$, by the harmonicity of h in Ω^1, using Lemma 6.1, we have

$$\mathcal{E}_{\Omega^1}[h]\lesssim\sum_{x,y\in\Lambda\Omega^1}(h(x)-h(y))^2=\sum_{\xi^-=\eta^-=\emptyset}\ \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}}\left(f_{\xi,p}-f_{\eta,q}\right)^2. \tag{6.22}$$

For $m\geq 2$, we estimate

$$\begin{aligned}\mathcal{E}_{\Omega^m}[h]&\lesssim\sum_{\gamma\in\Gamma_{m-1}}\frac{1}{r_\gamma}\sum_{x,y\in\Lambda\Omega^m\cap\Omega_\gamma}(h(x)-h(y))^2\\&\lesssim\sum_{\gamma\in\Gamma_{m-1}}\frac{1}{r_\gamma}\sum_{\xi^-=\eta^-=\gamma}\ \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}}\left(f_{\xi,p}-f_{\eta,q}\right)^2+\sum_{\gamma\in\Gamma_{m-1}}\frac{1}{r_\gamma}\sum_{\eta^-=\gamma}\ \sum_{p\in V^{(\gamma)},\ q\in V^{(\eta)}}\left(f_{\gamma,p}-f_{\eta,q}\right)^2.\end{aligned} \tag{6.23}$$

Observe that by Proposition 5.3, for each γ, $f_{\gamma,p}$ is a linear combination of those $f_{\eta,q},\eta^-=\gamma$ with the sum of weights equal to 1, and the weights are some constants independent of f, hence the second term on the RHS of (6.23) can be absorbed into the first term. We obtain

$$\mathcal{E}_{\Omega^m}[h]\lesssim\sum_{\gamma\in\Gamma_{m-1}}\frac{1}{r_\gamma}\sum_{\xi^-=\eta^-=\gamma}\ \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}}\left(f_{\xi,p}-f_{\eta,q}\right)^2. \tag{6.24}$$

By summing up the estimates (6.22), (6.24), we obtain

$$\mathcal{E}_\Omega[u]\leq\mathcal{E}_\Omega[h]=\sum_{m=1}^{\infty}\mathcal{E}_{\Omega^m}[h]\lesssim\sum_{m=0}^{\infty}\sum_{\gamma\in\Gamma_m}\frac{1}{r_\gamma}\sum_{\xi^-=\eta^-=\gamma}\ \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}}\left(f_{\xi,p}-f_{\eta,q}\right)^2,$$

which proves "$\lesssim$" in (6.21).

We then prove the "$\gtrsim$" of (6.21). For any word $\xi\in\Gamma_m$ with $m\geq 1$, we use Φ_ξ^{-1} to pull back the restriction of u on Ω_ξ to get a function on $\Omega_{T(\xi)}$ and denote it by $u^{(\xi)}$. For

$p \in V^{(\xi)}$, denote $v^{(\xi,p)} = R_{\Omega_{T(\xi)}}(\partial\Omega_{T(\xi)}, p)\varphi$ with φ being the realization of $R_{\Omega_{T(\xi)}}(\partial\Omega_{T(\xi)}, p)$. We apply (5.17) to obtain

$$f_{\xi,p} = u^{(\xi)}(p) - \sum_{q\in V^{(\xi)}} v^{(\xi,p)}(q)(du^{(\xi)})_q. \tag{6.25}$$

Now for a pair $\xi \sim \eta$ in Γ_m (i.e. $\xi^- = \eta^- = \gamma$ for some $\gamma \in \Gamma_{m-1}$) and $p \in V^{(\xi)}$, $p' \in V^{(\eta)}$, noticing that $0 \le v^{(\xi,p)} \le R_{\Omega_{T(\xi)}}(\partial\Omega_{T(\xi)}, p)$ and $0 \le v^{(\eta,p')} \le R_{\Omega_{T(\eta)}}(\partial\Omega_{T(\eta)}, p')$ which are bounded by a universal constant, we obtain

