# A UNIFORM TRACE THEOREM FOR DIRICHLET FORMS ON SIERPINSKI FRACTALS

JIEJIE CAI, HUA QIU, AND YIZHOU WANG

ABSTRACT. We establish trace theorems for the self-similar Dirichlet forms on the Sierpinski gasket and the Sierpinski carpet to their subsets generated by cutting with a straight line. For the Sierpinski gasket, the straight line can be in any direction. For the Sierpinski carpet, we require the straight line parallel to an edge of the carpet. The trace forms are expressed in term of values of functions along the cut, in a uniform way independent of the choices of the line.

## 1. INTRODUCTION

In this paper we consider traces of self-similar Dirichlet forms on two typical fractals, the Sierpinski gasket and Sierpinski carpet, to their subsets generated by cutting with a straight line, see Figure 1.1.

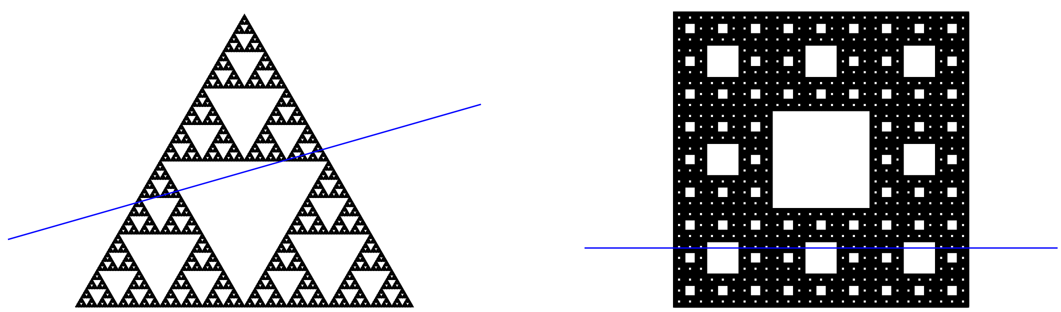

FIGURE 1.1. The Sierpinski gasket (left) and carpet (right) cut by a straight line.

In history, the study of trace theorem of Dirichlet forms on fractals was poineered by Jonsson [10], where he characterized the trace of the standard Dirichlet form on the Sierpinski gasket $\mathcal{SG}$ to its bottom line $\mathcal{I}$ as a $B^{2,2}$-type Besov space. Later, the trace theorem was extended by Hino and Kumagai [9] for self-similar Dirichlet forms on certain self-similar sets to certain self-similar subsets, including on the Sierpinski carpet $\mathcal{SC}$, a typical infinitely ramified fractal, to its bottom line. In these works, the formulas of the trace forms are expressed as discrete approximations. Recently, Kigami and Takahashi provided a trace

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theorem on $\mathcal{SG}\setminus\mathcal{I}$ to its "boundary" [15], a Cantor set from the viewpoint of analysis, and obtained precise expressions of the trace form, as well as the jump kernel of the trace.

It is known that domains of Dirichlet forms on such fractals are $B^{2,\infty}$-type Besov spaces. A general question is to ask what is the trace of a Besov-type space on fractals to certain subsets. In [6], Cao etc. obtained the traces of $B^{2,2}_{\alpha}$-type Besov spaces with lower exponents $\alpha$ from $\mathcal{SG}$ to $\mathcal{I}$, extending that of Jonsson [10].

In this paper, we will use $\mathcal{K}$ to denote either $\mathcal{SG}$ or $\mathcal{SC}$. Let $\rho>1$ be the reciprocal of the contraction ratio of the standard iterated functions generating $\mathcal{K}$, $(\mathcal{E},\mathcal{F})$ be the standard resistance form on $\mathcal{K}$, and $0<r<1$ be the associated renormalization factor. It is known that $\rho=2$ for $\mathcal{SG}$, $\rho=3$ for $\mathcal{SC}$; $r=3/5$ for $\mathcal{SG}$, $r\approx 0.1251$ for $\mathcal{SC}$; and $(\mathcal{E},\mathcal{F})$ will turn out to be a Dirichlet form, providing that we associate a Radon measure $\mu$ on $\mathcal{K}$, which by standard theory [8] will induce an infinitesimal generator, the Laplacian $\Delta_\mu$ on $\mathcal{K}$.

Let $L$ be a straight line intersecting $\mathcal{K}$ and denote $\mathcal{L}:=L\cap\mathcal{K}$. Denote $(\mathcal{E}_\mathcal{L},\mathcal{F}_\mathcal{L})$ the trace of $(\mathcal{E},\mathcal{F})$ to $\mathcal{L}$. We will define

$$\Lambda_\mathcal{L}(u):=\sum_{(x,y)\in E_\mathcal{L}} r^{\frac{\log|x-y|}{\log\rho}}\big(u(x)-u(y)\big)^2,\qquad \textit{for any } u\in C(\mathcal{L}),$$

where $E_\mathcal{L}$ is a countable subset of $\mathcal{L}^2$ generated from cells of $\mathcal{K}$ cut by $L$, see (1.1), (1.2) for the exact definition. The following are the main results in this paper.

**Theorem 1.1.** *Suppose $\mathcal{K}$ is the Sierpinski gasket and $L$ is a straight line cutting $\mathcal{K}$. Then $\mathcal{F}_\mathcal{L}=\{u\in C(\mathcal{L}):\Lambda_\mathcal{L}(u)<\infty\}$, and there is a constant $c>0$ independent of the choice of $L$ satisfying*

$$c^{-1}\Lambda_\mathcal{L}(u)\le\mathcal{E}_\mathcal{L}(u)\le c\Lambda_\mathcal{L}(u),\qquad \textit{for any } u\in\mathcal{F}_\mathcal{L}.$$

**Theorem 1.2.** *Suppose $\mathcal{K}$ is the Sierpinski carpet and $L$ is a straight line cutting $\mathcal{K}$ parallel to the bottom line of $\mathcal{K}$. Then $\mathcal{F}_\mathcal{L}=\{u\in C(\mathcal{L}):\Lambda_\mathcal{L}(u)<\infty\}$, and there is a constant $c>0$ independent of the choice of $L$ satisfying*

$$c^{-1}\Lambda_\mathcal{L}(u)\le\mathcal{E}_\mathcal{L}(u)\le c\Lambda_\mathcal{L}(u),\qquad \textit{for any } u\in\mathcal{F}_\mathcal{L}.$$

*Backgrounds and notations.* For a point $x\in\mathbb{R}^2$, denote $x=(x_1,x_2)$ the *coordinate* of $x$. For two points $x,y\in\mathbb{R}^2$, denote $|x-y|$ the *Euclidean distance* between $x$ and $y$, and denote $\overline{x,y}$ the line segment with endpoints $x,y$. For a set $A\subset\mathbb{R}^2$, we denote the *interior* of $A$ by $int(A)$ and the *closure* of $A$ by $cl(A)$.

Let $\triangle$ and $\square$ be the unit equilateral triangle and square in the plane $\mathbb{R}^2$. Denote the common symbols $\{q^{(i)}\}_{i\in S}$, $\{F_i\}_{i\in S}$ for endpoints of edges of $\triangle$ (or half-edges of $\square$) and iterated function systems, where $\#S=3$ (or 8), and each $F_i$ is a contraction map in term of $F_i(x):=\rho^{-1}(x-q^{(i)})+q^{(i)}$ for $x\in\mathbb{R}^2$. Then $\mathcal{SG}$ (or $\mathcal{SC}$) is the unique nonempty compact set in $\mathbb{R}^2$, denoted as $\mathcal{K}$, satisfying $\mathcal{K}=\bigcup_{i\in S}F_i(\mathcal{K})$.

For $m\ge1$, let $W_m:=S^m$ the collection of *words* of *length* $m$; also, for $m=0$, we define $W_0:=\{\emptyset\}$ where $\emptyset$ is the *empty word* with length 0. Write $W_*:=\bigcup_{m=0}^\infty W_m$ and denote the length of $w\in W_*$ by $|w|$. For $w=w_1w_2\cdots w_m\in W_*\setminus W_0$, we write $w^-:=w_1w_2\cdots w_{m-1}$, $F_w:=F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_m}$ for short; also, for $w=\emptyset$, we write $\emptyset^-:=\emptyset$ and $F_\emptyset$ the identity map. For $w\in W_m$, call $\mathcal{K}_w:=F_w(\mathcal{K})$ an *$m$-level cell* in $\mathcal{K}$.

For a set $A$, denote $\mathbb{R}^A$ the vector space of real-valued functions on $A$, and for $A' \subset A, X \subset \mathbb{R}^A$, denote $X|_{A'} := \{u|_{A'} \in \mathbb{R}^{A'} : u \in X\}$ the *restriction* of $X$ to $A'$. For $A \subset \mathbb{R}^2$, denote $C(A)$ the space of continuous functions on $A$.

The following proposition on the standard self-similar resistance forms on $\mathcal{SG}$ and $\mathcal{SC}$ is well known [1–3, 11, 13, 16–18] (see [13, 14] for knowledge of resistance forms).

**Proposition 1.3.** *Suppose $\mathcal{K}$ is either $\mathcal{SG}$ or $\mathcal{SC}$. There is a resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{K}$ satisfying:*

*1) for each $w \in W_*$, if $u \in \mathcal{F}$, then $u \circ F_w \in \mathcal{F}$,*
*2) for each $m \geq 0$, if $u \in C(\mathcal{K})$ and $u \circ F_w \in \mathcal{F}$ for any $w \in W_m$, then $u \in \mathcal{F}$ and*

$$\mathcal{E}(u) = r^{-m} \sum_{w \in W_m} \mathcal{E}(u \circ F_w),$$

*where $0 < r < 1$ is a constant independent of the choice of $u$ and $m$, called the* renormalization factor *of $\mathcal{E}$,*
*3) there is a constant $c_0 > 0$ such that*

$$\sup_{x,y \in \mathcal{K}} |u(x) - u(y)| \leq c_0 \sqrt{\mathcal{E}(u)}, \quad \text{for any } u \in \mathcal{F}.$$

*In particular $\mathcal{F} \subset C(\mathcal{K})$.*

For a closed set $A \subset \mathcal{K}$, we denote $(\mathcal{E}|_A, \mathcal{F}|_A)$ the *trace form* of $(\mathcal{E},\mathcal{F})$ to $A$, i.e.

$$\mathcal{E}|_A(u) := \inf\big\{\mathcal{E}(\tilde{u}) : \tilde{u} \in \mathcal{F}, \tilde{u}|_A = u\big\}, \quad \text{for any } u \in \mathcal{F}|_A.$$

We only pay attention to the case $A = \mathcal{L}$ for straight lines $L$ that cut $\mathcal{K}$, and abbreviate $(\mathcal{E}|_{\mathcal{L}}, \mathcal{F}|_{\mathcal{L}})$ as $(\mathcal{E}_{\mathcal{L}}, \mathcal{F}_{\mathcal{L}})$.

When $\mathcal{K}$ is $\mathcal{SG}$, we denote the largest up-side down triangle in $\triangle$ as $\bigtriangledown := cl\big(\triangle \setminus \bigcup_{i \in S} F_i(\triangle)\big)$ for convention. Denote $\triangle_w := F_w(\triangle), \bigtriangledown_w := F_w(\bigtriangledown)$ for $w \in W_*$. Define

$$E_{\mathcal{L}} := \{(x,y) \in \mathcal{L}^2 : x \neq y, \text{ there is } w \in W_* \text{ such that } \{x,y\} = \partial(\triangle_w \cap L) \text{ or } \partial(\bigtriangledown_w \cap L)\}, \tag{1.1}$$

where $\partial(A \cap L)$ denotes the endpoints of the line segment $A \cap L$ for $A = \triangle_w$ or $\bigtriangledown_w$.

When $\mathcal{K}$ is $\mathcal{SC}$, we denote $\square_w := F_w(\square)$ for $w \in W_*$. Define

$$E_{\mathcal{L}} := \{(x,y) \in \mathcal{L}^2 : x \neq y, \text{ there is } w \in W_* \text{ such that } \{x,y\} = \partial(\square_w \cap L)\}, \tag{1.2}$$

where $\partial(\square_w \cap L)$ denotes the endpoints of the line segment $\square_w \cap L$.

*Organization.* We will achieve Theorems 1.1 and 1.2 by dividing the proofs into restriction direction and extension direction respectively. The organization of this paper is as follows. For $\mathcal{SG}$, we will prove the restriction theorem in Section 2 and the extension theorem in Section 3. For $\mathcal{SC}$, we will prove the restriction theorem in Section 4 and the extension theorem in Section 5. In Section 6 we will provide some remarks.

## 2. The restriction theorem for the Sierpinski gasket

We specify some nations that will be used throughout Sections 2 and 3. Let $S = \{0,1,2\}$, $q^{(0)} = (0,0), q^{(1)} = (1,0), q^{(2)} = (1/2, \sqrt{3}/2)$ and $\rho = 2$. Then $\mathcal{K}$ is the Sierpinski gasket $\mathcal{SG}$. Denote the edges of $\triangle$ by $l^{(0)} := \overline{q^{(1)}, q^{(2)}}, l^{(1)} := \overline{q^{(0)}, q^{(2)}}$ and $l^{(2)} := \overline{q^{(0)}, q^{(1)}}$.

By the symmetry of $\mathcal{K}$, without loss of generality we only consider the case that the straight cutting line $L$ is the graph of a function $l(x) = bx + a$ with $0 \leq b \leq \sqrt{3}/3$. Note that $L$ is parallel to the edge $l^{(2)}$ when $b = 0$, and orthogonal with the edge $l^{(0)}$ when $b = \sqrt{3}/3$.

In this section, we will prove the following restriction theorem.

**Proposition 2.1.** *Suppose $\mathcal{K}$ is the Sierpinski gasket and $L$ is a straight line cutting $\mathcal{K}$. Then*

$$\Lambda_{\mathcal{L}}(u|_{\mathcal{L}}) \leq c_1 \mathcal{E}(u) \quad \text{for any } u \in \mathcal{F},$$

*where $c_1 > 0$ is a constant independent of the choice of $L$ and $u$.*

Our basic idea is roughly as follows. For a pair $(x, y)$ in $E_{\mathcal{L}}$, we will introduce a sequence of cells $\{\mathcal{K}_w\}_{w \in \mathcal{A}_{x,y}}$ connecting $x$ and $y$ for some index set $\mathcal{A}_{x,y}$, and control the term $\big(u(x) - u(y)\big)^2$ in $\Lambda_{\mathcal{L}}(u)$ from above by the sum of the oscillations of $u$ in $\mathcal{K}_w$ along this sequence, where the latter is further controlled by the energy $\mathcal{E}(u \circ F_w)$. Proposition 2.1 then follows by summing up the above consideration among all pairs $(x, y)$ in $E_{\mathcal{L}}$.