$$\begin{aligned}\left(f_{\xi,p} - f_{\eta,p'}\right)^2 &\lesssim \sum_{q\in V^{(\xi)}} \left|(du^{(\xi)})_q\right|^2 + \sum_{q\in V^{(\eta)}} \left|(du^{(\eta)})_q\right|^2 + (u^{(\xi)}(p) - u^{(\eta)}(p'))^2 \\ &\lesssim \begin{cases} r_\gamma \mathcal{E}_{\Omega^m\cap\Omega_\gamma}[u], & \text{if } \xi \neq \eta, \\ r_\xi \mathcal{E}_{\Omega^{m+1}\cap\Omega_\xi}[u], & \text{if } \xi = \eta, \end{cases}\end{aligned} \tag{6.26}$$

where we use Lemma 6.1 in the last estimate in (6.26). Summing up (6.26) with all the pairs $\xi \sim \eta$ in Γ_m and all possible p, p', we get

$$\frac{1}{r_\gamma} \sum_{\gamma\in\Gamma_{m-1}} \sum_{\xi^-=\eta^-=\gamma} \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}} \left(f_{\xi,p} - f_{\eta,q}\right)^2 \lesssim \mathcal{E}_{\Omega^m}[u] + \mathcal{E}_{\Omega^{m+1}}[u]. \tag{6.27}$$

Summing up the inequalities (6.27) for $m \ge 0$, we obtain

$$\sum_{m=0}^{\infty} \sum_{\gamma\in\Gamma_m}^{\infty} \frac{1}{r_\gamma} \sum_{\xi^-=\eta^-=\gamma} \sum_{p\in V^{(\xi)},\ q\in V^{(\eta)}} \left(f_{\xi,p} - f_{\eta,q}\right)^2 \lesssim \mathcal{E}_\Omega[u],$$

proving "$\gtrsim$" in (6.21).

□

7. Examples

In this section, we present several examples. We will first consider the Sierpinski gasket (SG) as a typical example. There is a large class of domains in SG which are constructed by using a straight line to "cut" the SG. We prove that these domains will satisfy the BGD condition if the line is passing through two points in V_* of SG. Then for some typical cases in this class, we compute the corresponding flux transfer matrices which generate the hitting probability measures, see [24, 10, 20, 6, 18] for several previous works. We also present some other examples satisfying the BGD as well as some calculations.

7.1. **Example: Sierpinski gasket.** Let K be the Sierpinski gasket in $\mathbb{R}^2$, generated by the IFS $\{F_i\}_{i=1}^3$ with $F_i(x) = \frac{1}{2}(x - p_i) + p_i$, $i = 1, 2, 3$, and $V_0 = \{p_1, p_2, p_3\}$ is the three vertices of an equilateral triangle. The standard resistance form $(\mathcal{E}, \mathcal{F})$ on K satisfies the self-similar identity [14]

$$\mathcal{E}[u] = \frac{5}{3} \sum_{i=1}^{3} \mathcal{E}[u \circ F_i], \quad \forall u \in \mathcal{F}.$$

Pick arbitrarily two distinct points $p, q \in V_*$. Let L be the straight line passing through p and q. Then L separates the plane into two disjoint (open) parts, say H_1 and H_2. The two sets $\Omega_1 = H_1 \cap K$ and $\Omega_2 = H_2 \cap K$ (if nonempty) are two connected open subsets of K.

Proposition 7.1. *Let $L, p, q, \Omega_1, \Omega_2$ be as above. Both Ω_1 and Ω_2 satisfy the BGD condition.*

Proof. We only prove the proposition for Ω_1, since the Ω_2 case is similar. In the following, we write $\Omega = \Omega_1$ and $H = H_1$.

For convenience, we write $\mathbf{e}_1 = \overrightarrow{p_1p_2}$, $\mathbf{e}_2 = \overrightarrow{p_1p_3}$ for two unit vectors, where $p_1 = O = (0,0)$, $p_2 = (1,0)$ and $p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. By symmetry, we may assume that the line L has the slope in $[0, \sqrt{3})$. For $k \geq 0$ and $\mathbf{x} = \frac{x_1}{2^k}\mathbf{e}_1 + \frac{x_2}{2^k}\mathbf{e}_2$ with integers x_1, x_2, define a map $\varphi_{k,\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varphi_{k,\mathbf{x}}(z) = \frac{z}{2^k} + \mathbf{x}.$$

Let

$$C_k = \{\varphi_{k,\mathbf{x}}(K) : \ \mathbf{x} = \frac{x_1}{2^k}\mathbf{e}_1 + \frac{x_2}{2^k}\mathbf{e}_2, x_1, x_2 \in \mathbb{Z}, \varphi_{k,\mathbf{x}}(K) \cap H \neq \emptyset, \varphi_{k,\mathbf{x}}(K) \cap L \neq \emptyset\}.$$

For $\alpha = \varphi_{k,\mathbf{x}}(K) \in C_k$, denote $\Omega_\alpha = \varphi_{k,\mathbf{x}}^{-1}(\varphi_{k,\mathbf{x}}(K) \cap H)$ and $D_\alpha = \varphi_{k,\mathbf{x}}^{-1}(\varphi_{k,\mathbf{x}}(K) \cap L)$.

Assume $p, q \in V_n$ for some $n \geq 0$.

Claim. The collection $\{(\Omega_\alpha, D_\alpha) : \alpha \in C_n\}$ is finite and satisfy the BGD condition.

Indeed, noting that $p, q \in \frac{\mathbb{Z}}{2^n}\mathbf{e}_1 + \frac{\mathbb{Z}}{2^n}\mathbf{e}_2$, by periodicity, the collection $\{(\Omega_\alpha, D_\alpha) : \alpha \in C_n\}$ is determined by those $\varphi_{n,\mathbf{x}}(K)$ with $\varphi_{n,\mathbf{x}}(K) \cap \overline{pq} \neq \emptyset$ (where $\overline{pq}$ is the line segment connecting p, q), hence is finite.

It suffices to check that for any $\beta \in C_{n+1}$, $\Omega_\beta = \Omega_\alpha$ for some $\alpha \in C_n$. Assume $\alpha = \varphi_{n,\mathbf{x}}(K) \in C_n$ for some $\mathbf{x} = \frac{x_1}{2^n}\mathbf{e}_1 + \frac{x_2}{2^n}\mathbf{e}_2$. Then

$$\begin{aligned} \Omega_\alpha &= 2^n\left(\left(\frac{1}{2^n}K + \mathbf{x}\right) \cap H - \mathbf{x}\right) \\ &= (K + 2^n\mathbf{x}) \cap 2^n H - 2^n\mathbf{x} \\ &= K \cap 2^n(H - \mathbf{x}) \\ &= K \cap (2^n H - x_1\mathbf{e}_1 - x_2\mathbf{e}_2). \end{aligned}$$

Similarly, for $\beta = \varphi_{n+1,\mathbf{y}}(K) \in C_{n+1}$ with $\mathbf{y} = \frac{y_1}{2^{n+1}}\mathbf{e}_1 + \frac{y_2}{2^{n+1}}\mathbf{e}_2$, we have

$$\Omega_\beta = K \cap (2^{n+1}H - y_1\mathbf{e}_1 - y_2\mathbf{e}_2).$$

Since H is determined by the line $L = \{z \in \mathbb{R}^2 : \ \overrightarrow{Oz} = t\overrightarrow{pq} + \overrightarrow{Op}, \ t \in \mathbb{R}\}$, we have $2^k H$ is determined by the line $\{z \in \mathbb{R}^2 : \ \overrightarrow{Oz} = t\overrightarrow{pq} + 2^k\overrightarrow{Op}, \ t \in \mathbb{R}\}$, for any $k \geq 0$. Now for $k = n$ or $k = n+1$, $2^k\overrightarrow{Op} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2$ for some integers k_1, k_2 since $p \in V_n$. Then the half-plane $2^{n+1}H - y_1\mathbf{e}_1 - y_2\mathbf{e}_2$ is determined by some line $L' = \{z \in \mathbb{R}^2 : \ \overrightarrow{Oz} = t\overrightarrow{pq} + k_1'\mathbf{e}_1 + k_2'\mathbf{e}_2, \ t \in \mathbb{R}\}$, for some integers k_1', k_2'. Hence we see that for any $\beta \in C_{n+1}$, there is some $\alpha \in C_n$ such that $\Omega_\beta = \Omega_\alpha$. The claim holds.