2.1. **Cell array.** From now on, we always fix a straight line $L$, and omit the subscript $\mathcal{L}$ when introduce new notations since no confusion will caused. Define

$$\widehat{W}_* := \{w \in W_* : \triangle_w \cap L = \emptyset, \triangle_{w^-} \cap L \neq \emptyset\}. \tag{2.1}$$

Clearly we have $int(\triangle_w) \cap int(\triangle_v) = \emptyset$ for any $w \neq v$ in $\widehat{W}_*$.

**Definition 2.2.** 1) For a subset $E \subset E_{\mathcal{L}}$, and a map $\mathcal{A} : E \times \mathbb{Z} \to \widehat{W}_*$, call $\mathcal{A}$ a *cell array* of $E$. Denote $w_{x,y}^{(j)} := \mathcal{A}((x, y), j)$ when no confusion caused.

2) Say a cell array $\mathcal{A}$ of $E$ satisfies conditions A1)-A4) with constants $C, N, M > 0$, if:

A1) for any $(x, y) \in E$, there is a connected curve $\gamma \subset \mathcal{K}$ such that $\{x, y\} \subset \gamma \subset cl\big(\bigcup_{j \in \mathbb{Z}} \mathcal{K}_{w_{x,y}^{(j)}}\big)$,

A2) for any $(x, y) \in E$, it holds that $2^{-|w_{x,y}^{(j)}|} \leq C|x - y|$ for all $j \in \mathbb{Z}$,

A3) for any $(x, y) \in E$, it holds that $|w_{x,y}^{(j+1)}| \geq |w_{x,y}^{(j)}|$ and $|w_{x,y}^{(j+N)}| > |w_{x,y}^{(j)}|$ for $j \geq 1$; $|w_{x,y}^{(j-1)}| \geq |w_{x,y}^{(j)}|$ and $|w_{x,y}^{(j-N)}| > |w_{x,y}^{(j)}|$ for $j \leq 0$,

A4) for any $j \in \mathbb{Z}$ and $w \in \widehat{W}_*$, it holds that $\#\{(x, y) \in E : w_{x,y}^{(j)} = w\} \leq M$.

**Remark 2.3.** Condition A1) is to connect each pair $(x, y)$ in $E$ with an infinite sequence of cells (a connected *chain*). Conditions A2) and A3) require that the size of cells in the sequence to be controlled from above by a multiple of the distance between $x$ and $y$, and decreases with a certain ratio when approaches to $x$ or $y$. Condition A4) is a technical requirement of the arrangement of cells in the chains, to guarantee that for each index $j \in \mathbb{Z}$, the repetition of a cell $\mathcal{K}_w$ in $\{w_{x,y}^{(j)}\}_{(x,y) \in E}$ is controlled by a common constant from above.

The following lemma for cell arrays can be checked directly from definition.

**Lemma 2.4.** *1) Suppose $E' \subset E \subset E_{\mathcal{L}}$, and $\mathcal{A}$ is a cell array of $E$ satisfying A1)-A4) with constants $C, N, M$. Then the restriction $\mathcal{A}' := \mathcal{A}|_{E' \times \mathbb{Z}}$ is also a cell array of $E'$ satisfying A1)-A4) with the same constants $C, N, M$.*

*2) Suppose $\{E_i\}_{1 \leq i \leq n}$ is a collection of disjoint subsets in $E_{\mathcal{L}}$, and for each $i$, $\mathcal{A}_i$ is a cell array of $E_i$ satisfying A1)-A4) with constants $C_i, N_i, M_i$. Define $E := \bigcup_{i=1}^n E_i$ and*

$\mathcal{A} : E \times \mathbb{Z} \to \widehat{W}_*$ *such that* $\mathcal{A}|_{E_i \times \mathbb{Z}} = \mathcal{A}_i$ *for all* $i$. *Then* $\mathcal{A}$ *is a cell array of* $E$ *satisfying A1)-A4) with constants* $C := \max_{1\le i\le n} C_i, N := \max_{1\le i\le n} N_i$ *and* $M := \sum_{i=1}^n M_i$.

The following is the main aim in this subsection.

**Lemma 2.5.** *If there is a cell array* $\mathcal{A}$ *of* $E_{\mathcal{L}}$ *satisfying A1)-A4) with universal (independent of the choice of* $L$*) constants* $C, N, M$*, then Proposition 2.1 holds.*

For each $u \in \mathcal{F}, w \in W_*$, denote

$$\mu_{\langle u\rangle}(\mathcal{K}_w) := r^{-|w|}\mathcal{E}(u\circ F_w)$$

and

$$osc_{\langle u\rangle}(\mathcal{K}_w) := \sup_{x,y\in\mathcal{K}_w} |u(x) - u(y)|.$$

Before proving Lemma 2.5, we claim some well known properties of $(\mathcal{E}, \mathcal{F})$.

**Lemma 2.6.** *1) It holds that*

$$\sum_{w\in\widehat{W}_*} \mu_{\langle u\rangle}(\mathcal{K}_w) \le \mu_{\langle u\rangle}(\mathcal{K}), \quad \text{for any } u \in \mathcal{F}. \tag{2.2}$$

*2) For each* $w \in W_*$*, it holds that*

$$osc_{\langle u\rangle}(\mathcal{K}_w) \le c_0 r^{\frac{|w|}{2}} \mu^{\frac{1}{2}}_{\langle u\rangle}(\mathcal{K}_w), \quad \text{for any } u \in \mathcal{F}. \tag{2.3}$$

*where* $c_0$ *is the same constant in Proposition 1.3.*

*Proof.* 1) Note that by Proposition 1.3-2, for each $m \ge 1$,

$$\sum_{w\in\widehat{W}_*:|w|\le m} r^{-|w|}\mathcal{E}(u\circ F_w) \le \sum_{w\in W_m} r^{-|w|}\mathcal{E}(u\circ F_w) = \mathcal{E}(u).$$

The desired inequality follows by letting $m \to \infty$.

2) This follows by $osc_{\langle u\rangle}(\mathcal{K}_w) = osc_{\langle u\circ F_w\rangle}(\mathcal{K}) \le c_0\mu^{\frac{1}{2}}_{\langle u\circ F_w\rangle}(\mathcal{K}) = c_0 r^{\frac{|w|}{2}}\mu^{\frac{1}{2}}_{\langle u\rangle}(\mathcal{K}_w)$. □

*Proof of Lemma 2.5.* By A2) and A3), for any $j \in \mathbb{Z}$ and $(x, y) \in E_{\mathcal{L}}$, we have

$$|w^{(j)}_{x,y}| \ge [\frac{|j|-1}{N}] - \frac{\log(C|x-y|)}{\log\rho}, \tag{2.4}$$

where $[a] := \max\{n \in \mathbb{Z} : n \le a\}$ for $a \in \mathbb{R}$. Combining equation (2.2) and A4), for any $j \in \mathbb{Z}$ and $u \in \mathcal{F}$, we have

$$\sum_{(x,y)\in E_{\mathcal{L}}} \mu_{\langle u\rangle}(\mathcal{K}_{w^{(j)}_{x,y}}) = \sum_{w\in\widehat{W}_*} \#\{(x,y)\in E_{\mathcal{L}} : w^{(j)}_{x,y} = w\}\cdot\mu_{\langle u\rangle}(\mathcal{K}_w) \le M\mu_{\langle u\rangle}(\mathcal{K}). \tag{2.5}$$

Then it holds that

$$\begin{aligned}
\Lambda_{\mathcal{L}}^{\frac{1}{2}}(u|_{\mathcal{L}}) &= \Big(\sum_{(x,y)\in E_{\mathcal{L}}} r^{\frac{\log|x-y|}{\log\rho}}\big(u(x)-u(y)\big)^2\Big)^{\frac{1}{2}}\\
&\le \Big(\sum_{(x,y)\in E_{\mathcal{L}}} r^{\frac{\log|x-y|}{\log\rho}}\big(\sum_{j\in\mathbb{Z}} osc_{\langle u\rangle}(\mathcal{K}_{w_{x,y}^{(j)}})\big)^2\Big)^{\frac{1}{2}}\\
&\le \Big(\sum_{(x,y)\in E_{\mathcal{L}}} r^{\frac{\log|x-y|}{\log\rho}}\big(\sum_{j\in\mathbb{Z}} c_0 r^{\frac{|w_{x,y}^{(j)}|}{2}}\mu_{\langle u\rangle}^{\frac{1}{2}}(\mathcal{K}_{w_{x,y}^{(j)}})\big)^2\Big)^{\frac{1}{2}}\\
&\le c_0\Big(\sum_{(x,y)\in E_{\mathcal{L}}}\big(\sum_{j\in\mathbb{Z}} r^{\frac{1}{2}([\frac{|j|-1}{N}]-\frac{\log C}{\log\rho})}\mu_{\langle u\rangle}^{\frac{1}{2}}(\mathcal{K}_{w_{x,y}^{(j)}})\big)^2\Big)^{\frac{1}{2}}\\
&\le c_0\sum_{j\in\mathbb{Z}}\big(\sum_{(x,y)\in E_{\mathcal{L}}} r^{[\frac{|j|-1}{N}]-\frac{\log C}{\log\rho}}\mu_{\langle u\rangle}(\mathcal{K}_{w_{x,y}^{(j)}})\big)^{\frac{1}{2}}\\
&\le c_0\sum_{j\in\mathbb{Z}} r^{\frac{1}{2}([\frac{|j|-1}{N}]-\frac{\log C}{\log\rho})}M^{\frac{1}{2}}\mu_{\langle u\rangle}^{\frac{1}{2}}(\mathcal{K}) = \sqrt{c_1\mathcal{E}(u)},
\end{aligned}$$

where the second line follows by A1), the third line follows by (2.3), the fourth line follows by (2.4), the fifth line follows by Minkowski inequality, the sixth line follows by (2.5), and the constant $c_1 := \big(c_0\sum_{j\in\mathbb{Z}} r^{\frac{1}{2}([\frac{|j|-1}{N}]-\frac{\log C}{\log\rho})}M^{\frac{1}{2}}\big)^2 < \infty$ is independent of $L$ and $u$. □

2.2. **Proof of Proposition 2.1.** By Lemma 2.5, it amounts to construct a cell array of $E_{\mathcal{L}}$ satisfying conditions A1)-A4) with constants independent of $L$. Clearly, it suffices to consider the cell array of $E := \{(x,y)\in E_{\mathcal{L}} : x_1 < y_1\}$. We divide $E$ into three parts:

$$E_i := \{(x,y)\in E : \textit{there is } w\in W_* \textit{ such that } \{x,y\} = \partial(\triangledown_w\cap L), y\in F_{wi}(l^{(i)})\} \textit{ for } i=1,2,$$

and

$$E_3 := \{(x,y)\in E : \textit{there is } w\in W_* \textit{ such that } \{x,y\} = \partial(\triangle_w\cap L)\}.$$

See Figure 2.1 for an illustration. Then $E = \bigcup_{i=1}^3 E_i$. Once we construct a cell array $\mathcal{A}_i$ of each $E_i$ satisfying A1)-A4) with constants $C_i, N_i, M_i$, then we will derive the desired cell array $\mathcal{A}$ of $E$ by combining $\mathcal{A}_i$ together using Lemma 2.4.

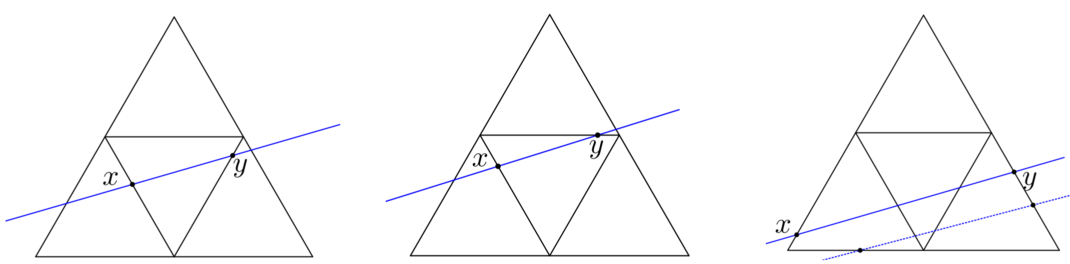


FIGURE 2.1. A pair $(x,y)$ in $E_1$ (left), $E_2$ (middle), or $E_3$ (right).

First, we construct cell arrays of $E_1, E_2$. For each $(x,y)\in E_i, i\in\{1,2\}$, there is a unique $w_{x,y}\in W_*$ such that $\{x,y\}=\partial(\triangledown_{w_{x,y}}\cap L)$. Denote $p_{x,y}:=F_{w0}(q^{(i)})$ and a zigzag line $\gamma_{x,y}:=\overline{x,p_{x,y}}\cup\overline{p_{x,y},y}\setminus\{x,y\}$, see Figure 2.2. Put

$$\mathcal{A}_{x,y}:=\{w\in\widehat{W}_*:\triangle_w\cap\gamma_{x,y}\neq\emptyset\}$$

as the sequence of cells connecting $x,y$ (with orders to be determined later).

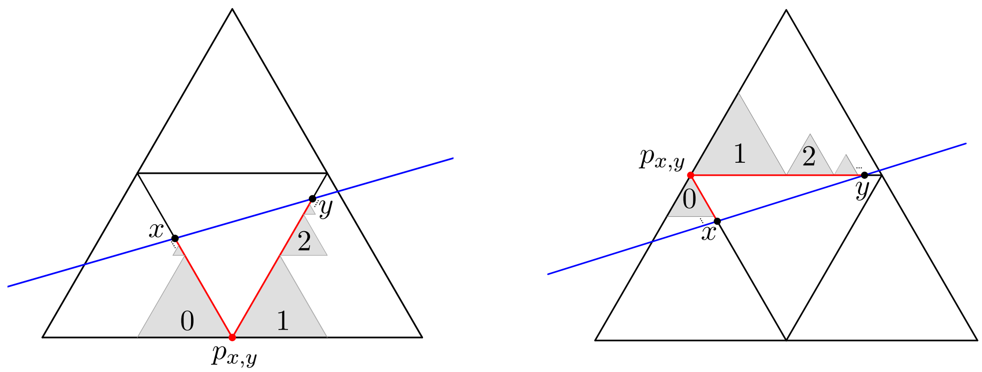


FIGURE 2.2. $\gamma_{x,y}$ (red line) and orders of cells in $\mathcal{A}_{x,y}$ for $(x,y)$ in $E_1$ (left) or $E_2$ (right).