From the claim, we immediately see that $\{\Omega_\alpha : \ \alpha \in C_k, \ 0 \leq k \leq n\}$ satisfies the BGD condition. In particular, $\Omega \in \{\Omega_\alpha : \ \alpha \in C_0\}$ satisfies also. □

Now we illustrate two particular situations and calculate their flux transfer matrices.

1. $p = p_1, q = p_2, \Omega = K \setminus \overline{p_1p_2}, D = \overline{p_1p_2}$ (see Figure 3). This is an example in [24] by Owen and Strichartz. By using a Haar basis expansion method, they proved that for this domain Ω, the hitting probability from p_3 to $\partial\Omega$ is the normalized uniform measure on

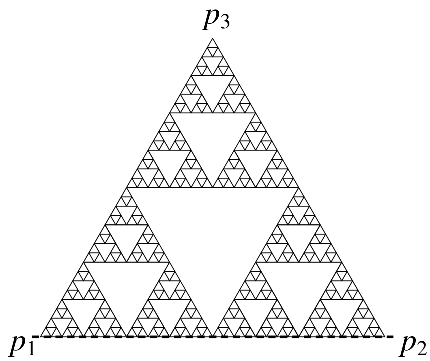

FIGURE 3. a domain in SG

$\partial\Omega$. We refer to [10, 6, 18] for further discussions. Under the general setting here, we can reformulate their result as follows. The boundary $D = \overline{p_1 p_2}$ can be viewed as a self-similar set generated by the IFS $\{F_1, F_2\}$, and this example satisfies the BGD condition, which has the directed graph $(\mathcal{A}, \Gamma)$ with only one vertex $\mathcal{A} = \{1\}$ and two edges $\Gamma = \{\gamma_1, \gamma_2\}$, each of which is from 1 to itself, where γ_i is associated with the contraction map F_i for $i = 1, 2$ respectively. By using that the renormalizing factor $r = \frac{3}{5}$ together with the self-similarity, it is not hard to compute the effective resistance $R_\Omega(\partial\Omega, p_3) = \frac{3}{7}$, and the unit flow from $\partial\Omega$ to p_3 flows outwards $F_i(\Omega)$ through $F_i(p_3)$ with flux $\frac{1}{2}$ for $i = 1, 2$. Thus the flux transfer matrices associated with γ_1 and γ_2 are

$$M_{\gamma_1} = M_{\gamma_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Then by Theorem 5.4, we can compute by using the product of $M_{\gamma_1}, M_{\gamma_2}$ to obtain that the hitting probability from p_3 is the $(\frac{1}{2}, \frac{1}{2})$-self-similar measure on $\partial\Omega$.

2. $p = p_3$, $q = (\frac{1}{2}, 0)$, $\Omega = \{x = (x_1, x_2) \in K : \ x_1 < \frac{1}{2}\}$. Note that $D = L \cap K$ consists of countably many points. By solving systems of countably infinite linear equations, Li and Strichartz [20] computed explicitly the hitting probability from p_1 to $\partial\Omega$ (homeomorphic to D). See also [6] for generalizations by Cao and the second author.

Write $\Omega_1 = \Omega$ with boundary $D_1 = D$, and $\Omega_2 = K \setminus \{p_2\}$ with boundary $D_2 = \{p_2\}$, see Figure 4. Then $\{(\Omega_i, D_i)\}_{i=1}^2$ satisfies the BGD condition with a directed graph $(\mathcal{A}, \Gamma)$:

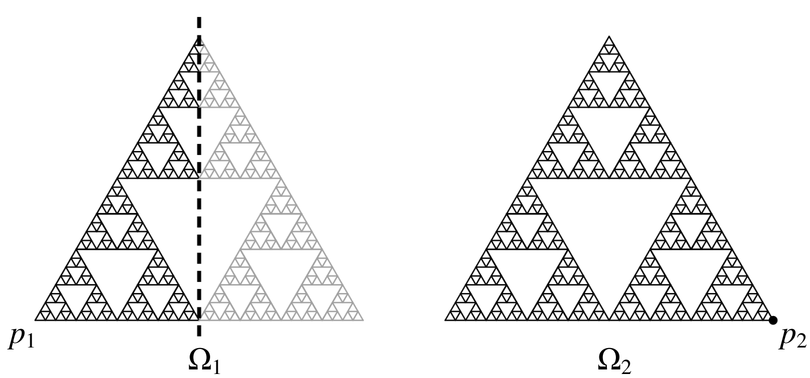

FIGURE 4. domains in SG

$\mathcal{A} = \{1, 2\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, where γ_1 is from 1 to 1 associated with F_3, γ_2 is from 1 to 2 associated with F_1, and γ_3 is from 2 to 2 associated with F_2, see Figure 5.

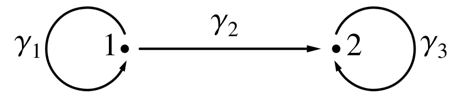

FIGURE 5. The directed graph $(\mathcal{A}, \Gamma)$

Then by a direct calculation, the associated flux transfer matrices are

$$M_{\gamma_1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ M_{\gamma_2} = \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_{\gamma_3} = \begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 0 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

By Theorem 5.4, we see that the hitting probability from p_1 to the boundary $\partial\Omega$ is

$$\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \delta_{F_{3^n 1}(p_2)},$$

where δ_x is the Dirac measure at x.

7.2. **Example: hexagasket.** The hexagasket is a p.c.f. self-similar set generated by the IFS $\{F_i\}_{i=1}^6$, with $F_i(x) = \frac{1}{3}(x - p_i) + p_i$, where $V_0 = \{p_i\}_{i=1}^6$ are the six vertices of a regular hexagon in $\mathbb{R}^2$. The standard resistance form $(\mathcal{E}, \mathcal{F})$ on K satisfies [27]

$$\mathcal{E}[u] = \frac{7}{3} \sum_{i=1}^{6} \mathcal{E}[u \circ F_i], \quad \forall u \in \mathcal{F}.$$

Set $p_1 = (-1, 0)$, $p_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $p_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $p_4 = (1, 0)$, $p_5 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $p_6 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $D = \overline{p_1 p_4} \cap K$, which is a middle-third Cantor set. Let $H = \{x = (x_1, x_2) : \ x_2 > 0\}$ be the (open) upper half plane. We define the domain $\Omega = K \cap H$, with boundary D, see Figure 6. Then (Ω, D) satisfies the BGD condition with the directed graph $(\mathcal{A}, \Gamma)$ given by $\mathcal{A} = \{1\}$ and $\Gamma = \{\gamma_i\}_{i=1}^2$, where both γ_1 and γ_2 are from 1 to itself. The associated flux

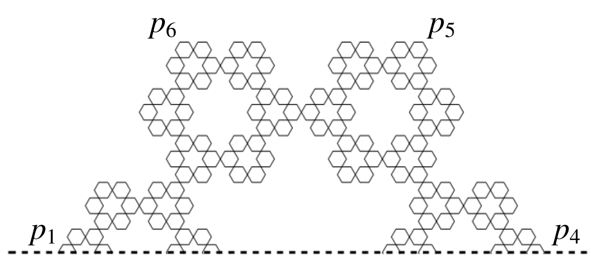

FIGURE 6. a half domain in the hexagasket

transfer matrices are

$$M_{\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 \end{pmatrix}, M_{\gamma_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{pmatrix}.$$

By Theorem 5.4, we obtain the hitting probability from p_5 (or p_6) is a twisted $(1/3, 2/3)$-self-similar measures on $\partial\Omega$.