For $(x,y)\in E_1$, we arrange $\mathcal{A}_{x,y}=\{w_{x,y}^{(j)}\}_{j\in\mathbb{Z}}$ such that

1) $p_{x,y}=F_{w_{x,y}^{(0)}}(q^{(1)})=F_{w_{x,y}^{(1)}}(q^{(0)})$,

2) denoting $z^{(j)}:=F_{w_{x,y}^{(j)}}(q^{(2)})$, $z_1^{(j)}<z_1^{(j+1)}$ for $j\in\mathbb{Z}$, where $z_1^{(j)}$ is the first coordinate of $z^{(j)}=(z_1^{(j)},z_2^{(j)})$.

Define $\mathcal{A}_1((x,y),j):=w_{x,y}^{(j)}$ for $j\in\mathbb{Z}$ and $(x,y)\in E_1$, see Figure 2.2 (left).

For $(x,y)\in E_2$, we arrange $\mathcal{A}_{x,y}=\{w_{x,y}^{(j)}\}_{j\in\mathbb{Z}}$ such that

1) $p_{x,y}=F_{w_{x,y}^{(0)}}(q^{(2)})=F_{w_{x,y}^{(1)}}(q^{(0)})$,

2) denoting $z^{(j)}:=F_{w_{x,y}^{(j)}}(q^{(2)})$, $z_1^{(j)}<z_1^{(j+1)}$ for $j\geq 1$ and $z_1^{(j)}<z_1^{(j-1)}$ for $j\leq 0$.

Define $\mathcal{A}_2((x,y),j):=w_{x,y}^{(j)}$ for $j\in\mathbb{Z},(x,y)\in E_2$, see Figure 2.2 (right).

In both two cases, $\triangle_{w_{x,y}^{(0)}},\triangle_{w_{x,y}^{(1)}}$ are the two cells adjacent to $p_{x,y}$, and cells in $\mathcal{A}_{x,y}$ are ordered along the zigzag line $\gamma_{x,y}$ from $x$ to $y$ passing $p_{x,y}$.

**Lemma 2.7.** *For $i=1,2$, $\mathcal{A}_i$ is a cell array of $E_i$ satisfying A1)-A4) with constants $C_i=\frac{2}{\sqrt{3}},N_i=1$ and $M_i=1$.*

*Proof.* We prove the lemma for $\mathcal{A}_1$, since that for $\mathcal{A}_2$ is same.

A1) By taking $\gamma:=cl(\gamma_{x,y})$, it holds.

A2) For $j\geq 1$, we have

$$2^{-|w_{x,y}^{(j)}|}=|F_{w_{x,y}^{(j)}}(q^{(0)})-F_{w_{x,y}^{(j)}}(q^{(2)})|\leq|p_{x,y}-y|\leq\frac{2}{\sqrt{3}}|x-y|,$$

and for $j \le 0$,

$$2^{-|w_{x,y}^{(j)}|} = |F_{w_{x,y}^{(j)}}(q^{(1)}) - F_{w_{x,y}^{(j)}}(q^{(2)})| \le |x - p_{x,y}| \le |x-y|.$$

A3) For $j \ge 1$, take $v \in W_*$ such that $|v| = |w_{x,y}^{(j+1)}|$ and $F_v(q^{(2)}) = F_{w_{x,y}^{(j+1)}}(q^{(0)})$. We claim that $int(\triangle_{v^-}) \cap int(\triangle_{w_{x,y}^{(j+1)}}) = \emptyset$. Otherwise $(w_{x,y}^{(j+1)})^- = v^-$, which implies $\triangle_{(w_{x,y}^{(j+1)})^-} \cap L = \emptyset$, a contradiction to (2.1). This claim implies $|w_{x,y}^{(j)}| \le |v^-| < |v| = |w_{x,y}^{(j+1)}|$. The case $j \le 0$ follows by a similar discussion.

A4) Note that for any $(x,y) \neq (x',y') \in E_1$, we have

$$w_{x,y} \neq w_{x',y'} \Rightarrow p_{x,y} \neq p_{x',y'} \Rightarrow F_{w_{x,y}^{(1)}}(q^{(0)}) \neq F_{w_{x',y'}^{(1)}}(q^{(0)}) \Rightarrow w_{x,y}^{(1)} \neq w_{x',y'}^{(1)}.$$

Then inductively, for $j \ge 1$,

$$w_{x,y}^{(j)} \neq w_{x',y'}^{(j)} \Rightarrow F_{w_{x,y}^{(j)}}(q^{(2)}) \neq F_{w_{x',y'}^{(j)}}(q^{(2)}) \Rightarrow F_{w_{x,y}^{(j+1)}}(q^{(0)}) \neq F_{w_{x',y'}^{(j+1)}}(q^{(0)}) \Rightarrow w_{x,y}^{(j+1)} \neq w_{x',y'}^{(j+1)}.$$

The case $j \le 0$ follows by a same argument. □

Next, we construct the cell array of $E_3$. Define

$$\widehat{W}_*^+ := \{w \in \widehat{W}_* : x_2 > l(x_1) \text{ for any } x \in \triangle_w\}.$$

For $(x,y)$ in $E_3$, we will choose a sequence of cells in $\widehat{W}_*^+$. Take the unique $w_{x,y} \in W_*$ such that $\{x,y\} = \partial(\triangle_{w_{x,y}} \cap L)$ and $\{x,y\} \neq \partial(\triangle_{w_{x,y}i} \cap L)$ for any $i \in S$. Then $x \in F_{w_{x,y}}(l^{(1)} \cup l^{(2)})$ and $y \in F_{w_{x,y}}(l^{(0)})$, see Figure 2.1(right). Denote $p_{x,y} := F_{w_{x,y}}(q^{(2)})$. We will construct a cell sequence connecting $x$ and $y$ through two possible ways according to the location of $x$ in $F_{w_{x,y}}(l^{(1)} \cup l^{(2)})$.

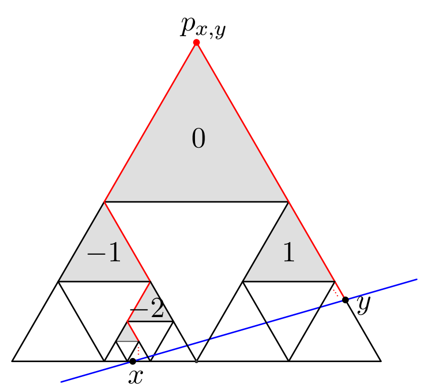


FIGURE 2.3. Orders of cells in $\mathcal{A}_{x,y}$ for $(x,y)$ in $E_3$ with $x \in F_{w_{x,y}}(l^{(2)})$.

For $x \in F_{w_{x,y}}(l^{(1)})$, we take

$$\mathcal{A}_{x,y} := \{w \in \widehat{W}_*^+ : \triangle_w \cap \gamma_{x,y} \neq \emptyset, \triangle_w \subset \triangle_{w_{x,y}}\},$$

where $\gamma_{x,y} := \overline{x, p_{x,y}} \cup \overline{p_{x,y}, y} \setminus \{x,y\}$. We arrange $\mathcal{A}_{x,y} = \{w_{x,y}^{(j)}\}_{j \in \mathbb{Z}}$ such that

1) $p_{x,y} = F_{w_{x,y}^{(0)}}(q^{(2)})$,

2) denoting $z^{(j)} := F_{w^{(j)}_{x,y}}(q^{(2)})$, $z^{(j)}_1 < z^{(j+1)}_1$ for $j \in \mathbb{Z}$.

For $x \in F_{w_{x,y}}(l^{(2)}) \setminus F_{w_{x,y}}(l^{(1)})$, we take $w^{(0)}_{x,y} \in \widehat{W}^+_*$ such that $F_{w^{(0)}_{x,y}}(q^{(2)}) = p_{x,y}$. For $j \geq 1$, we inductively take $w^{(j)}_{x,y} \in \widehat{W}^+_*$ such that $F_{w^{(j)}_{x,y}}(q^{(2)}) = F_{w^{(j-1)}_{x,y}}(q^{(1)})$. For $j \leq 0$, denote the unique $v^{(j)}_{x,y} \in W_*$ such that $F_{v^{(j)}_{x,y}}(q^{(2)}) = F_{w^{(j)}_{x,y}}(q^{(2)})$ and $x \in F_{v^{(j)}_{x,y}}(l^{(2)})$. Then inductively take $w^{(j-1)}_{x,y} \in \widehat{W}^+_*$ such that,

$$F_{w^{(j-1)}_{x,y}}(q^{(2)}) = \begin{cases} F_{w^{(j)}_{x,y}}(q^{(0)}) & \text{if } x \in F_{v^{(j)}_{x,y}0}(l^{(2)}) \\ F_{w^{(j)}_{x,y}}(q^{(1)}) & \text{if } x \in F_{v^{(j)}_{x,y}1}(l^{(2)}) \setminus F_{v^{(j)}_{x,y}0}(l^{(2)}). \end{cases}$$

This procedure gives also a sequence $\mathcal{A}_{x,y} := \{w^{(j)}_{x,y}\}_{j\in\mathbb{Z}}$ connecting $x$ and $y$, see Figure 2.3.

Combining these two cases, we can define a cell array $\mathcal{A}_3$ of $E_3$ by $\mathcal{A}_3((x,y),j) := w^{(j)}_{x,y}$ for $(x,y) \in E_3$ and $j \in \mathbb{Z}$.

**Lemma 2.8.** *$\mathcal{A}_3$ is a cell array of $E_3$ satisfying A1)-A4) with constants $C_3 = \frac{4}{\sqrt{3}}, N_3 = 2$ and $M_3 = 1$.*

*Proof.* A1) For the case $x \in F_{w_{x,y}}(l^{(1)})$, we may take $\gamma = cl(\gamma_{x,y})$. For the case $x \in F_{w_{x,y}}(l^{(2)}) \setminus F_{w_{x,y}}(l^{(1)})$, we have $y \in \overline{p_{x,y}, y} \subset cl\big(\bigcup_{j\geq 0} \mathcal{K}_{w^{(j)}_{x,y}}\big)$. For $j \leq -1$, we have $x \in \triangle_{v^{(j)}_{x,y}}$, which gives that $\lim_{j\to-\infty} |x - F_{w^{(j)}_{x,y}}(q^{(2)})| \leq \lim_{j\to-\infty} 2^{-|v^{(j)}_{x,y}|} = 0$. Thus

$$x \in cl\Big(\bigcup_{j\leq 0} \overline{F_{w^{(j-1)}_{x,y}}(q^{(2)}), F_{w^{(j)}_{x,y}}(q^{(2)})}\Big) \subset cl\big(\bigcup_{j\leq 0} \mathcal{K}_{w^{(j)}_{x,y}}\big).$$

A2) Note that we have $x \in F_{w_{x,y}0}(l^{(1)} \cup l^{(2)})$ by the choice of $w_{x,y}$. This gives $2^{-|w_{x,y}|} \leq \frac{4}{\sqrt{3}}|x-y|$. Since $\triangle_{w^{(j)}_{x,y}} \subset \triangle_{w_{x,y}}$ for any $j \in \mathbb{Z}$, we have

$$2^{-|w^{(j)}_{x,y}|} \leq 2^{-|w_{x,y}|} \leq \frac{4}{\sqrt{3}}|x-y|.$$

A3) For $j \geq 0$, by a same argument in the proof of Lemma 2.7-A3, we have $|w^{(j+1)}_{x,y}| > |w^{(j)}_{x,y}|$.

For $j \leq 0$, take $v \in W_*$ such that $|v| = |w^{(j-2)}_{x,y}|$ and $F_{w^{(j-2)}_{x,y}}(q^{(2)}) \in \{F_v(q^{(0)}), F_v(q^{(1)})\}$. We have three possible cases.

1) $F_{w^{(j-2)}_{x,y}}(q^{(2)}) = F_v(q^{(1)})$. In this case $|w^{(j-1)}_{x,y}| < |w^{(j-2)}_{x,y}|$ follows a same argument as above.

2) $F_{w^{(j-2)}_{x,y}}(q^{(2)}) = F_v(q^{(0)})$ *and* $(w^{(j-2)}_{x,y})^- \neq v^-$. Note that $F_{w^{(j-2)}_{x,y}}(q^{(1)}) \notin L$ implies $\triangle_{v^-} \cap L = \emptyset$. So $|w^{(j-1)}_{x,y}| \leq |v^-| < |v| = |w^{(j-2)}_{x,y}|$.

3) $F_{w^{(j-2)}_{x,y}}(q^{(2)}) = F_v(q^{(0)})$ *with* $(w^{(j-2)}_{x,y})^- = v^-$. In this case $\triangle_{(w^{(j-2)}_{x,y})^-} \cap L \neq \emptyset$ and $\triangle_v \cap L = \emptyset$, which implies $v = w^{(j-1)}_{x,y}$. Take $v' \in W_*$ such that $|v'| = |v|$ and $F_v(q^{(2)}) \in \{F_{v'}(q^{(0)}), F_{v'}(q^{(1)})\}$. Then either $F_v(q^{(2)}) = F_{v'}(q^{(1)})$, or $F_v(q^{(2)}) = F_{v'}(q^{(0)})$ with $v^- \neq (v')^-$. Then by the discussion in Cases 1, 2, we have $|w^{(j)}_{x,y}| < |w^{(j-1)}_{x,y}| = |w^{(j-2)}_{x,y}|$.

A4) Denote $\hat{w} \in \widehat{W}_*^+$ such that $F_{\hat{w}}(q^{(2)}) = q^{(2)}$. In a clear way, we can assign a map $\sigma : \widehat{W}_*^+ \setminus \{\hat{w}\} \to \widehat{W}_*^+$ such that $F_w(q^{(2)}) \in \{F_{\sigma(w)}(q^{(0)}), F_{\sigma(w)}(q^{(1)})\}$ for any $w \in \widehat{W}_*^+ \setminus \{\hat{w}\}$. Clearly $\sigma(w_{x,y}^{(j+1)}) = w_{x,y}^{(j)}$ for $j \geq 0$, $\sigma(w_{x,y}^{(j-1)}) = w_{x,y}^{(j)}$ for $j \leq 0$.

Then for each $(x,y) \neq (x',y') \in E_3$, we have

$$w_{x,y} \neq w_{x',y'} \Rightarrow F_{w_{x,y}}(q^{(2)}) \neq F_{w_{x',y'}}(q^{(2)}) \Rightarrow F_{w_{x,y}^{(0)}}(q^{(2)}) \neq F_{w_{x',y'}^{(0)}}(q^{(2)}) \Rightarrow w_{x,y}^{(0)} \neq w_{x',y'}^{(0)}.$$

Then inductively,

$$w_{x,y}^{(j)} \neq w_{x',y'}^{(j)} \Rightarrow w_{x,y}^{(j+1)} \neq w_{x',y'}^{(j+1)} \quad \text{for } j \geq 0,$$

and

$$w_{x,y}^{(j)} \neq w_{x',y'}^{(j)} \Rightarrow w_{x,y}^{(j-1)} \neq w_{x',y'}^{(j-1)} \quad \text{for } j \leq 0,$$

by the definition of $\sigma$.

Thus the lemma follows. □

*Proof of Proposition 2.1.* Define $\mathcal{A} : E_{\mathcal{L}} \times \mathbb{Z} \to \widehat{W}_*$ by

$$\mathcal{A}((x,y),j) = \begin{cases} \mathcal{A}_1((x,y),j), & \text{if } (x,y) \in E_1 \\ \mathcal{A}_2((x,y),j), & \text{if } (x,y) \in E_2 \setminus E_1 \\ \mathcal{A}_3((x,y),j), & \text{if } (x,y) \in E_3 \setminus (E_1 \cup E_2) \\ \mathcal{A}((y,x),j), & \text{if } x_1 > y_1. \end{cases}$$

Then by Lemmas 2.4 2.7 and 2.8, $\mathcal{A}$ is a cell array of $E_{\mathcal{L}}$ satisfying A1)-A4) with constants $C = \frac{4}{\sqrt{3}}, N = 2$ and $M = 6$. So the proposition follows by Lemma 2.5. □

## 3. The extension theorem for the Sierpinski gasket

For a cell collection $\{w^{(j)}\}_{j=1}^J \subset W_*, J \in \mathbb{N} \cup \{\infty\}$ with $int(\triangle_{w^{(i)}}) \cap int(\triangle_{w^{(j)}}) = \emptyset$ for any $i \neq j$, we denote $\mu_{\langle u \rangle}(\bigcup_{j=1}^J \mathcal{K}_{w^{(j)}}) := \sum_{j=1}^J \mu_{\langle u \rangle}(\mathcal{K}_{w^{(j)}})$ for any $u \in \mathcal{F}$. Clearly this notation is well defined. In this section, we prove the following extension theorem.

**Proposition 3.1.** *Suppose $\mathcal{K}$ is the Sierpinski gasket and $L$ is a straight line cutting $\mathcal{K}$. For any $u \in C(\mathcal{L})$ with $\Lambda_{\mathcal{L}}(u) < \infty$, there is $\tilde{u} \in \mathcal{F}$ such that $\tilde{u}|_{\mathcal{L}} = u$, and*

$$\mathcal{E}(\tilde{u}) \leq c_2 \Lambda_{\mathcal{L}}(u),$$

*where $c_2 > 0$ is a constant independent of $L$ and $u$.*

The proof of Proposition 3.1 consists of five steps.

*Step 1.* For each $u \in C(\mathcal{L})$ with $\Lambda_{\mathcal{L}}(u) < \infty$, define a sequence of functions $\tilde{u}_m \in \mathcal{F}, m \geq 0$.

Fix $m \geq 0$ throughout Steps 2-4.

*Step 2.* Give a finite partition $\tilde{\mathcal{P}}_m = \{\tilde{Q}_w\}_{w \in I}$ of $\mathcal{K}$ such that

$$\mathcal{E}(\tilde{u}_m) = \sum_{w \in I} \mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w).$$

*Step 3.* For each $w \in I$, take $E_w \subset E_{\mathcal{L}}$ such that

$$\mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w) \leq C \sum_{(x,y)\in E_w} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x)-u(y)\big)^2$$

for some constant $C > 0$ independent of the choice of $L, m$ and $u$.

*Step 4.* Prove that there is a constant $N > 0$ independent of the choice of $L, m$ and $u$ such that

$$\#\{w \in I : (x,y) \in E_w\} \leq N, \quad \textit{for any } (x,y) \in E_{\mathcal{L}}.$$

*Step 5.* Prove that there is $\tilde{u} \in \mathcal{F}$ such that $\lim_{m\to\infty} \mathcal{E}(\tilde{u}_m) = \mathcal{E}(\tilde{u})$ and $\tilde{u}|_{\mathcal{L}} = u$.

**Lemma 3.2.** *If Steps 1-5 are fulfilled, then Proposition 3.1 holds.*

*Proof.* Once Steps 1-4 are fulfilled, for any $m \geq 0$ we have

$$\begin{aligned}\mathcal{E}(\tilde{u}_m) = \sum_{w\in I} \mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w) &\leq C \sum_{w\in I} \sum_{(x,y)\in E_w} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x)-u(y)\big)^2 \\ &\leq CN \sum_{(x,y)\in E_{\mathcal{L}}} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x)-u(y)\big)^2 = CN\Lambda_{\mathcal{L}}(u).\end{aligned} \tag{3.1}$$

Moreover by Step 5, $\mathcal{E}(\tilde{u}) = \lim_{m\to\infty} \mathcal{E}(\tilde{u}_m) \leq CN\Lambda_{\mathcal{L}}(u)$. Then Proposition 3.1 holds with $c_2 = CN$. □

The rest part of this section is to fulfill Steps 1-5.

**Step 1.** First for $m \geq 0$, we define a partition $\mathcal{P}_m$ of $\mathcal{K}$. Denote

$$\widehat{W}_m^0 := \{w \in W_m : \triangle_w \cap L \neq \emptyset\}.$$

For each $w \in \bigcup_{m\geq 0} \widehat{W}_m^0$, denote $Q_w := \bigcup_i \mathcal{K}_{wi}$, where $i$ is taken from $S$ such that $\mathcal{K}_{wi} \cap L = \emptyset$. Then there are 5 possible cases of $Q_w$, see Figure 3.1.

1) $J_1 := \{w \in \bigcup_{m\geq 0} \widehat{W}_m^0 : Q_w = \mathcal{K}_{w0} \cup \mathcal{K}_{w1}\}$,
2) $J_2 := \{w \in \bigcup_{m\geq 0} \widehat{W}_m^0 : Q_w = \mathcal{K}_{w1}\}$,
3) $J_3 := \{w \in \bigcup_{m\geq 0} \widehat{W}_m^0 : Q_w = \emptyset\}$,
4) $J_4 := \{w \in \bigcup_{m\geq 0} \widehat{W}_m^0 : Q_w = \mathcal{K}_{w2}\}$,
5) $J_5 := \{w \in \bigcup_{m\geq 0} \widehat{W}_m^0 : Q_w = \mathcal{K}_{w0} \cup \mathcal{K}_{w2}\}$.

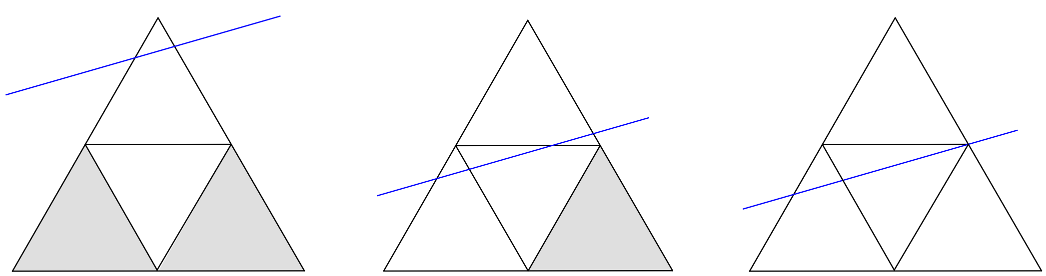

FIGURE 3.1. An illustration for $w$ in $J_1$ (left), $J_2$ (middle) or $J_3$ (right).

For each $m \geq 0$, let

$$\mathcal{P}_m := \{Q_w : w \in \bigcup_{n\leq m-1} \widehat{W}_n^0\} \cup \{\mathcal{K}_w : w \in \widehat{W}_m^0\}.$$

Clearly $\mathcal{P}_m$ is a partition of $\mathcal{K}$.

Next, for each $w \in \bigcup_{m\geq 0} \widehat{W}_m^0$, denote the *boundary* of $Q_w$ by

$$V_w := \big(Q_w \cap cl(\mathcal{K} \setminus Q_w)\big) \cup (Q_w \cap V_0),$$

where $V_0 := \{q^{(0)}, q^{(1)}, q^{(2)}\}$. Clearly we have

$$V_{\mathcal{P}_m} := \Big(\bigcup_{n\leq m-1} \bigcup_{w\in\widehat{W}_n^0} V_w\Big) \cup \Big(\bigcup_{w\in\widehat{W}_m^0} F_w(V_0)\Big) = \bigcup_{n\leq m} \bigcup_{w\in\widehat{W}_n^0} F_w(V_0), \tag{3.2}$$

so $V_{\mathcal{P}_m}$ is collection of junction points of the partition $\mathcal{P}_m$.

Then we define $\tilde{u}_m$ on $V_{\mathcal{P}_m}$:

1) denote

$$V'_{\mathcal{P}_m} := \bigcup_{n\leq m} \bigcup_{w\in\widehat{W}_n^0} \{F_w(q^{(1)}), F_w(q^{(2)})\} \tag{3.3}$$

and define

$$\tilde{u}_m(F_w(q^{(1)})) = \tilde{u}_m(F_w(q^{(2)})) := u(x), \quad \forall w \in \bigcup_{n\leq m} \widehat{W}_n^0,$$

where $x \in \mathcal{L}$ satisfying $\{x\} = L \cap F_w(l^{(0)})$. This gives the values of $\tilde{u}_m$ on $V'_{\mathcal{P}_m}$,

2) for each $w \in \bigcup_{n\leq m} \widehat{W}_n^0$ satisfying $F_w(q^{(0)}) \in V_{\mathcal{P}_m} \setminus V'_{\mathcal{P}_m}$, define

$$\tilde{u}_m(F_w(q^{(0)})) := u(x),$$

where $x \in \mathcal{L}$ such that $\{x\} = L \cap F_w(l^{(1)} \cup l^{(2)})$. This gives the values of $\tilde{u}_m$ on $V_{\mathcal{P}_m} \setminus V'_{\mathcal{P}_m}$.

Finally, we extend $\tilde{u}_m$ from $V_{\mathcal{P}_m}$ to $\mathcal{K}$ such that

$$\mu_{\langle\tilde{u}_m\rangle}(\mathcal{K}_w) = \inf\{\mu_{\langle f\rangle}(\mathcal{K}_w) : f \in \mathcal{F}, f|_{F_w(V_0)} = \tilde{u}_m|_{F_w(V_0)}\}, \quad \forall w \in \widehat{W}_m^0,$$
$$\mu_{\langle\tilde{u}_m\rangle}(Q_w) = \inf\{\mu_{\langle f\rangle}(Q_w) : f \in \mathcal{F}, f|_{V_w} = \tilde{u}_m|_{V_w}\}, \quad \forall w \in \bigcup_{n\leq m-1} \widehat{W}_n^0.$$

So that by Proposition 1.3, $\tilde{u}_m$ is a well defined function in $\mathcal{F}$.

**Step 2.** We now fix an $m \geq 0$, and define a new partition $\tilde{P}_m$ by composing elements in $\mathcal{P}_m$. Denote

$$J^+ := \{w \in \bigcup_{n\leq m-1} \widehat{W}_n^0 : Q_w \neq \emptyset \textit{ and } x_2 > l(x_1) \textit{ for any } x \in Q_w\},$$

$$J^- := \{w \in \bigcup_{n\leq m-1} \widehat{W}_n^0 : Q_w \neq \emptyset \textit{ and } x_2 < l(x_1) \textit{ for any } x \in Q_w\}.$$

Clearly, $J^- \subset J_1 \cup J_2$ and $J^+ \subset J_4 \cup J_5$.

Define an equivalence relation on $J^- \cap J_1$ by $w \sim w'$ if there is $k \geq 0$ such that $F_w = F_{w'} \circ F_2^k$ or $F_{w'} = F_w \circ F_2^k$. Similarly, define an equivalence relation on $J^+ \cap J_5$ by $w \sim w'$ if there is $k \geq 0$ such that $F_w = F_{w'} \circ F_1^k$ or $F_{w'} = F_w \circ F_1^k$, see Figure 3.2.

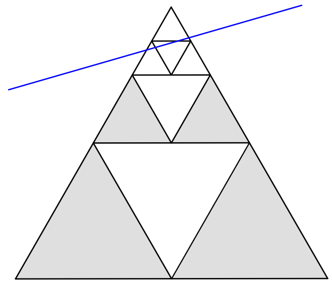

FIGURE 3.2. An examples for $w \sim w'$ in $J^- \cap J_1$.

Define an index set

$$I := \{w \in (J^- \cap J_1) \cup (J^+ \cap J_5) : |w| \leq |v| \text{ for any } v \sim w\} \cup (J^- \cap J_2) \cup (J^+ \cap J_4) \cup \widehat{W}^0_m.$$

For $w \in I \cap (J_1 \cup J_5)$, denote $\tilde{Q}_w := \bigcup_{v \sim w} Q_v$; for $w \in I \cap (J_2 \cup J_4)$, denote $\tilde{Q}_w := Q_w$; and for $w \in I \cap \widehat{W}^0_m$, denote $\tilde{Q}_w := \mathcal{K}_w$. Then $\tilde{\mathcal{P}}_m := \{\tilde{Q}_w\}_{w \in I}$ is a partition of $\mathcal{K}$. Clearly

$$\mathcal{E}(\tilde{u}_m) = \sum_{w \in I} \mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w).$$

**Step 3.** We define $E_w$ for $w \in I$ in 5 possible cases according to $Q_w$ and the value of $\tilde{u}_m$ at $F_w(q^{(0)})$, see Figures 3.3 and 3.4.