7.3. **Example: Vicsek set.** The Vicsek set is a p.c.f. self-similar set generated by the IFS $\{F_i\}_{i=1}^5$, with $F_i(x) = \frac{1}{3}(x - p_i) + p_i$, where $V_0 = \{p_i\}_{i=1}^4$ are the four corner vertices of a square and p_5 is its center. The standard resistance form $(\mathcal{E}, \mathcal{F})$ on K satisfies [27]

$$\mathcal{E}[u] = 3 \sum_{i=1}^{5} \mathcal{E}[u \circ F_i], \quad \forall u \in \mathcal{F}.$$

Let $D_1 = \overline{p_1 p_2} \cap K$ and $D_2 = (\overline{p_1 p_2} \bigcup \overline{p_2 p_3}) \cap K$. Then D_1 is a middle-third Cantor set and D_2 is a union of two copies of D_1. Let $\Omega_1 = K \setminus D_1$, $\Omega_2 = K \setminus D_2$ with boundaries D_1, D_2 respectively, see Figure 7. Then $\{(\Omega_i, D_i)\}_{i=1}^2$ satisfies the BGD condition with the directed

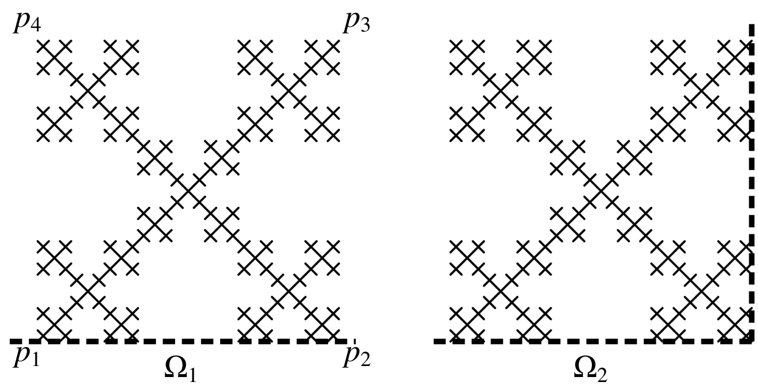

FIGURE 7. domains in the Vicsek set with Cantor boundaries

graph $(\mathcal{A}, \Gamma)$ given by $\mathcal{A} = \{1, 2\}$ and $\Gamma = \{\gamma_i\}_{i=1}^5$ as illustrated in Figure 8, where for brevity we treat domains modulo symmetry. The associated contraction maps of $\{\gamma_i\}_{i=1}^5$

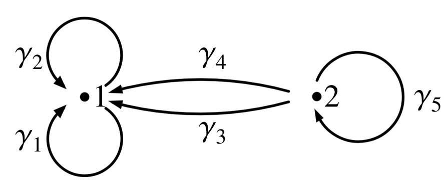

FIGURE 8. The directed graph $(\mathcal{A}, \Gamma)$ in Example 7.3

are F_1, F_2, F_1, $F_3 \circ \kappa$ and F_2, where κ is the counterclockwise rotation by $\frac{\pi}{2}$ around the center p_5. By a direct computation, we obtain the associated flux transfer matrices are

$$M_{\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, M_{\gamma_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, M_{\gamma_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13+t}{26+14t} & 0 \end{pmatrix},$$

$$M_{\gamma_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{13+t}{26+14t} \end{pmatrix}, M_{\gamma_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6t}{13+7t} \end{pmatrix}, \quad \text{where } t = \frac{\sqrt{69} - 2}{5}.$$

For Ω_1, the hitting probability from p_3 (or p_4) is the $(1/2, 1/2)$-self-similar measures on $\partial\Omega_1$. For Ω_2, the hitting probability μ from p_4 to $\partial\Omega_2$ can be described as: for any $k \geq 0$, the restriction of μ on the boundary of $F_{2^k 1}(\Omega_1)$ is a $(1/2, 1/2)$-self-similar measure with total weight $\left(\frac{6t}{13+7t}\right)^k (\frac{13+t}{26+14t})$.

References

[1] M. Barlow, *Diffusions on fractals*, vol. **1690** of Lect. Notes Math., Springer, 1998, pp. 1–121.

[2] M. Barlow, *Random walks and heat kernels on graphs*, London Mathematical Society Lecture Note Series **438**, Cambridge University Press, Cambridge, 2017.