1) $w \in I \cap (J_1 \cup J_5)$.

*Case 1.* $Q_w = \mathcal{K}_{w0} \cup \mathcal{K}_{wi}$ *for* $i \in \{1, 2\}$, *and there is* $\tau \in W_*$ *such that* $|\tau| = |w|$, $F_\tau(q^{(i)}) = F_w(q^{(0)})$ *and* $L \cap \triangle_\tau \neq \emptyset$. Take the unique $v \in W_*$ such that $F_{v0}(q^{(i)}) = F_w(q^{(0)})$, and let

$$E_w := \{(x, y) \in \mathcal{L}^2 : x \neq y, \{x, y\} = \partial(\triangledown_v \cap L) \text{ or } \partial(\triangle_w \cap L)\}.$$

*Case 2.* $Q_w = \mathcal{K}_{w0} \cup \mathcal{K}_{wi}$ *for* $i \in \{1, 2\}$ *and* $F_w(q^{(0)}) \in \mathcal{P}_{V_m} \setminus \mathcal{P}'_{V_m}$ (See equations (3.2),(3.3)). Let

$$E_w := \{(x, y) \in \mathcal{L}^2 : x \neq y, \{x, y\} = \partial(\triangle_w \cap L)\}.$$

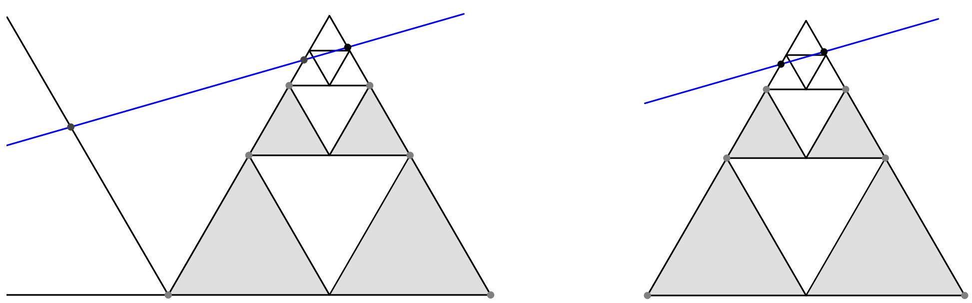

FIGURE 3.3. An illustration for $E_w$ in Cases 1(left) and 2(right) with $i = 1$.

2) $w \in I \cap (J_2 \cup J_4)$.

*Case 3.* $Q_w = \mathcal{K}_{wi}$ *for* $i \in \{1,2\}$. Take $j \in \{1,2\} \setminus \{i\}$, and let

$$E_w := \{(x,y) \in \mathcal{L}^2 : x \neq y, \{x,y\} = \partial(\bigtriangledown_w \cap L) \text{ or } \partial(\bigtriangleup_{wj} \cap L)\}.$$

3) $w \in I \cap \widehat{W}_m^0$.

*Case 4. There is* $\tau \in W_*, i \in \{1,2\}$ *such that* $|\tau| = |w|, F_\tau(q^{(i)}) = F_w(q^{(0)})$ *and* $L \cap \bigtriangleup_\tau \neq \emptyset$. Take the unique $v \in W_*$ such that $F_{v0}(q^{(i)}) = F_w(q^{(0)})$, and let

$$E_w := \{(x,y) \in \mathcal{L}^2 : x \neq y, \{x,y\} = \partial(\bigtriangledown_v \cap L) \text{ or } \partial(\bigtriangleup_w \cap L)\}.$$

*Case 5.* $F_w(q^{(0)}) \in \mathcal{P}_{V_m} \setminus \mathcal{P}'_{V_m}$. Let

$$E_w := \{(x,y) \in \mathcal{L}^2 : x \neq y, \{x,y\} = \partial(\bigtriangleup_w \cap L)\}.$$

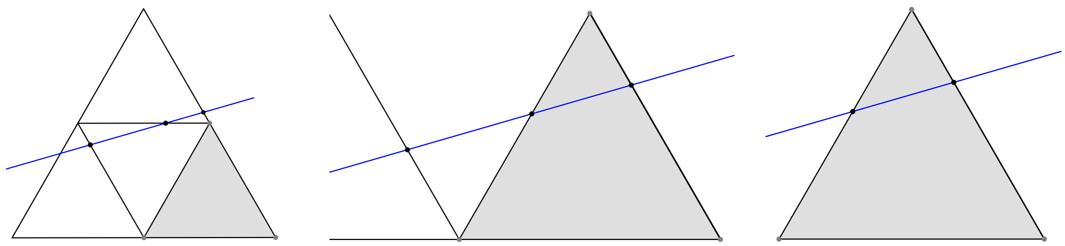

FIGURE 3.4. An illustration for $E_w$ in Cases 3(left), 4(middle) and 5(right) with $i = 1$.

It remains to prove the following lemma.

**Lemma 3.3.** *For any* $L, m, u$ *and* $w \in I$, *it holds that*

$$\mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w) \leq \frac{35}{6} \sum_{(x,y) \in E_w} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x) - u(y)\big)^2.$$

*Proof.* We only prove Cases 1-5 for $i = 1$, since that for $i = 2$ is same.

*Case 1.* Denote $\{x,y\} = \partial(\bigtriangledown_v \cap L)$ and $\{y,z\} = \partial(\bigtriangleup_w \cap L)$ with $x_1 \leq y_1 \leq z_1$. Note that $\bigtriangleup_w \cap L \neq \emptyset$ implies $2^{-|w|} = |F_w(q^{(0)}) - F_w(q^{(2)})| \geq |x-y|$. Take $w'' \sim w$ such that $|w''| \geq |w'|$ for any $w' \sim w$. Then $2^{-|w''|} \geq 2|y-z|$, which gives

$$\begin{aligned}
\mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w) =& \mu_{\langle \tilde{u}_m \rangle}(Q_w) + \sum_{w' \sim w, w' \neq w} \mu_{\langle \tilde{u}_m \rangle}(Q_{w'}) \\
\leq& 2r^{-|w|-1}\big(u(x) - u(y)\big)^2 + 2r^{-|w|-1}\big(u(y) - u(z)\big)^2 + \sum_{j \geq 0} r^{-|w''|-1+j}\big(u(y) - u(z)\big)^2 \\
\leq& \frac{10}{3} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x) - u(y)\big)^2 + \frac{35}{6} r^{\frac{\log|y-z|}{\log \rho}} \big(u(y) - u(z)\big)^2.
\end{aligned}$$

*Case 2.* Denote $\{x,y\} = \partial(\bigtriangleup_w \cap L)$ with $x_1 \leq y_1$. Take $w'' \sim w$ such that $|w''| \geq |w'|$ for any $w' \sim w$. Then we have $2^{-|w''|} \geq 2|x-y|$, which gives

$$\mu_{\langle \tilde{u}_m \rangle}(\tilde{Q}_w) = \sum_{w' \sim w} \mu_{\langle \tilde{u}_m \rangle}(Q_{w'}) \leq \sum_{j \geq 0} r^{-|w''|-1+j}\big(u(x) - u(y)\big)^2 \leq \frac{5}{2} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x) - u(y)\big)^2.$$

*Case 3.* Denote $\{x,y\}=\partial(\bigtriangledown_w\cap L)$ and $\{y,z\}=\partial(\triangle_{w2}\cap L)$ with $x_1\le y_1\le z_1$. Then $2^{-|w|}\ge 2|x-z|=2|x-y|+2|y-z|$, which gives

$$\begin{aligned}\mu_{\langle\tilde u_m\rangle}(\tilde Q_w)=&\mu_{\langle\tilde u_m\rangle}(\mathcal K_{w1})=2r^{-|w|-1}\big(u(x)-u(z)\big)^2\\ \le& 4r^{\frac{\log|x-y|}{\log\rho}}\big(u(x)-u(y)\big)^2+4r^{\frac{\log|y-z|}{\log\rho}}\big(u(y)-u(z)\big)^2.\end{aligned}$$

*Case 4.* Denote $\{x,y\}=\partial(\bigtriangledown_v\cap L)$ and $\{y,z\}=\partial(\triangle_w\cap L)$ with $x_1\le y_1\le z_1$. Then $2^{-|w|}\ge |x-z|=|x-y|+|y-z|$, which gives

$$\mu_{\langle\tilde u_m\rangle}(\tilde Q_w)=2r^{-|w|}\big(u(x)-u(z)\big)^2\le 4r^{\frac{\log|x-y|}{\log\rho}}\big(u(x)-u(y)\big)^2+4r^{\frac{\log|y-z|}{\log\rho}}\big(u(y)-u(z)\big)^2.$$

*Case 5.* Denote $\{x,y\}=\partial(\triangle_w\cap L)$ with $x_1\le y_1$. Then $2^{-|w|}\ge|x-y|$, which gives

$$\mu_{\langle\tilde u_m\rangle}(\tilde Q_w)=2r^{-|w|}\big(u(x)-u(y)\big)^2\le 2r^{\frac{\log|x-y|}{\log\rho}}\big(u(x)-u(y)\big)^2.$$

In summary, the lemma follows. □

**Step 4.** It suffices to prove the following lemma.

**Lemma 3.4.** *For any $(x,y)\in E_{\mathcal L}$, it holds that $\#\{w\in I:(x,y)\in E_w\}\le 4$.*

*Proof.* Suppose $(x,y)=\partial(\triangle_v\cap L)$ with $x\ne y$ for some $v\in W_*$. Then for each $w\in I$, $(x,y)\in E_w$ implies $x\in\triangle_w$. Since $\#\{w\in I: z\in\triangle_w\}\le 2$ for any $z\in\mathcal K$, we have $\#\{w\in I:(x,y)\in E_w\}\le 2$.

Suppose $(x,y)=\partial(\bigtriangledown_v\cap L)$ with $x\ne y$ for some $v\in W_*$. Then for each $w\in I\cap(J_1\cup J_5\cup\widehat W^0_m)$, $(x,y)\in E_w$ implies $F_w(q^{(0)})\in\{F_{v0}(q^{(i)}):i=1,2\}$, so $\#\{w\in I\cap(J_1\cup J_5\cup\widehat W^0_m):(x,y)\in E_w\}\le 2$. For each $w\in I\cap(J_2\cup J_4)$, $(x,y)\in E_w$ implies $x\in\triangle_w$, so $\#\{w\in I\cap(J_2\cup J_4):(x,y)\in E_w\}\le 2$. Thus we have $\#\{w\in I:(x,y)\in E_w\}\le 4$. □

**Step 5.** Note that for any $k\ge m$, we have $\tilde u_k(x)=\tilde u_m(x)$ for $x\in\mathcal K\setminus\bigcup_{w\in\widehat W^0_m}\mathcal K_w$. Define $\tilde u$ on $V_*:=\bigcup_{m\ge0}V_{\mathcal P_m}$ by $\tilde u(p)=\lim_{m\to\infty}\tilde u_m(p)$ for $p\in V_*$. Then $\tilde u\in\mathcal F|_{V_*}$ if and only if $\lim_{m\to\infty}\mathcal E(\tilde u_m)<\infty$. Since we have fulfilled Steps 2-4, by (3.1),

$$\lim_{m\to\infty}\mathcal E(\tilde u_m)\le\frac{70}{3}\Lambda_{\mathcal L}(u)<\infty.$$

So $\tilde u\in\mathcal F|_{V_*}$ and $\lim_{m\to\infty}\mathcal E(\tilde u_m)=\mathcal E(\tilde u)$. Then we can extend $\tilde u$ to $\mathcal K$ continuously by a standard argument.

To see $\tilde u(x)=u(x)$ for any $x\in\mathcal L$, we take $w^{(m)}\in\widehat W^0_m$ such that $x\in\mathcal K_{w^{(m)}}$ for each $m\ge0$. Denote $x^{(m)}\in\mathcal L$ such that $\{x^{(m)}\}=L\cap F_{w^{(m)}}(l^{(0)})$. Then we have

$$\tilde u(x)=\lim_{m\to\infty}\tilde u(F_{w^{(m)}}(q^{(2)}))=\lim_{m\to\infty}\tilde u_m(F_{w^{(m)}}(q^{(2)}))=\lim_{m\to\infty}u(x^{(m)})=u(x).$$

*Proof of Proposition 3.1.* Since Steps 1-5 are fulfilled, Proposition 3.1 follows by Lemma 3.2. □

*Proof of Theorem 1.1.* Combining Proposition 2.1 and 3.1, Theorem 1.1 follows. □

## 4. The restriction theorem for the Sierpinski carpet

From now on, we turn to consider the trace theorem for the Sierpinski carpet. Throughout Sections 4 and 5, let $S = \{0,1,\cdots,7\}$, $q^{(0)} = (0,0), q^{(1)} = (1/2,0), q^{(2)} = (1,0), q^{(3)} = (1,1/2), q^{(4)} = (1,1), q^{(5)} = (1/2,1), q^{(6)} = (0,1), q^{(7)} = (0,1/2)$ and $\rho = 3$. Then $\mathcal{K}$ is the Sierpinski carpet $\mathcal{SC}$. Let $L$ be a straight line cutting $\mathcal{K}$ parallel to an edge of $\square$. By the symmetry of $\mathcal{K}$, without loss of generality we require that $L$ is parallel to the edge $\overline{q^{(0)}, q^{(2)}}$.

In this section, we will prove the following restriction theorem.