[3] M. Barlow and R. Bass, *On the resistances of the Sierpinski carpet*, Proc. Roy. Soc. London Ser. A **431** (1990), pp. 345–360.

[4] M. Barlow and R. Bass, *The construction of Brownian motion on the Sierpinski carpet*, Ann. Inst. Henri Poincaré **25** (1989), no. 3, pp. 225–257.

[5] M. Barlow and E. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields, **79** (1988), pp. 543–623.

[6] S. Cao and H. Qiu, *Boundary value problems for harmonic functions on domains in Sierpinski gaskets*, Commun. Pure Appl. Anal., **19** (2020), no. 7, pp. 3901–3916.

[7] P. Doyle and J. Snell, *Random walks and electric networks*, Carus Mathematical Monographs Vol. **22**, Mathematical Association of America, Washington, DC 1984.

[8] P. Fitzsimmons, B. Hambly and T. Kumagai, *Transition density estimates for Brownian motion on affine nested fractals*, Comm. Math. Phys., **165** (1994), pp. 595–620.

[9] S. Goldstein, *Random walks and diffusions on fractals*, Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), 121–129, IMA Vol. Math. Appl., 8, Springer, New York, 1987.

[10] Z. Guo, R. Kogan, H. Qiu and R. S. Strichartz, *Boundary value problems for a family of domains in the Sierpinski gasket*, Illinois J. Math., **58** (2014), pp. 497–519.

[11] B. Hambly, V. Metz and A. Teplyaev, *Self-similar energies on post-critically finite self-similar fractals*, J. London Math. Soc. (2), **74** (2006), pp. 93–112.

[12] M. Hino and T. Kumagai, *A trace theorem for Dirichlet forms on fractals*, J. Funct. Anal., **238** (2006), pp. 578–611.

[13] A. Jonsson, *A trace theorem for the Dirichlet form on the Sierpinski gasket*, Math. Zeit., **250** (2005), pp. 599–609.

[14] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math., **6** (1989), no. 2, pp. 259–290.

[15] J. Kigami, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Amer. Math. Soc., **335** (1993), no. 2, pp. 721–755.

[16] J. Kigami, *Analysis on Fractals*, Cambridge Univ. Press, 2001.

[17] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, Mem. Amer. Math. Soc., **216** (2012), no. 1015.

[18] J. Kigami and K. Takahashi, *"The Sierpinski gasket minus its bottom line" as a tree of Sierpinski gaskets*, preprint.

[19] S. Kusuoka, *A diffusion process on a fractal,* in "Probabilistic Methods in Mathematical Physics, Pro. Taniguchi Intern. Symp. (Katata/Kyoto, 1985)", Ito, K., Ikeda, N. (eds.). pp. 251-274, Academic Press, Boston, 1987.

[20] W. Li and R. S. Strichartz, *Boundary value problems on a half Sierpinski gasket*, J. Fractal Geom., **1** (2014), pp. 1–43.

[21] T. Lindstrøm, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc., **83** (1990), no. 420.

[22] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc., **309** (1988), pp. 811–829.

[23] V. Metz, *Hilbert's projective metric on cones of Dirichlet forms*, J. Funct. Anal., **127** (1995), pp. 438–455.

[24] J. Owen and R. S. Strichartz, *Boundary value problems for harmonic functions on a domain in the Sierpinski gasket*, Indiana Univ. Math., **61** (2012), pp. 319–335.

[25] R. Peirone, *Existence of self-similar energies on finitely ramified fractals*, J. Anal. Math., **123** (2014), pp. 35–94.
[26] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals*, Ann. Sci. École Norm. Sup. (4), **30** (1997), pp. 605–673.
[27] R. Strichartz, *Differential equations on fractals: a tutorial*, Princeton University Press, 2006.

Department of Mathematics, Nanjing University, Nanjing, 210093, China
Email address: qingsonggu@nju.edu.cn

Department of Mathematics, Nanjing University, Nanjing, 210093, China
Email address: huaqiu@nju.edu.cn