**Proposition 4.1.** *Suppose $\mathcal{K}$ is the Sierpinski carpet and $L$ is a straight line cutting $\mathcal{K}$ parallel to an edge of $\square$. Then*

$$\Lambda_{\mathcal{L}}(u|_{\mathcal{L}}) \leq c_3 \mathcal{E}(u), \quad \text{for any } u \in \mathcal{F},$$

*where $c_3 > 0$ is a constant independent of the choice of $L$ and $u$.*

The strategy for the proof of Proposition 4.1 is similar to that in Section 2. For each $w \in W_*$, we write

$$\tilde{\mathcal{K}}_{w,1} := \mathcal{K}_{w4} \cup \mathcal{K}_{w5} \cup \mathcal{K}_{w6},\ \tilde{\mathcal{K}}_{w,2} := \mathcal{K}_{w7},\ \tilde{\mathcal{K}}_{w,3} := \mathcal{K}_{w3},\ \tilde{\mathcal{K}}_{w,4} := \mathcal{K}_{w0} \cup \mathcal{K}_{w1} \cup \mathcal{K}_{w2}.$$

For each $w \in W_*, i \in \{1,2,3,4\}$, denote

$$top(\tilde{\mathcal{K}}_{w,i}) := \{x \in \tilde{\mathcal{K}}_{w,i} : x_2 \geq y_2 \text{ for any } y \in \tilde{\mathcal{K}}_{w,i}\},$$
$$bot(\tilde{\mathcal{K}}_{w,i}) := \{x \in \tilde{\mathcal{K}}_{w,i} : x_2 \leq y_2 \text{ for any } y \in \tilde{\mathcal{K}}_{w,i}\}$$

the top and bottom line in $\tilde{\mathcal{K}}_{w,i}$. Define

$$\widehat{W}_* := \{(w,i) \in W_* \times \{1,2,3,4\} : \tilde{\mathcal{K}}_{w,i} \cap L = \emptyset, \square_w \cap L \neq \emptyset\}.$$

Clearly we have $int(A) \cap int(A') = \emptyset$ for any $(w,i) \neq (w',i') \in \widehat{W}_*$, where $A$ is the convex hull of $\tilde{\mathcal{K}}_{w,i}$, $A'$ is the convex hull of $\tilde{\mathcal{K}}_{w',i'}$.

We decompose $\widehat{W}_*$ into two parts according to the relative position of $\tilde{\mathcal{K}}_{w,i}$ and $L$:

$$\widehat{W}_*^+ := \{(w,i) \in \widehat{W}_* : x_2 > y_2 \text{ for any } x \in \tilde{\mathcal{K}}_{w,i}, y \in L\},$$
$$\widehat{W}_*^- := \{(w,i) \in \widehat{W}_* : x_2 < y_2 \text{ for any } x \in \tilde{\mathcal{K}}_{w,i}, y \in L\}.$$

Denote

$$\widehat{Z}_*^+ := \{(w,i) \in \widehat{W}_*^+ : top(\tilde{\mathcal{K}}_{w,i}) \subset bot(\tilde{\mathcal{K}}_{v,j}) \text{ for some } (v,j) \in \widehat{W}_*^+\},$$
$$\widehat{Z}_*^- := \{(w,i) \in \widehat{W}_*^- : bot(\tilde{\mathcal{K}}_{w,i}) \subset top(\tilde{\mathcal{K}}_{v,j}) \text{ for some } (v,j) \in \widehat{W}_*^-\}.$$

Then we assign a map $\sigma_+ : \widehat{Z}_*^+ \to \widehat{W}_*^+$ such that $\sigma_+(w,i) = (v,j)$ if $top(\tilde{\mathcal{K}}_{w,i}) \subset bot(\tilde{\mathcal{K}}_{v,j})$. Similarly assign $\sigma_- : \widehat{Z}_*^- \to \widehat{W}_*^-$ such that $\sigma_-(w,i) = (v,j)$ if $bot(\tilde{\mathcal{K}}_{w,i}) \subset top(\tilde{\mathcal{K}}_{v,j})$. Clearly, the maps $\sigma_+$, $\sigma_-$ are well defined.

The following lemma is similar to the construction of cell arrays in Section 2.

**Lemma 4.2.** *There is a map $\mathcal{A} : E_{\mathcal{L}} \times \mathbb{Z} \to \widehat{W}_*$ satisfying the following conditions. Denoting $w_{x,y}^{(j)} \in W_*$ such that $\mathcal{A}((x,y),j) = (w_{x,y}^{(j)}, i)$ for some $i \in \{1,2,3,4\}$,*

*1) for any $(x,y) \in E_{\mathcal{L}}$, there is a connected curve $\gamma_{x,y} \subset \mathcal{K}$ such that $\{x,y\} \subset \gamma_{x,y} \subset cl\big(\bigcup_{j\in\mathbb{Z}} \tilde{\mathcal{K}}_{\mathcal{A}((x,y),j)}\big)$,* *2) for any $(x,y)\in E_{\mathcal{L}}$, it holds that $3^{-|w^{(j)}_{x,y}|}\leq |x-y|$ for all $j\in\mathbb{Z}$,*

*3) for any $(x,y)\in E_{\mathcal{L}}$, it holds that $|w^{(j+2)}_{x,y}|>|w^{(j)}_{x,y}|$ for $j\geq 0$ and $|w^{(j-2)}_{x,y}|>|w^{(j)}_{x,y}|$ for $j\leq 0$,*

*4) for any $j\in\mathbb{Z}$ and $(w,i)\in\widehat{W}_*$, it holds that $\#\{(x,y)\in E_{\mathcal{L}}:\mathcal{A}((x,y),j)=(w,i)\}\leq 4$.*

*Proof.* We decompose $E:=\{(x,y)\in E_{\mathcal{L}}:x_1<y_1\}$ into two parts:

$$E_1:=\{(x,y)\in E: \textit{there is } w\in W_* \textit{ such that } \{x,y\}=\partial(\square_w\cap L) \textit{ and } x\notin\square_{w6}\},$$

$$E_2:=\{(x,y)\in E: \textit{there is } w\in W_* \textit{ such that } \{x,y\}=\partial(\square_w\cap L) \textit{ and } x\in\square_{w6}\}.$$

Then $E=E_1\cup E_2$ is a disjoint union.

For each $(x,y)\in E_1$, take the unique $w_{x,y}\in W_*$ such that $\{x,y\}=\partial(\square_{w_{x,y}}\cap L)$ and $x\notin\square_{w_{x,y}6}$. Denote $\gamma_{x,y}:=\overline{x,F_{w_{x,y}}(q^{(6)})}\cup\overline{F_{w_{x,y}}(q^{(6)}),F_{w_{x,y}}(q^{(4)})}\cup\overline{F_{w_{x,y}}(q^{(4)}),y}\setminus\{x,y\}$ and

$$\mathcal{A}_{x,y}:=\{(w,i)\in\widehat{W}^+_*:\tilde{\mathcal{K}}_{w,i}\cap\gamma_{x,y}\neq\emptyset,\tilde{\mathcal{K}}_{w,i}\subset\square_{w_{x,y}}\}.$$

Arrange $\mathcal{A}_{x,y}=\{\mathcal{A}((x,y),j)\}_{j\in\mathbb{Z}}$ such that

1) $\mathcal{A}((x,y),0)=(w_{x,y},1)$,

2) $\sigma_+\big(\mathcal{A}((x,y),j+1)\big)=\mathcal{A}((x,y),j)$ for all $j\geq 0$ and $\sigma_+\big(\mathcal{A}((x,y),j-1)\big)=\mathcal{A}((x,y),j)$ for all $j\leq 0$.

For each $(x,y)\in E_2$, take the unique $w_{x,y}\in W_*$ such that $\{x,y\}=\partial(\square_{w_{x,y}}\cap L)$ and $x\in\square_{w_{x,y}6}$. Denote $\gamma_{x,y}:=\overline{x,F_{w_{x,y}}(q^{(0)})}\cup\overline{F_{w_{x,y}}(q^{(0)}),F_{w_{x,y}}(q^{(2)})}\cup\overline{F_{w_{x,y}}(q^{(2)}),y}\setminus\{x,y\}$ and

$$\mathcal{A}_{x,y}:=\{(w,i)\in\widehat{W}^-_*:\tilde{\mathcal{K}}_{w,i}\cap\gamma_{x,y}\neq\emptyset,\tilde{\mathcal{K}}_{w,i}\subset\square_{w_{x,y}}\}.$$

Arrange $\mathcal{A}_{x,y}=\{\mathcal{A}((x,y),j)\}_{j\in\mathbb{Z}}$ such that

1) $\mathcal{A}((x,y),0)=(w_{x,y},4)$,

2) $\sigma_-\big(\mathcal{A}((x,y),j+1)\big)=\mathcal{A}((x,y),j)$ for all $j\geq 0$ and $\sigma_-\big(\mathcal{A}((x,y),j-1)\big)=\mathcal{A}((x,y),j)$ for all $j\leq 0$.

Now we have defined $\mathcal{A}$ on $E\times\mathbb{Z}$. For $(x,y)\in E_{\mathcal{L}}\setminus E$, we define $\mathcal{A}((x,y),j)=\mathcal{A}((y,x),j)$. This fulfills the definition of $\mathcal{A}$ on $E_{\mathcal{L}}\times\mathbb{Z}$.

It remains to check condition 4 for such $\mathcal{A}$, since conditions 1-3 are obvious.

For each $(x,y)\neq(x',y')\in E_1$, we have

$$w_{x,y}\neq w_{x',y'}\Rightarrow w^{(0)}_{x,y}\neq w^{(0)}_{x',y'}\Rightarrow\mathcal{A}((x,y),0)\neq\mathcal{A}((x',y'),0).$$

Then inductively,

$$\mathcal{A}((x,y),j)\neq\mathcal{A}((x',y'),j)\Rightarrow\mathcal{A}((x,y),j+1)\neq\mathcal{A}((x',y'),j+1)\quad\text{ for } j\geq 0,$$

and

$$\mathcal{A}((x,y),j)\neq\mathcal{A}((x',y'),j)\Rightarrow\mathcal{A}((x,y),j-1)\neq\mathcal{A}((x',y'),j-1)\quad\text{ for } j\leq 0,$$

by the definition of $\sigma_+$, $\sigma_-$. This gives $\#\{(x,y)\in E_1:\mathcal{A}((x,y),j)=(w,i)\}\leq 1$ for any $(w,i)\in\widehat{W}_*$ and $j\in\mathbb{Z}$.

A similar argument gives $\#\{(x,y)\in E_2:\mathcal{A}((x,y),j)=(w,i)\}\leq 1$. Thus $\#\{(x,y)\in E:\mathcal{A}((x,y),j)=(w,i)\}\leq 2$, and so $\#\{(x,y)\in E_{\mathcal{L}}:\mathcal{A}((x,y),j)=(w,i)\}\leq 4$. The condition 4 follows. $\square$

For each $u\in\mathcal{F}, w\in W_*$, similarly as the case for $\mathcal{SG}$, we denote $\mu_{\langle u\rangle}(\mathcal{K}_w):=r^{-|w|}\mathcal{E}(u\circ F_w)$. Moreover, for $\{w^{(j)}\}_{j=1}^J\subset W_*, J\in\mathbb{N}\cup\{\infty\}$ with $int(\square_{w^{(i)}})\cap int(\square_{w^{(j)}})=\emptyset$ for $i\neq j$, we denote $\mu_{\langle u\rangle}(\bigcup_{j=1}^J\mathcal{K}_{w^{(j)}}):=\sum_{j=1}^J\mu_{\langle u\rangle}(\mathcal{K}_{w^{(j)}})$. For $A\subset\mathcal{K}$, $u\in\mathbb{R}^A$, denote $osc_{\langle u\rangle}(A):=\sup_{x,y\in A}|u(x)-u(y)|$. The following lemma is similar to Lemma 2.6.

**Lemma 4.3.** *1) It holds that*

$$\sum_{(w,i)\in\widehat{W}_*}\mu_{\langle u\rangle}(\tilde{\mathcal{K}}_{w,i})\leq\mu_{\langle u\rangle}(\mathcal{K}),\qquad \textit{for any } u\in\mathcal{F}. \tag{4.1}$$

*2) For each $(w,i)\in\widehat{W}_*$, it holds that*

$$osc_{\langle u\rangle}(\tilde{\mathcal{K}}_{w,i})\leq c_0r^{\frac{|w|+1}{2}}\mu^{\frac12}_{\langle u\rangle}(\tilde{K}_{w,i}),\qquad \textit{for any } u\in\mathcal{F}, \tag{4.2}$$

*where $c_0$ is the same constant in Proposition 1.3.*

*Proof.* 1) This follows by a similar argument of the proof of equation (2.2).

2) Let $(w,i)\in\widehat{W}_*$. For the case $i=1$, we have

$$osc_{\langle u\rangle}(\tilde{\mathcal{K}}_{w,i})\leq\sum_{j=4}^{6}osc_{\langle u\rangle}(\mathcal{K}_{wj})\leq\sum_{j=4}^{6}c_0r^{\frac{|w|+1}{2}}\mu^{\frac12}_{\langle u\rangle}(\mathcal{K}_{wj})=c_0r^{\frac{|w|+1}{2}}\mu^{\frac12}_{\langle u\rangle}(\tilde{\mathcal{K}}_{w,i}).$$

The cases $i=2,3,4$ are similar. □

*Proof of Proposition 4.1.* With Lemmas 4.2, 4.3 in hand, the proposition follows in a same way as that for Proposition 2.5. □

## 5. The extension theorem for the Sierpinski carpet

In this section we prove the following extension theorem.

**Proposition 5.1.** *Suppose $\mathcal{K}$ is the Sierpinski carpet and $L$ is a straight line cutting $\mathcal{K}$ parallel to an edge of $\square$. Then for any $u\in C(\mathcal{L})$ with $\Lambda_{\mathcal{L}}(u)<\infty$, there is $\tilde{u}\in\mathcal{F}$ such that $\tilde{u}|_{\mathcal{L}}=u$, and*

$$\mathcal{E}(\tilde{u})\leq c_4\Lambda_{\mathcal{L}}(u),$$

*where $c_4>0$ is a constant independent of $L$ and $u$.*

The strategy for the proof of Proposition 5.1 is similar to that in Section 3. The following lemma is useful.

**Lemma 5.2.** *1) Denote $E_1:=\{(q^{(0)},q^{(2)}),(q^{(2)},q^{(4)}),(q^{(4)},q^{(6)}),(q^{(6)},q^{(0)})\}$, $\partial\mathcal{K}:=\bigcup_{(p,q)\in E_1}\overline{p,q}$ and $U_1:=\{q^{(0)},q^{(2)},q^{(4)},q^{(6)}\}$. Let*

$$X_1:=\big\{u\in C(\partial\mathcal{K}):u(x)=|x-p|\cdot u(q)+|x-q|\cdot u(p) \textit{ for any } (p,q)\in E_1,x\in\overline{p,q}\big\}.$$

*There is a constant $C_1>0$ and a linear operator $H_1:\mathbb{R}^{U_1}\to\mathcal{F}$ such that for $u\in\mathbb{R}^{U_1}$, $(H_1u)|_{U_1}=u$, $(H_1u)|_{\partial\mathcal{K}}\in X_1$ and*

$$\mathcal{E}(H_1u)\leq C_1\sum_{(p,q)\in E_1}\big(u(p)-u(q)\big)^2.$$

*2) Denote* $\mathcal{B} := \bigcup_{i=3}^{7} \mathcal{K}_i$,

$$E_2 := \big\{\big(F_7(q^{(0)}), F_7(q^{(2)})\big), \big(F_3(q^{(0)}), F_3(q^{(2)})\big), \big(F_3(q^{(2)}), q^{(4)}\big), \big(q^{(4)}, q^{(6)}\big), \big(q^{(6)}, F_7(q^{(0)})\big)\big\},$$

$\partial\mathcal{B} := \bigcup_{(p,q)\in E_2} \overline{p,q}$ *and* $U_2 := \big\{p \in \partial\mathcal{B} : \text{there is } (p,q) \text{ or } (q,p) \text{ in } E_2\big\}$*, see Figure 5.1 (left). Let*

$$X_2 := \big\{u \in C(\partial\mathcal{B}) : u(x) = \frac{|x-p|}{|p-q|}u(q) + \frac{|x-q|}{|p-q|}u(p) \text{ for any } (p,q) \in E_2, x \in \overline{p,q}\big\}.$$

*There is a constant* $C_2 > 0$ *and a linear operator* $H_2 : \mathbb{R}^{U_2} \to \mathcal{F}|_{\mathcal{B}}$ *such that for* $u \in \mathbb{R}^{U_2}$, $(H_2u)|_{U_2} = u$, $(H_2u)|_{\partial\mathcal{B}} \in X_2$ *and*

$$\mu_{\langle H_2u\rangle}(\mathcal{B}) \le C_2 \sum_{(p,q)\in E_2} \big(u(p) - u(q)\big)^2.$$

*3) For each* $m \ge 0$*, denote*

$$E_3^{(m)} := \big\{\big(F_w(q^{(0)}), F_w(q^{(2)})\big)\big\}_{w\in\{0,1,2\}^m} \cup \big\{\big(q^{(2)}, q^{(4)}\big), \big(q^{(4)}, q^{(6)}\big), \big(q^{(6)}, q^{(0)}\big)\big\}$$

*and* $U_3^{(m)} := \big\{p \in \partial\mathcal{K} : \text{there is } (p,q) \in E_3^{(m)}\big\}$*, see Figure 5.1 (right). Let*

$$X_3^{(m)} := \big\{u \in C(\partial\mathcal{K}) : u(x) = \frac{|x-p|}{|p-q|}u(q) + \frac{|x-q|}{|p-q|}u(p) \text{ for any } (p,q) \in E_3^{(m)}, x \in \overline{p,q}\big\}.$$

*There is a constant* $C_3 > 0$ *and a linear operator* $H_3^{(m)} : \{u \in \mathbb{R}^{U_3^{(m)}} : u(q^{(0)}) = u(q^{(6)}), u(q^{(2)}) = u(q^{(4)})\} \to \mathcal{F}$ *such that for each* $u$, $(H_3^{(m)}u)|_{U_3^{(m)}} = u$, $(H_3^{(m)}u)|_{\partial\mathcal{K}} \in X_3^{(m)}$ *and*

$$\mathcal{E}(H_3^{(m)}u) \le C_3 \sum_{(p,q)\in\bigcup_{n=0}^{m} E_3^{(n)}} r^{\frac{\log|p-q|}{\log\rho}} \big(u(p) - u(q)\big)^2.$$

*Proof.* Statement (1) follows from [5, Section 4.2] using a building brick technique, see also [9, Section 5.3]. Statement (2) follows from first assigning values to $F_i(U_1)$ for each $i \in \{3, \cdots, 7\}$ linearly according to the values of $u$ on $U_2$, then applying the statement (1) to each $\mathcal{K}_i$. Statement (3) follows in a similar way as above by locally using statements (1) and (2). □

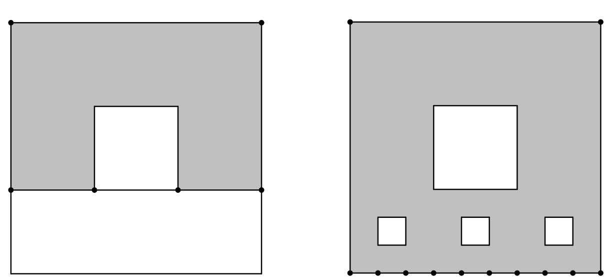

FIGURE 5.1. An illustration of $\mathcal{B}, U_2$ (left) and $U_3^{(2)}$ (right).

For each $u \in C(\mathcal{L})$ with $\Lambda_{\mathcal{L}}(u) < \infty$, we will define a sequence of functions $\tilde{u}_m \in \mathcal{F}, m \ge 0$.

For $m \geq 0$, denote

$$\widehat{W}_m^0 := \{w \in W_m : \square_w \cap L \neq \emptyset\},$$

and for $w \in \bigcup_{m\geq 0} \widehat{W}_m^0$, denote $P_w := \{F_w(q^{(i)})\}_{i\in\{0,2,4,6\}}$. For each $m \geq 0$, denote $P_m := \bigcup_{n=0}^{m} \bigcup_{w\in\widehat{W}_n^0} P_w$.

Define $u'_m \in \mathbb{R}^{P_m}$ such that for each $w \in \bigcup_{n=0}^{m} \widehat{W}_n^0$, $u'_m(F_w(q^{(0)})) = u'_m(F_w(q^{(6)})) := u(x)$ for $x \in F_w\big(\overline{q^{(0)}, q^{(6)}}\big) \cap L$, and $u'_m(F_w(q^{(2)})) = u'_m(F_w(q^{(4)})) := u(y)$ for $y \in F_w\big(\overline{q^{(2)}, q^{(4)}}\big) \cap L$.

For $w \in \bigcup_{n\geq 0} \widehat{W}_n^0$, denote $Q_w^+ := \bigcup_i \mathcal{K}_{wi}$, where $i$ is taken from $S$ such that $x_2 > y_2$ for any $x \in \mathcal{K}_{wi}$ and $y \in L$; denote $Q_w^- := \bigcup_i \mathcal{K}_{wi}$, where $i$ is taken from $S$ such that $x_2 < y_2$ for any $x \in \mathcal{K}_{wi}$ and $y \in L$; denote $Q_w^0 := \mathcal{K}_w$. Let

1) $J_{m,1}^+ := \{(w,+) : w \in \bigcup_{n=0}^{m-1} \widehat{W}_n^0 : Q_w^+ = \bigcup_{i=4}^{6} \mathcal{K}_{wi}\}$,
2) $J_{m,2}^+ := \{(w,+) : w \in \bigcup_{n=0}^{m-1} \widehat{W}_n^0 : Q_w^+ = \bigcup_{i=3}^{7} \mathcal{K}_{wi}\}$,
3) $J_{m,1}^- := \{(w,-) : w \in \bigcup_{n=0}^{m-1} \widehat{W}_n^0 : Q_w^- = \bigcup_{i=0}^{2} \mathcal{K}_{wi}\}$,
4) $J_{m,2}^- := \{(w,-) : w \in \bigcup_{n=0}^{m-1} \widehat{W}_n^0 : Q_w^- = \bigcup_{i=0}^{3} \mathcal{K}_{wi} \cup \mathcal{K}_{w7}\}$,
5) $J_m^0 := \{(w,0) : w \in \widehat{W}_m^0\}$.

Define an index set $I_m := J_{m,1}^+ \cup J_{m,2}^+ \cup J_{m,1}^- \cup J_{m,2}^- \cup J_m^0$, and a partition of $\mathcal{K}$ by $\mathcal{P}_m := \{Q_w^s\}_{(w,s)\in I_m}$, which satisfies

$$\mathcal{E}(v) = \sum_{(w,s)\in I_m} \mu_{\langle v\rangle}(Q_w^s), \quad \textit{for any } v \in \mathcal{F}. \tag{5.1}$$

We will define the function $\tilde{u}_m$ on each $Q_w^s$ and define a subset $E_{m,w}^s$ of $E_{\mathcal{L}}$ for all $(w,s) \in I_m$.

*1) The construction for* $(w,+) \in J_{m,1}^+$. Define

$$m_w := \sup\big\{n \in \mathbb{N} : n \leq m,\ \textit{there is } v \in \widehat{W}_n^0 \textit{ such that } F_v\big(\overline{q^{(4)}, q^{(6)}}\big) \subset F_{w6}\big(\overline{q^{(0)}, q^{(2)}}\big)\big\} - (|w|+1).$$

If $m_w = 0$, let $P_w^+ := \bigcup_{i\in\{4,5,6\}} P_{wi}$. Define $u''_m$ on $P_w^+$ by

$$u''_m(p) := \begin{cases} u'_m(p) & \textit{if } p \in P_w^+ \cap P_m \\ \frac{2}{3}u'_m(F_w(q^{(6)})) + \frac{1}{3}u'_m(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(6)}) \\ \frac{1}{3}u'_m(F_w(q^{(6)})) + \frac{2}{3}u'_m(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(4)}), \end{cases}$$

and define $\tilde{u}_m$ on $Q_w^+$ such that $\tilde{u}_m \circ F_{wi} = H_1[u''_m \circ F_{wi}]$ for $i \in \{4,5,6\}$. Denote

$$E_{m,w}^+ := \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_v \cap L), v \in \{w, w3, w7\}\big\}.$$

If $m_w \geq 1$, let

$$P_w^+ := \Big(\bigcup_{i\in\{4,5,6\}} P_{wi}\Big) \cup \Big(\bigcup_{i\in\{4,5,6\}} \bigcup_{j\in\{0,1,2\}} P_{wij}\Big) \cup \Big(\bigcup_{i\in\{4,6\}} \bigcup_{j\in\{0,1,2\}^{m_w}} \{F_{wij}(q^{(0)}), F_{wij}(q^{(2)})\}\Big).$$

See Figure 5.2 for an illustration.

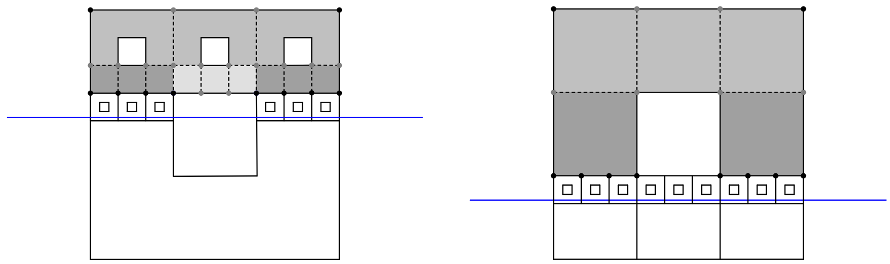

FIGURE 5.2. Examples of $P_w^+ \cap P_m$ (black points) and $P_w^+ \setminus P_m$ (grey points) with $m_w = 1$ for $(w,+) \in J_{m,1}^+$(left) and $J_{m,2}^+$(right).

Define $u_m''$ on each $P_w^+$ by

$$u_m''(p) := \begin{cases} u_m'(p) & \textit{if } p \in P_w^+ \cap P_m \\ \frac{2}{3}u_m'(F_w(q^{(6)})) + \frac{1}{3}u_m'(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(6)}) \\ \frac{1}{3}u_m'(F_w(q^{(6)})) + \frac{2}{3}u_m'(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(4)}) \\ u_m'(F_{wij}(q^{(0)})) & \textit{if } i \in \{4,6\}, j \in \{0,1,2\}, p = F_{wij}(q^{(6)}) \\ u_m'(F_{wi2}(q^{(2)})) & \textit{if } i \in \{4,6\}, p = F_{wi2}(q^{(4)}) \\ u_m'(F_{w6}(q^{(2)})) & \textit{if } p \in P_{w52}, \end{cases}$$

and define $\tilde{u}_m$ on $Q_w^+$ such that

$$\begin{cases} (\tilde{u}_m \circ F_{wi})|_{\mathcal{B}} = H_2[u_m'' \circ F_{wi}] & \textit{if } i \in \{4,5,6\} \\ \tilde{u}_m \circ F_{wij} = H_3^{(m_w)}[u_m'' \circ F_{wij}] & \textit{if } i \in \{4,6\}, j \in \{0,1,2\} \\ \tilde{u}_m \circ F_{w5j} = H_1[u_m'' \circ F_{w5j}] & \textit{if } j \in \{0,1,2\}. \end{cases}$$

Denote

$$E_{m,w}^+ := \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_v \cap L), \ \textit{where } v \in \bigcup_{n=0}^{m} \widehat{W}_n^0 \ \textit{such that}$$
$$F_v\big(\overline{q^{(4)}, q^{(6)}}\big) \subset \bigcup_{i\in\{4,6\}} F_{wi}\big(\overline{q^{(0)}, q^{(2)}}\big)\big\} \cup \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_w \cap L)\big\}.$$

*2) The construction for $(w,+) \in J_{m,2}^+$.* Define

$$m_w := \sup\big\{n \in \mathbb{N} : n \le m, \ \textit{there is } v \in \widehat{W}_n^0 \ \textit{such that } F_v\big(\overline{q^{(4)}, q^{(6)}}\big) \subset F_{w7}\big(\overline{q^{(0)}, q^{(2)}}\big)\big\} - (|w|+1),$$

and let

$$P_w^+ := \Big(\bigcup_{i\in\{4,5,6\}} P_{wi}\Big) \cup \Big(\bigcup_{i\in\{3,7\}} \bigcup_{j\in\{0,1,2\}^{m_w}} P_{wij}\Big).$$

See Figure 5.2 for an illustration.

Define $u''_m$ on each $P_w^+$ by

$$u''_m(p) := \begin{cases} u'_m(p) & \textit{if } p \in P_w^+ \cap P_m \\ \frac{2}{3}u'_m(F_w(q^{(6)})) + \frac{1}{3}u'_m(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(6)}) \\ \frac{1}{3}u'_m(F_w(q^{(6)})) + \frac{2}{3}u'_m(F_w(q^{(4)})) & \textit{if } p = F_{w5}(q^{(4)}) \\ u'_m(F_{wi}(q^{(0)})) & \textit{if } i \in \{3,7\}, p = F_{wi}(q^{(6)}) \\ u'_m(F_{wi}(q^{(2)})) & \textit{if } i \in \{3,7\}, p = F_{wi}(q^{(4)}), \end{cases}$$

and define $\tilde{u}_m$ on $Q_w^+$ such that

$$\begin{cases} \tilde{u}_m \circ F_{wi} = H_1[u''_m \circ F_{wi}] & \textit{if } i \in \{4,5,6\} \\ \tilde{u}_m \circ F_{wi} = H_3^{(m_w)}[u''_m \circ F_{wi}] & \textit{if } i \in \{3,7\}. \end{cases}$$

Denote

$$E_{m,w}^+ := \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_v \cap L), \ \textit{where } v \in \bigcup_{n=0}^{m} \widehat{W}_n^0 \ \textit{such that}$$
$$F_v\big(\overline{q^{(4)}, q^{(6)}}\big) \subset \bigcup_{i\in\{3,7\}} F_{wi}\big(\overline{q^{(0)}, q^{(2)}}\big)\big\} \cup \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_w \cap L)\big\}.$$

*3-4) The construction for* $(w,-) \in J_{m,1}^- \cup J_{m,2}^-$. In this case, we denote $\tilde{u}_{m,\mathcal{L}} := \tilde{u}_m$, $J_{m,i,\mathcal{L}}^s := J_{m,i}^s$ and $E_{m,w,\mathcal{L}}^s := E_{m,w}^s$ to emphasize the dependence on $L$. Let $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be the axial symmetry along $\overline{q^{(7)}, q^{(3)}}$. Note that $\tilde{u}_{m,\Gamma(\mathcal{L})}^+$ has been constructed on each $Q_w^+$ with $(w,+) \in J_{m,1,\Gamma(\mathcal{L})}^+ \cup J_{m,2,\Gamma(\mathcal{L})}^+$ by constructions 1 and 2.

Define $\tilde{u}_{m,\mathcal{L}}$ on $Q_w^-$ for $(w,-) \in J_{m,1,\mathcal{L}}^- \cup J_{m,2,\mathcal{L}}^-$ by $\tilde{u}_{m,\mathcal{L}}|_{Q_w^-} := \big(\tilde{u}_{m,\Gamma(\mathcal{L})} \circ \Gamma\big)|_{Q_w^-}$ and let $E_{m,w,\mathcal{L}}^- := \Gamma\big(E_{m,\Gamma(w),\Gamma(\mathcal{L})}^+\big)$.

*5) The construction for* $(w,0) \in J_m^0$. Define $\tilde{u}_m$ on $Q_w^0$ such that $\tilde{u}_m \circ F_w = H_1[u'_m \circ F_w]$, and let

$$E_{m,w}^0 := \big\{(x,y) \in E_{\mathcal{L}} : \{x,y\} = \partial(\square_w \cap L)\big\}.$$

Through constructions 1-5, we have defined $\tilde{u}_m$ on $\mathcal{K}$ and $E_{m,w}^s \subset E_{\mathcal{L}}$ for all $(w,s) \in I_m$. By Lemma 5.2, there is a constant $C > 0$ independent of the choice of $L, m, u$ and $(w,s) \in I_m$ such that

$$\mu_{\langle \tilde{u}_m \rangle}(Q_w^s) \le C \sum_{(x,y)\in E_{m,w}^s} r^{\frac{\log|x-y|}{\log \rho}} \big(u(x) - u(y)\big)^2. \tag{5.2}$$

Clearly, for each $(x,y) \in E_{\mathcal{L}}$, we have

$$\#\{(w,s) \in I_m : (x,y) \in E_{m,w}^s\} \le 3,$$

which combining with (5.1),(5.2) gives

$$\mathcal{E}(\tilde{u}_m) = \sum_{(w,s)\in I_m} \mu_{\langle \tilde{u}_m \rangle}(Q_w^s) \le 3C\Lambda_{\mathcal{L}}(u), \quad \textit{for all } m \ge 0. \tag{5.3}$$

*Proof of Proposition 5.1.* With equation (5.3) in hand, the proposition follows by a same argument as in Step 5 in Section 3. □

*Proof of Theorem 1.2.* Combining Proposition 4.1 and 5.1, Theorem 1.2 follows. □

## 6. Some remarks

**Remark 6.1.** We have proved

$$\mathcal{F}|_{\mathcal{L}} = \{u \in C(\mathcal{L}) : \Lambda_{\mathcal{L}}(u) < \infty\}.$$

Denote $\mathcal{L}_0 := \{x, y \in \mathcal{L} : (x, y) \in E_{\mathcal{L}}\}$, which is dense in $\mathcal{L}$. By definition, $\Lambda_{\mathcal{L}}(u)$ only depends on the values of $u$ on $\mathcal{L}_0$. We point out that for any $u \in \mathbb{R}^{\mathcal{L}_0}$, the condition $\Lambda_{\mathcal{L}}(u) < \infty$ is sufficient to imply that $u$ can be continuously extended to $\mathcal{L}$. Consequently, we have

$$\mathcal{F}|_{\mathcal{L}_0} = \{u \in \mathbb{R}^{\mathcal{L}_0} : \Lambda_{\mathcal{L}}(u) < \infty\}.$$

In fact, with a slight adjustment of the proofs for Propositions 3.1 and 5.1, we have

*for any $u \in \mathbb{R}^{\mathcal{L}_0}$ with $\Lambda_{\mathcal{L}}(u) < \infty$, there is $\tilde{u} \in \mathcal{F}$ such that $\tilde{u}|_{\mathcal{L}_0} = u$, and $\mathcal{E}(\tilde{u}) \leq c\Lambda_{\mathcal{L}}(u)$, where $c > 0$ is a constant independent of the choice of $L$ and $u$.*

We only provide the proof for the Sierpinski gasket, since that for the carpet is similar. We follow the same constructions and estimates in Steps 1-4 in Section 3. The only difference occurs in Step 5, where now we only need to prove $\tilde{u}(x) = u(x)$ for each $x \in \mathcal{L}_0$.

For $x \in \mathcal{L}_0$, we take a sequence $w^{(m)} \in \widehat{W}_m^0$ such that $x \in \partial(L \cap \triangle_{w^{(m)}})$, and let $y^{(m)} \in L \cap F_{w^{(m)}}(l^{(0)}), p^{(m)} = F_{w^{(m)}}(q^{(2)})$. Note that

$$\lim_{m\to\infty} |u(x) - u(y^{(m)})| \leq \lim_{m\to\infty} r^{-\frac{\log|x-y^{(m)}|}{2\log\rho}} \Lambda_{\mathcal{L}}^{\frac{1}{2}}(u) = 0,$$

which gives $\tilde{u}(x) = \lim_{m\to\infty} \tilde{u}_m(p^{(m)}) = \lim_{m\to\infty} u(y^{(m)}) = u(x)$. □

**Remark 6.2.** When $\mathcal{K}$ is the Sierpinski gasket or carpet, denote $R(x, y)$ the *resistance distance* between $x, y \in \mathcal{K}$ [12,13]. It's well known that $R(x, y)^{-1} \asymp r^{\frac{\log|x-y|}{\log\rho}}$ for $x, y \in \mathcal{K}$. So the trace theorems can also be expressed as

$$\mathcal{E}_{\mathcal{L}}(u) \asymp \sum_{(x,y)\in E_{\mathcal{L}}} R(x, y)^{-1}\big(u(x) - u(y)\big)^2.$$

**Remark 6.3.** For closed sets $A, A' \subset \mathcal{K}$, we denote the *Hausdorff distance* between $A, A'$ by

$$d_H(A, A') := \max\big\{\sup_{x\in A}\inf_{y\in A'} |x - y|, \sup_{y\in A'}\inf_{x\in A} |x - y|\big\}.$$

Suppose $A_n, A, n \geq 1$ are closed sets in $\mathcal{K}$ satisfying $\lim_{n\to\infty} d_H(A_n, A) = 0$. For functions $f_n \in \mathbb{R}^{A_n}, f \in \mathbb{R}^A$, we say $f_n$ *converges to* $f$ and denote $f_n \rightarrowtail f$ [4], if $\lim_{n\to\infty} f_n(x^{(n)}) = f(x)$ for any $x^{(n)} \in A_n, x \in A$ with $\lim_{n\to\infty} |x^{(n)} - x| = 0$. For resistance forms $(\mathcal{E}_n, \mathcal{F}_n)$ on $A_n$, $(\mathcal{E}, \mathcal{F})$ on $A$, we say $\mathcal{E}_n$ $\Gamma$-*converges to* $\mathcal{E}$ and denote $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ [4,7], if

Γ1) for any $f_n \in \mathcal{F}_n, f \in \mathcal{F}$ with $f_n \rightarrowtail f$, it holds that $\liminf_{n\to\infty} \mathcal{E}_n(f_n) \geq \mathcal{E}(f)$,
Γ2) for any $f \in \mathcal{F}$, there is a sequence $f_n \in \mathcal{F}_n$ such that $f_n \rightarrowtail f$ and $\lim_{n\to\infty} \mathcal{E}_n(f_n) = \mathcal{E}(f)$.

The trace forms have the following continuity property.

*C) For closed sets $A_n, A \subset \mathcal{K}, n \geq 1$ with $\lim_{n\to\infty} d_H(A_n, A) = 0$, it holds that $\mathcal{E}|_{A_n} \xrightarrow{\Gamma} \mathcal{E}|_A$.*

The proof consists of two steps. First we claim that $\liminf_{n\to\infty} \mathcal{E}|_A(f_n) \geq \mathcal{E}|_A(f)$ for any $f_n, f \in \mathcal{F}|_A$ with $f_n(x) \to f(x)$ for any $x \in A$. Take a sequence of finite sets $V_m \subset A, m \geq 1$ satisfying $V_m \subset V_{m+1}$, and $\bigcup_{m\geq 1} V_m$ dense in $A$. Then we have

$$\liminf_{n\to\infty} \mathcal{E}|_A(f_n) \geq \liminf_{n\to\infty} \mathcal{E}|_{V_m}(f_n|_{V_m}) = \mathcal{E}|_{V_m}(f|_{V_m}), \quad \textit{for any } m \geq 1. \tag{6.1}$$

By [13, Section 2.3], it holds $\mathcal{E}|_A(f) = \lim_{m\to\infty} \mathcal{E}|_{V_m}(f|_{V_m})$, which combining (6.1) gives the claim.

Next we turn to prove (C).

For ($\Gamma$1), we suppose $\liminf_{n\to\infty} \mathcal{E}|_{A_n}(f_n) = M < \infty$ and take a subsequence $n_k$ such that $\lim_{k\to\infty} \mathcal{E}|_{A_{n_k}}(f_{n_k}) = M$ with $\mathcal{E}|_{A_{n_k}}(f_{n_k}) \leq M+1$ for all $k$. Let $\tilde{f}_n \in \mathcal{F}$ such that $\tilde{f}_n|_{A_n} = f_n$ and $\mathcal{E}(\tilde{f}_n) = \mathcal{E}|_{A_n}(f_n)$. Since

$$M = \lim_{k\to\infty} \mathcal{E}|_{A_{n_k}}(f_{n_k}) = \lim_{k\to\infty} \mathcal{E}(\tilde{f}_{n_k}) \geq \liminf_{k\to\infty} \mathcal{E}|_A(\tilde{f}_{n_k}|_A),$$

by the previous step, it remains to prove $\tilde{f}_{n_k}(x) \to f(x)$ for any $x \in A$. Take a sequence $x^{(k)} \in A_{n_k}$ such that $\lim_{k\to\infty} x^{(k)} = x$. Then

$$\begin{aligned}
\lim_{k\to\infty} |\tilde{f}_{n_k}(x) - f(x)| &\leq \lim_{k\to\infty} |\tilde{f}_{n_k}(x) - \tilde{f}_{n_k}(x^{(k)})| + \lim_{k\to\infty} |\tilde{f}_{n_k}(x^{(k)}) - f(x)| \\
&\leq \lim_{k\to\infty} R(x, x^{(k)})^{\frac{1}{2}} \mathcal{E}|_{A_{n_k}}(f_{n_k})^{\frac{1}{2}} + \lim_{k\to\infty} |f_{n_k}(x^{(k)}) - f(x)| \\
&\leq \lim_{k\to\infty} R(x, x^{(k)})^{\frac{1}{2}} (M+1)^{\frac{1}{2}} + \lim_{k\to\infty} |f_{n_k}(x^{(k)}) - f(x)| = 0,
\end{aligned}$$

where $R$ is the resistance distance on $\mathcal{K}$.

For ($\Gamma$2), take $\tilde{f} \in \mathcal{F}$ such that $\tilde{f}|_A = f$ and $\mathcal{E}(\tilde{f}) = \mathcal{E}|_A(f)$. Let $f_n := \tilde{f}|_{A_n}$ for each $n \geq 1$. For any $x^{(n)} \in A_n, x \in A$ with $x^{(n)} \to x$, we have

$$|f_n(x^{(n)}) - f(x)| = |\tilde{f}(x^{(n)}) - \tilde{f}(x)| \leq R(x^{(n)}, x)^{\frac{1}{2}} \mathcal{E}|_A(f)^{\frac{1}{2}} \to 0,$$

which gives $f_n \rightarrowtail f$. Then ($\Gamma$2) follows by $\mathcal{E}|_{A_n}(f_n) \leq \mathcal{E}(\tilde{f}) = \mathcal{E}|_A(f)$ for all $n$. $\square$

Suppose $\mathcal{K}$ is the Sierpinski gasket $\mathcal{SG}$. For $a, b \in \mathbb{R}$, denote $L_{a,b}$ as the graph of the function $l(x) = bx + a$, and $\mathcal{L}_{a,b} := L_{a,b} \cap \mathcal{K}$. Fix $b$ such that $L_{0,b}$ is not parallel to any edge of $\triangle$. Then clearly $\mathcal{L}_{a,b}$ is continuous with respect to $a$ in Hausdorff distance, so $\mathcal{E}_{\mathcal{L}_{a,b}}$ is also continuous with respect to $a$ in the sense of $\Gamma$-convergence.

In a same way, we can define the concept "$\Gamma$-*convergence*" for the collection of forms $\{\Lambda_{\mathcal{L}}\}$.

**Question 6.4.** For $\mathcal{L}_n, \mathcal{L}, n \geq 1$ with $\lim_{n\to\infty} d_H(\mathcal{L}_n, \mathcal{L}) = 0$, do we have $\Lambda_{\mathcal{L}_n} \xrightarrow{\Gamma} \Lambda_{\mathcal{L}}$?

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Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China.
*Email address*: jiejiec@smail.nju.edu.cn

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China.
*Email address*: huaqiu@nju.edu.cn

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China.
*Email address*: 652022210012@smail.nju.edu.cn